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ON INNER DERIVATIONS OF MALCEV ALGEBRAS

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1. **Introduction.** This paper generalizes a result due originally to Sagle [4] on inner derivations. In 1955, A. I. Malcev introduced a new product defined by a commutator in an alternative algebra. He called this structure a Moufang-Lie algebra. Sagle [3] developed some of the structure theory of these algebras and named them Malcev algebras. A Malcev algebra A is defined to be a nonassociative algebra which satisfies the identities:

(i) $x^2 = 0$ for x in A,

(ii) $(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$ for x, y, z in A.

Throughout this paper A will denote a finite-dimensional Malcev algebra over a field F of arbitrary characteristics unless otherwise specified. The product of any two elements x, y of A will be denoted by juxtaposition, xy. For x in A let $R(x)$ denote the linear map $a \rightarrow ax$ for every a in A and let $R(B)$ be the linear space spanned by all $R(y)$ for y in B. Let $J(A, A, A)$ be the linear space spanned by all elements of the form $J(x, y, z) = (xy)z + (yz)x + (zx)y$ for x, y, z in A. Recall that the J-nucleus N of A is defined by $N = \{x \in A : J(x, A, A) = 0\}$. Schafer [5] defines the Lie multiplication algebra $L(A)$ of an arbitrary nonassociative algebra. Let $[R(x), R(y)]$ be the commutator of any two elements $R(x)$, $R(y)$ where x and y are in A. Sagle [3] shows $L(A) = R(A) + [R(A), R(A)]$ if A is a Malcev algebra. A derivation of an algebra A is a linear map D of A such that $(xy)D = (xD)y + x(yD)$ for every x, y in A. A derivation D of a Malcev algebra is inner if D is in $L(A)$. The main result is: If A is a Malcev algebra over a field F of characteristic unequal to 2 or 3 and the Killing form on A and $L(A)$ is nondegenerate then every derivation of A is inner. From this result we obtain the fact that if F has zero characteristic, then every derivation of A, where A is a semisimple Malcev algebra, is an inner derivation.

2. **Inner derivations of Malcev algebras.** Recall that if A is a semisimple Malcev algebra, then A is a direct sum of ideals which are simple algebras.

LEMMA 1. *If A is a semisimple Malcev algebra over a field F of characteristic unequal to 2 or 3 and $f(x, y) =$ Tr $R(x)R(y)$, for x, y in*

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A, is a nondegenerate bilinear form on A, then f(,) is nondegenerate on N.

Proof. Let $R(x)|N$ be the restriction map of the linear map $R(x)$ to the subspace N. Since A is semisimple, then by [3, Theorem 5.17, p. 441], $A = N \oplus J(A, A, A)$ is a direct sum of ideals N and $J(A, A, A)$. Thus for x in A all the nonzero entries of the matrix of $R(x)$ are the same as the matrix of $R(x)|N$ and $\mathrm{Tr}\, R(x)R(y) = \mathrm{Tr}(R(x)|N)(R(y)|N)$ for every x, y in N.

Now suppose $f(x, y)|N = \mathrm{Tr}(R(x)|N)(R(y)|N) = 0$ for each x, y in N. For $a \in A$ and $x, y \in N$, $f(xy, a) = f(x, ya) = f(x, ya)|N = 0$. Therefore $f(xy, a) = 0$ for all a in A and x, y in N. Since $f(,)$ is nondegenerate on A, $xy = 0$ for $x, y \in N$. Consequently $N^2 = 0$ and N is a commutative ideal of A. This contradicts [3, Lemma 7.18, p. 452].

Let I be the linear space spanned by all derivations of the form

$$R(n) + \sum_{i=1}^{k} D(x_i, y_i) \quad \text{for } n \in N \text{ and } x_i, y_i \in A,$$

where $D(x_i, y_i) = [R(x_i), R(y_i)] + R(x_i y_i)$, and let $R(J(A, A, A))$ be the linear space spanned by all $R(z)$ for $z \in J(A, A, A)$.

Lemma 2. *If A is a semisimple Malcev algebra over a field F of characteristic unequal to 2 or 3, $L(A) = I \oplus R(J(A, A, A))$ as a linear space direct sum.*

Proof. Since A is semisimple $A = N \oplus J(A, A, A)$ by [3, Theorem 5.17, p. 441]. Recall that if $n, m \in N$, $[R(n), R(m)] = R(nm)$. Also if $x, y \in J(A, A, A)$ then $[R(x), R(y)] = D(x, y) - R(xy) \in I + J(A, A, A)$. Thus $L(A) = I + R(J(A, A, A))$. Clearly, $I \cap R(J(A, A, A)) = 0$.

Theorem 3. *Let A be a finite-dimensional Malcev algebra over a field F of characteristic unequal to 2 or 3. If the Killing form on A and $L(A)$ is nondegenerate then every derivation of A is an inner derivation.*

Proof. Dieudonné's theorem (see [5, p. 24]) and the nondegeneracy of the Killing form on A imply that A is semisimple. By [3, Theorem 5.17, p. 441], $A = N \oplus J(A, A, A)$ is a direct sum of ideals.

