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A CONDITION ALLOWING THE REDUCTION OF THE GENUS OF A HEEGAARD SPLITTING

LLOYD G. ROELING

In [1] it is shown that every 3-manifold can be given a combinatorial triangulation. It follows from this that any orientable, closed 3-manifold M can be represented as $H \cup H'$ where H and H' are solid tori of genus n (i.e., homeomorphic to regular neighborhoods of compact connected graphs with Euler characteristic $1 - n$) and $H \cap H' = \partial H = \partial H' = T$ is an orientable surface of genus n. This is called a Heegaard splitting of genus n for M. It is known that the 3-sphere S^3 is the only simply connected such manifold with a Heegaard splitting of genus 1. (Manifolds with Heegaard splittings of genus 1 are called lens spaces.) Thus, a possible approach to the Poincaré conjecture is to find conditions under which the genus of a Heegaard splitting for any homotopy 3-sphere might be reduced. We give here (Theorem 2) one such set of conditions.

All spaces considered will be polyhedra and all maps will be piecewise linear. The following characterization is an easy consequence of Dehn's Lemma [3], the loop theorem [4] and Poincaré duality.

Proposition 1. *Let H be a compact, connected 3-manifold with connected boundary. Assume $\pi_1(H)$ is a free group of rank n. Then H is a solid torus of genus n if and only if H can be embedded in R^3.*

Now consider a compact, orientable surface F with nonempty boundary. Let $r = g(F)$ be the genus of F and s be the number of boundary components. Then F has Euler characteristic $\chi(F) = 2 - 2r - s$ and $\pi_1(F)$ is free of rank $2r + s - 1$. Hence, $F \times [0, 1]$ is a solid torus of genus $2r + s - 1$.

Theorem 2. *Suppose $H \cup H'$ is a Heegaard splitting of genus n for the closed 3-dimensional manifold M. Assume that F is a compact, connected, orientable surface with nonempty boundary and that $h : F \times [0, 1] \to H'$ is a homeomorphism such that $h(\partial F \times 0)$ bounds an orientable, not necessarily connected, surface G properly embedded in H with $\chi(G) > 1 - n$. Then there is a Heegaard splitting for M of genus less than n.*

Proof. Assume that G is chosen so that $\chi(G)$ is maximal. (Note that

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$\chi(G)$ is no greater than the number of components of ∂F.) Since $H_2(H; Z) = 0$, the boundary induced map $\partial_*\colon H_2(H, T; Z) \to H_1(T; Z)$ must be monic. However, ∂_* maps the element of $H_2(H, T; Z)$ represented by G onto the element of $H_1(T;Z)$ represented by $h(\partial F)$, which is zero. Therefore G must separate H.

Now we may choose a thickening $G \times [0, 1]$ of G in H so that G corresponds to $G \times 0$ and $\partial G \times [0, 1] = h(\partial F \times [0, 1])$ in T. This may be done in such a way that the two fiberings over ∂G are the same, for if not, then there is a simple closed curve in M whose regular neighborhood is a solid Klein bottle.

Let $F^* = G \cup h(F)$. Then $H' \cup (G \times [0, 1])$ is just $F^* \times [0, 1]$ and $g(F^*) = 1 - 1/2(1 - n + \chi(G)) < n$, since $\chi(G) > 1 - n$. Then $\mathrm{Cl}(M - (H' \cup G \times [0, 1]))$ has two boundary components and, in fact, has two components since G separates H. Let H_1 and H_1' be the closures of these components.

Now we claim that for some $x_0 \in G$ the inclusion induced homomorphism $\mu_*\colon \pi_1(H_1, x_0) \to \pi_1(H, x_0)$ is a monomorphism. If not, then by Dehn's Lemma and the loop theorem there is a simple closed curve J in G that bounds a disk D in H, but does not bound a disk in H_1. Let $N(J)$ be a regular neighborhood of J in G and let D_1 and D_2 be disjoint disks in H bounded by the boundary components of $N(J)$ so that $D_i \cap G = \partial D_i$ for $i = 1, 2$. Let $G' = \mathrm{Cl}(G - N(J)) \cup (D_1 \cup D_2)$. Then G' is an orientable surface properly embedded in H with $\partial G' = h(\partial F \times 0)$.

