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THE DOUBLE CENTRALIZER PROPERTY IS CATEGORICAL[1]

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The purpose of this note is to prove the result stated in the title. This answers a question raised by E. A. Walker at the summer symposium on ring theory at Appalachian State University in August, 1969, as to whether being QF-1 is categorical. This also seems timely since many papers have appeared recently studying the double centralizer property for modules, for instance [2], [4], and [6].

Let R and S be associative rings with identity such that the categories ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ of left R-modules and left S-modules respectively are equivalent. Then by [7, Theorem 3.5] or [1, Theorem 3.4, p. 62] there is a right R-progenerator P_R with $S \simeq \mathrm{End}_R(P_R)$ such that the functor $F = {}_SP_R \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ gives the equivalence; we say R and S are Morita equivalent via ${}_SP_R$.

Let M be a left R-module and let $C = \mathrm{End}_R({}_RM)$. Then the map $C \to \mathrm{End}_S({}_SP \otimes_R M)$ via $f \to 1_P \otimes f$ is a unital ring isomorphism so we identify C with $\mathrm{End}_S(P \otimes_R M)$. Let $D = \mathrm{End}_C(M_C)$ and $E = \mathrm{End}_C(P \otimes_R M_C)$ be the double centralizers of ${}_RM$ and ${}_SP \otimes_R M = {}_SF(M)$ respectively; note that we write homomorphisms opposite scalars. We say that M has the *double centralizer property* (DCP) if the natural map η: $R \to \mathrm{End}_C(M_C)$ via $\eta(r)(m) = rm$ is onto. Equivalently, if $\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$, M has the DCP if the natural map $R/\mathrm{Ann}_R(M) \to \mathrm{End}_C(M)$ is an isomorphism.

Finally, we say that two modules N and N' are *similar* if each is isomorphic to a direct summand of a finite direct sum of copies of the other and in this case we write $N \sim N'$.

THEOREM. *Let R and S be Morita equivalent via the module ${}_SP_R$. If a left R-module ${}_RM$ has the double centralizer property, so does ${}_SF(M) = {}_SP \otimes_R M$.*

PROOF. Since P_R is a progenerator, $P_R \sim R_R$, so $P \otimes_R M \sim R \otimes_R M \cong M$ as abelian groups, and the action of C on $P \otimes_R M$ makes

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[1]K. R. Fuller tells us this result also appears in an unpublished paper by Morita and Tachikawa. We feel our proof is worth recording since it was obtained independently and no proof is available in print.

$P \otimes_R M \sim M$ as right C-modules. Thus by [5, Theorem 1.5], D and E are Morita equivalent via the module ${}_E\mathrm{Hom}_C(M, P \otimes_R M)_D$. Our proof of the theorem will be in two parts.

(a) If ${}_RM$ is faithful and has the DCP we may identify R and D. Then the Morita equivalences of S and R and of R and E give a Morita equivalence of S and E via the module ${}_E\mathrm{Hom}_C(M, P \otimes_R M) \otimes_R P_S{}^*$ where $P^* = \mathrm{Hom}_R(P, R)$. Applying the Hom-Tensor identities [3, VI, Proposition 5.2] and [3, II, Proposition 5.2], recalling that P_R is a progenerator, we have

$$\begin{aligned}\mathrm{Hom}_C(M, P \otimes_R M) \otimes_R P^* &\simeq \mathrm{Hom}_C(\mathrm{Hom}_R(P^*, M), P \otimes_R M)\\ &\simeq \mathrm{Hom}_C(P \otimes_R M, P \otimes_R M) \simeq E\end{aligned}$$

as left E-right S-bimodules. Thus ${}_EE_S$ gives a Morita equivalence of E and S and we conclude by [1, Theorem 3.5, p. 65] that $S \simeq \mathrm{End}_E(E) \simeq E$. Since this isomorphism is the canonical mapping and ${}_SF(M)$ is faithful [1, Theorem 3.5, p. 65], ${}_SP \otimes_R M = {}_SF(M)$ has the DCP.

(b) If ${}_RM$ is not faithful, let $A = \mathrm{Ann}_R(M)$. Then the ideal A' of S corresponding to A under the lattice isomorphism of ideals of R and S is isomorphic to $\mathrm{Hom}_R(P, PA)$ and is the annihilator in S of ${}_SP \otimes_R M$ [1, Theorem 3.5, p. 65]. Let $R' = R/A$ and $S' = S/A'$. Clearly P/PA is a right R'-progenerator. Since P_R is projective, the natural ring homomorphism from $\mathrm{End}_R(P)$ to $\mathrm{End}_{R'}(P/PA)$ via $f \to \bar{f}$ where $\bar{f}(p + PA) = f(p) + PA$ is surjective with kernel $\mathrm{Hom}_R(P, PA)$. It follows as noted above that $S' \simeq \mathrm{End}_{R'}(P/PA)$ so that R' and S' are Morita equivalent via ${}_{S'}P/PA_{R'}$. Since M is faithful over R', we may apply the first part of the proof to conclude that ${}_SP \otimes_R M$ has the DCP provided we show that ${}_SP \otimes_R M \simeq {}_SP/PA \otimes_{R'} M$. But $0 \to PA \to P \to P/PA \to 0$ is an exact sequence of left S-right R-bimodules, so tensoring with ${}_RM$ we have the exact sequence of left S-modules

$$PA \otimes_R M \to P \otimes_R M \to P/PA \otimes_R M \to 0.$$

Since the image of $PA \otimes_R M$ in $P \otimes_R M$ is zero, this shows $P \otimes_R M \simeq P/PA \otimes_R M \simeq P/PA \otimes_{R'} M$ as left S- modules.

A ring R is called *left* QF-1 if every faithful left R-module has the DCP, while R is called *balanced* if every left R-module has the DCP. See [2] for further background on these types of rings.

COROLLARY. *Let R and S be Morita equivalent.*

(a) *If R is left QF-1, then S is left QF-1.*

(b) *If R is balanced, then S is balanced.*

In particular, if S is a full $n \times n$ matric ring over R, then S is left QF-1 or balanced if and only if R is.

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