Case 1. Suppose $N = 0$. By Lemma 2 above $L(A) = I \oplus R(J(A, A, A)) = I \oplus R(A)$ since $A = J(A, A, A)$. Consider the map G: $L(A) \to L(A)$ defined by $(L)G = [L, D]$ for L in $L(A)$ and D a derivation of A. The linear map G is a derivation of the Lie algebra $L(A)$. Since $N = 0$, [3, Theorem 5.9, p. 439] implies that $L(A) =$

$\Delta(A, A)$. By [3, Proposition 8.14, p. 454], $[\Delta(A, A), D(A)] \subset \Delta(A, A) = L(A)$. Therefore $[L, D] \in L(A)$ for each L in $L(A)$ and D in $D(A)$. Since $L(A)$ is a Lie algebra over F and the Killing form is nondegenerate on $L(A)$, [1, Theorem 6, p. 74] implies that every derivation of $L(A)$ is inner. Thus there exists S in $L(A)$ so that $G = R(S)$, i.e., $[L, D] = [L, S]$ for all L in $L(A)$. Now for S in $L(A)$ we have that $S = D_1 + R(z)$ where z is an element of A and D_1 is in I. Therefore, for x in A,

$$\begin{aligned}[R(x), R(z)] &= [R(x), S - D_1] = [R(x), S] - [R(x), D_1] \\ &= [R(x), D] - [R(x), D_1], \quad \text{for some } D \text{ in } D(A), \\ &= [R(x), D - D_1] \\ &= [R(x), \tilde{D}] = R(xD).\end{aligned}$$

Note that D is a derivation of A. Now $R(x(\tilde{D} + R(z))) = D(x, z)$. By [3, Proposition 8.3, p. 453], $D(x, z)$ is a derivation of A. Since $D(x, z)$ is a right multiplication and $N = 0$, [3, Theorem 8.5, p. 453] implies that $0 = x(\tilde{D} + R(z))$. Thus $z = 0$ and $D = S$ is an inner derivation of A.

Case 2. Let $A = N \oplus J(A, A, A)$ and N be unequal to zero. N is isomorphic to $A/J(A, A, A)$. Thus N is a semisimple Lie algebra. Also $B = J(A, A, A)$ is a semisimple Malcev non-Lie algebra, i.e., $N(B) = \{x \in B : J(x, B, B) = 0\} = 0$. Case 1 above implies that all derivations of $J(A, A, A)$ are inner. The Killing form is nondegenerate on N by Lemma 1. Therefore by [1, Theorem 6, p. 74] all derivations of N are inner derivations. [6, Theorem 4, p. 772] implies that all derivations of A are inner.

COROLLARY 4. *If A is a semisimple Malcev algebra over a field F of characteristic zero, then every derivation of A is inner.*

PROOF. It suffices to show that the Killing form on A and $L(A)$ is nondegenerate. [3, Corollary 5.32, p. 444] and [3, Corollary 7.3, p. 447] imply that $L(A)$ is a semisimple Lie algebra. Cartan's criterion [1, p. 69] proves that the Killing form on $L(A)$ is nondegenerate.

Since A is semisimple, A is a direct sum of simple ideals. Thus $A = A_1 \oplus \cdots \oplus A_n$ where A_i is a simple ideal. Let R be the unique maximal solvable ideal of A. [2, Lemma 4, p. 556] implies that $R = R_1 \oplus \cdots \oplus R_n$ where R_i, for $i = 1, \cdots, n$, is the unique maximal solvable ideal of A_i. Since A_i, for $i = 1, \cdots, n$, is simple, $R_i = 0$ for each i. Thus $R = 0$. [2, Theorem A] implies that the Killing form on A is nondegenerate.

THEOREM 5. *If A is a semisimple Malcev algebra over a field F of characteristic zero, then the Lie algebra D(A) of all derivations of A is completely reducible in A.*

PROOF. Corollary 4 and the fact that A is semisimple implies that all derivations of A are inner derivations. Thus $L(A) = D(A) \oplus R(J(A, A, A))$ as a linear space direct sum by Lemma 2. $L(A)$ is completely reducible in A. [1, Theorem 17, p. 100] implies that every nonzero element of $L(A)$ can be imbedded in a 3-dimensional split simple subalgebra of $L(A)$ and $L(A)$ is almost algebraic. Now $[R(J(A, A, A)), D(A)] \subset R(J(A, A, A))$. Thus by [1, Lemma 8, p. 99] every nonzero nilpotent element of $D(A)$ can be imbedded in a 3-dimensional split simple subalgebra of $D(A)$. The Lie algebra $D(A)$ of derivations of a finite-dimensional algebra is almost algebraic by [1, Exercise 8, p. 54]. Since $D(A)$ is almost algebraic, its center is almost algebraic. So by [1, Theorem 17, p. 100], $D(A)$ is completely reducible.

COROLLARY 6. *Let A be a semisimple Malcev algebra over a field F of characteristic zero. The Lie algebra* $D(A) = C(D(A)) \oplus D_1(A)$ *is a direct sum of ideals where* $C(D(A))$ *is the center of* $D(A)$ *and* $D_1(A)$ *is a semisimple ideal of* $D(A)$. *Also the elements of* $C(D(A))$ *are semisimple.*

PROOF. $D(A)$ is a completely reducible Lie algebra of linear transformations. By [1, Theorem 10, p. 81] the result follows.

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