If J does not separate G, then $g(G') < g(G)$, so $\chi(G') > \chi(G)$. Suppose J separates G into components A and B. If both A and B meet T, then $\chi(G') = \chi(G) + 2$. If one of A and B, say B, misses T, then the component of G' meeting B is a closed surface in int (H). This component may be removed from G' to get an orientable surface G'' properly embedded in H and bounded by $h(\partial F \times 0)$ with $\chi(G'') > \chi(G)$. So in any case we increase the Euler characteristic thereby contradicting the maximality of $\chi(G)$.

Similarly, we may show that $\nu_*\colon \pi_1(H_1', x_0) \to \pi_1(H, x_0)$ is monic. Hence, both $\pi_1(H_1)$ and $\pi_1(H_1')$ are free, so H_1 and H_1' must be solid tori by Proposition 1. Since $\mathrm{Cl}(M - (H_1 \cup H_1')) = F^* \times [0, 1]$, we get a Heegaard splitting of genus $g(F^*) < n$. $\square$

If H is a solid torus of genus n and $D_1, \cdots, D_k$ ($k \leqq n$) are properly embedded, pairwise disjoint disks in H so that $\mathrm{Cl}(H - \bigcup_{i=1}^k N(D_i))$ is a solid torus of genus $(n - k)$ ($N(D_i)$ is a regular neighborhood of D_i in H), then $D_1, \cdots, D_k$ is called a set of cutting disks for H.

LEMMA 3. *Let H be a solid torus of genus $2n$ and J be a simple closed curve in ∂H such that J separates ∂H into two components* *whose closures we denote by* F_1 *and* F_2. *For* $i = 1, 2$, *let* $\mu_i : F_i \to H$ *be inclusion and choose a point* x_0 *in* J. *Assume that* $\mu_{i*} : \pi_1(F_i, x_0) \to \pi_1(H, x_0)$ *is monic for* $i = 1, 2$. *Then* μ_{i*} *is an isomorphism for* $i = 1, 2$ *if and only if there is a set of cutting disks* $D_1, \cdots, D_{2n}$ *of* H *so that each* D_i *meets* J *exactly twice.*

Proof. Let N be a regular neighborhood of $J \cup (\bigcup_{i=1}^{n} \partial D_i)$ in ∂H. Since $\bigcup_{i=1}^{n} \partial D_i - J$ has $4n$ components, $\chi(N) = -4n$. But $\chi(\partial H) = 2 - 4n$ and $\mathrm{Cl}(\partial H - N)$ is a collection of r disjoint disks, for otherwise, the kernel of μ_{i*} is nontrivial. Then $\chi(\partial H) = \chi(N) + r$, so $r = 2$. Therefore N has exactly two boundary components. Hence for each $i = 1, 2$, $\mathrm{Cl}(F_i - N)$ is a 2-cell, so there are simple closed curves $J_1, \cdots, J_{2n}$ in F_i such that $J_j \cap \partial D_j$ is a point for $j = 1, \cdots, 2n$ and $J_j \cap \partial D_k = \emptyset$ if $k \neq j$. No two of these curves can be homotopic in F_i since no two are homotopic in H. Thus μ_{i*} is an epimorphism and hence an isomorphism.

Conversely, if each μ_{i*} is an isomorphism, then Brown [2] has shown that H is homeomorphic to $F_1 \times [0, 1]$ and F_1 is a surface of genus n with one boundary component. Then there are properly embedded, pairwise disjoint arcs $A_1, \cdots, A_{2n}$ in F_1 so that $\mathrm{Cl}(F_1 - \bigcup_{i=1}^{n} N(A_i))$ is a 2-cell. Then the 2-cells $A_1 \times [0, 1], \cdots, A_{2n} \times [0, 1]$ form the required cutting disks. □

Let M and N be closed, orientable 3-manifolds and let $B \subset M$, $E \subset N$ be 3-cells. Choose an orientation reversing homeomorphism $h : \partial B \to \partial E$. Then $M \# N = (M - \mathrm{int}(B)) \cup_h (N - \mathrm{int}(E))$ is called the connected sum of M and N. This is independent of the choices of B, E and h.

