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APPLICATIONS OF THE THEORY OF IMAGINARY POWERS OF OPERATORS

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ABSTRACT. Imaginary powers of directional derivatives and of certain other operators are used to study semigroups which arise in the analysis of singular integral operators. Imaginary powers of directional derivatives are used to estimate the maximal functions and the Littlewood-Paley g-function of the Poisson integral on a Hilbert space.

I. **Introduction.** The purpose of this paper is to study some of the implications of the existence as bounded operators of purely imaginary powers of the infinitesimal generators of certain semigroups. The setting of the paper will be Classical Analysis on Hilbert Space.

Let H be a real separable Hilbert space and let $L_p(H)$ denote the Banach space of p-power integrable functions with respect to the normal distribution with variance parameter 1. Let $y \to T_y$ denote the regular representation of the additive group of H as isometries on $L_p(H)$. Fix p in $1 < p < \infty$. Let B denote a one-one Hilbert-Schmidt operator on H and let n_t denote the normal distribution on H with variance parameter $t/2$. Then $n_t \circ B^{-1}$ is a Borel probability measure on H; for f in $L_p(H)$, set

$$H_t(f) = \int_H T_y f \, dn_t \circ B^{-1}(y),$$

$$P_z(f) = \int_0^\infty H_t(f) N_t(z) \, dt/t$$

where $N_t(z) = (\pi t)^{-1/2} z \exp(-t^{-1}z^2)$. $P_z(f)$ is the Poisson integral of f. If $(-D_h)$ denotes the infinitesimal generator of the translation semigroup T_{th}, $t > 0$, and if $(-T)$ denotes the infinitesimal generator of P_z, $z > 0$, then $(D_h)^{ic}$ and T^{ic} are strongly continuous groups of

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bounded operators on $L_p(H)$. In addition, the analytic semigroup

$$J_r^\alpha(f) = \Gamma(\alpha)^{-1}\int_0^\infty P_t(f)t^{\alpha-1}e^{-rt}\,dt, \quad r>0,\ \mathrm{Re}(\alpha)>0,$$

extends to a strongly continuous semigroup on $L_p(H)$ in $\mathrm{Re}(\alpha) \geqq 0$. By using these facts and an interpolation theorem due to E. M. Stein, we shall study the semigroups I^α of powers of the indefinite integral, $(D_{Bh})^\alpha J_r^\alpha$, $T^\alpha J_r^\alpha$, and $(D_{Bh})^\alpha T^{-\alpha}$. Results concerning these semigroups will be applied to the study of singular integral operators.

The boundedness of imaginary powers of certain operators will also be applied to the study of the maximal functions and the Littlewood-Paley g-function for the Poisson integral.

The following table summarizes the contents and organization of the paper.

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Throughout this paper A, $A(x)$, $A(x, y)$, etc. denote positive constants which depend only on the parameters shown; and K, $K(x)$, $K(x, y)$, etc. denote complex constants which depend only on the parameters shown. The value of these constants may vary with the occasion of their use. If T is an operator defined in a Banach space X to X, $D(T)$ denotes the domain of T and $R(T)$ denotes the range of T. If $1 < p < \infty$, q denotes the real number conjugate to p; $p^{-1} + q^{-1} = 1$. $\langle f, g\rangle = \int_S f(s)g(s)\,d\mu(s)$ denotes the dual pairing between $L_p(S, \mu)$ and $L_q(S, \mu)$. An operator $T : L_p \rightarrow L_p$ has norm $\|T\|_p$.

Some of the results reported in this paper appeared in other forms in the papers [7], [8], [9], [10]. They are repeated here in order to give a more complete picture of the uses of imaginary powers of operators.

II. Preliminaries.

1. *The normal distribution on Hilbert space.* To minimize the discussion of measure theory on Hilbert space we refer the reader to papers [11], [12], [13] of L. Gross and [19] of I. E. Segal.

DEFINITION. A weak distribution on a real Hilbert space, H, is an equivalence class of linear maps, F, from the conjugate space H^* of H to real valued measurable functions (modulo null functions) on a probability space (depending on F). Two such maps, F and F', are equivalent if for any finite set of vectors $y_1, \cdots, y_k$ in H^*, $F(y_1), \cdots, F(y_k)$ and $F'(y_1), \cdots, F'(y_k)$ have the same joint distribution in k-space. A weak distribution is continuous if a representative is a continuous linear map (the range space has the topology of convergence in measure).

In what follows we shall be most interested in the normal distribution with variance parameter $c/2 > 0$. This distribution is uniquely determined by the following properties: (1) for any y in H^*, $n_c(y)$ is normally distributed with mean zero and variance $(c/2)\|y\|^2$; (2) n_c maps orthogonal vectors to independent random variables. The normal distribution is continuous. There is an essentially unique (up to expectation preserving isomorphism) probability space (S, Σ, μ) and a continuous linear map, F, from H^* to the real valued measurable functions on (S, Σ, μ) such that F is a representative of the normal distribution. Σ has no proper sub-σ-fields with respect to which all of the $F(y)$, y in H^*, are measurable. The measurable functions on H are the measurable functions on (S, Σ, μ). $L_p(H, n_c) = L_p(S, \Sigma, \mu)$, by definition. When the variance parameter $c = 2$, we set $n = n_2$ and $L_p(H) = L_p(H, n)$. The expectation, $E(f)$, of a measurable function f is $E(f) = \int_S f \, d\mu$.