Corollary 4. *Let* M *be a closed 3-manifold with a Heegaard splitting* $H \cup H'$ *of genus 2. Let* $\{D_1, D_2\}$ *be a set of cutting disks for* H' *and* $J_i = \partial D_i \subset \partial H$ *for* $i = 1, 2$. *Suppose there is a properly embedded separating disk* D *in* H *such that* ∂D *does not contract in* ∂H *but meets each of* J_1 *and* J_2 *exactly twice. Then* M *is either a lens space or the connected sum of two lens spaces.*

Proof. Let $J = \partial D \subset \partial H'$. Since J_1 and J_2 bound cutting disks for H', there are free generators α and β for $\pi_1(H', x_0)$ ($x_0 \in J$) such that the conjugacy class in $\pi_1(H', x_0)$ determined by J, say $C(J)$, is either trivial or the class of $\alpha\beta\alpha^{-1}\beta^{-1}$. This is true since J separates $\partial H'$, and so J must cross each ∂D_i once from each side of D_i.

If $C(J)$ is trivial, then by Dehn's Lemma, J bounds a properly embedded 2-cell D' in H'. Since J separates $\partial H'$, D' must separate H'. Hence, $D \cup D'$ is a separating 2-sphere in M. Now the closure of each component of $M - (D \cup D')$ is the union of two solid tori of genus 1.

These two solid tori meet in the complement of an open 2-cell in their respective boundaries. Hence, the closure of each component of $M - (D \cup D')$ is a lens space with the interior of a 3-cell removed. Therefore, M is the connected sum of two lens spaces.

Now suppose $C(J)$ is the same as the conjugacy class of $\alpha\beta\alpha^{-1}\beta^{-1}$. Let F_1 and F_2 be the closures of the components of $\partial H' - J$ and let $\mu_{i*} : \pi_1(F_i, x_0) \to \pi_1(H', x_0)$ be the inclusion induced homomorphism for $i = 1, 2$. Suppose μ_{i*} is not monic. Then there is, by Dehn's Lemma and the loop theorem, a properly embedded disk D' in H' whose boundary lies in F_1, but does not bound a disk in F_1.

Since $\chi(\partial H') = \chi(F_1) + \chi(F_2)$, we have that $g(F_1) + g(F_2) = g(\partial H') = 2$. Since J does not contract in $\partial H'$, $g(F_1)$ and $g(F_2)$ are nonzero. So $g(F_1) = g(F_2) = 1$. Suppose $\partial D'$ separates F_1. Then $\partial D'$ is either contractible in F_1 or homotopic to J, both of which are impossible. So assume $\partial D'$ does not separate F_1. Then $\partial D'$ does not separate $\partial H'$, so D' is a cutting disk for H'. Hence J is a separating simple closed curve for the boundary of the solid torus of genus 1, $\mathrm{Cl}(H' - N(D'))$, where $N(D')$ is a regular neighborhood of D'. Thus J must bound a disk in $\mathrm{Cl}(H' - N(D'))$ and so in H'. But this cannot happen. Hence μ_{1*} must be monic. Similarly, we can show that μ_{2*} must be monic.

Hence, by Lemma 3, μ_{1*} and μ_{2*} are isomorphisms. Therefore, by [2], H' is homeomorphic to $F_1 \times [0, 1]$. But $\partial F_1 \times \{0\} = J$ bounds in H the orientable surface D with $\chi(D) = 1 > 1 - 2$. So by Theorem 2, M has a Heegaard splitting of genus less than 2. Thus M is a lens space.

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University of Southwestern Louisiana, Lafayette, Louisiana 70501