A function $f(x)$ on the points of H is a tame function if there is a Baire function g on a finite dimensional Euclidean space, E_k, and orthonormal vectors, $h_1, \cdots, h_k$, in H^* such that

$$f(x) = g((x, h_1), \cdots, (x, h_k)).$$

The span of the h_i, $i = 1, 2, \cdots, k$, in H is called the base space of f. If F is a representative of the normal distribution and $f(x) = g((x, h_1), \cdots, (x, h_k))$ is a tame function, then

$$\tilde{f}(s) = g(F(h_1)(s), \cdots, F(h_k)(s))$$

is a measurable function on H. The expectation of f is

$$E(f) = (\pi c)^{-k/2} \int \quad g(t)\exp\left[-\frac{\|t\|^2}{c}\right] dt$$

where k is the dimension of the base space of f. This equality holds in the sense that if either side exists and is finite then so is the other and the two are equal.

Several very useful representatives of the normal distribution are known. Of these the one in which we shall be most interested is the mapping studied by Gross in [**13**] from H^* to Borel measurable functions on an abstract Wiener space. We adopt the notation and terminology of [**13**]. Let B be a one-one Hilbert-Schmidt operator on a real separable Hilbert space H. Then $\|Bx\| = |x|_1$ is a measurable norm on H. Let H_B denote the completion of H in this norm. Let $\mathcal{S}$ denote the σ-field generated by the closed subsets of H_B. The normal distribution n_c induces a Borel probability measure N_c on H_B such that the extension of the identity map on $H_B{}^*$ ($\subset H^*$), regarded as a densely defined map on H^* to measurable functions on $(H_B, \mathcal{S}, N_c)$ to H^* is a representative of the normal distribution on H. Continuous functions, f, on H_B are measurable functions on H and if g denotes the restriction of f to H and if $\mathcal{P}$ denotes the directed set (ordered by inclusion of the ranges) of finite dimensional projections on H, the net $\{\tilde{g}(Qx) \mid Q \in \mathcal{P}\}$ of measurable tame functions converges in measure to f as Q tends strongly to the identity through $\mathcal{P}$.

Let N_c be as above. We may regard B as an isometry from H_B to H. Hence $N_c \circ B^{-1}$ is a Borel measure on H. This measure is usually denoted by $n_c \circ B^{-1}$. See [**12**] for a discussion of these measures. If f is a bounded and continuous function from H to a Banach space E, $\int_H f(x)\, dn_c \circ B^{-1}(x) = \int_{H_B} f(By)\, dN_c(y) = E((f \circ B)^{\sim})$.

If f, g, and fg are absolutely integrable tame functions on H, $(fg)^{\sim} = \tilde{f}\tilde{g}$, $(af + g)^{\sim} = a\tilde{f} + \tilde{g}$ for constants a, and if $f \leqq g$ on H, $\tilde{f} \leqq \tilde{g}$ almost everywhere. We shall use these properties often.

2. *Fractional powers of operators.* Early work on the theory of fractional powers of operators is surveyed in [**24**]. H. Komatsu [**17**] has developed an extensive theory of fractional powers of operators. In [**17**-I, II] it is assumed that A is a linear operator (not necessarily densely defined) such that the negative half line is in the resolvent set of A and $\|t(t + A)^{-1}\| \leqq N < \infty$ for all $t > 0$. A^α is defined for all complex α in §4 of [**17**-I]. For our purposes it will be sufficient to recall some of Komatsu's results for the case when $(-A)$ generates a bounded, strongly continuous semigroup on a reflexive Banach space X.

K-1. If $0 < \mathrm{Re}(\alpha) < \sigma < 1$, then

$$A^\alpha x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{\alpha-1} A(t + A)^{-1} x\, dt$$

when $x \in D(A)$, the domain of A.

K-2. If $0 < \mathrm{Re}(\alpha) < \sigma < n$, n a positive integer, then

$$A^{\alpha}x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{\infty} t^{\alpha-1}(A(t+A)^{-1})^m x \, dt$$

for $x \in D(A^N)$ when $N > m > n$ [17-II, p. 92].

K-3. If $(-A)$ generates a bounded strongly continuous semigroup T_t on X, $x \in D(A)$ and $0 < \mathrm{Re}(\alpha) < \sigma < 1$, then

$$A^{\alpha}x = \Gamma(-\alpha)^{-1} \int_{0^+}^{\infty} (T_t x - x)t^{-\alpha-1} \, dt$$

[17-I, p. 325].

More formally, K-1 and K-3 define an operator $A_{\sigma}{}^{\alpha}$ on a subspace D^{σ} of X; D^{σ} is defined in [17-I]. If $A_{+}{}^{\alpha}$ denotes the smallest closed extension of $A_{\sigma}{}^{\alpha}$, whose existence is proved in [17-I, Proposition 4.1], then $A^{\alpha} = A_{+}{}^{\alpha}$. Similarly K-2 defines an operator on a natural subspace of X and its smallest closed extension is $A_{+}{}^{\alpha} = A^{\alpha}$ as is shown in [17-II]. When $\mathrm{Re}(\alpha) < 0$, $A_{-\sigma}^{\alpha}$ is defined by equation 4.10 of [17-I, p. 304] and $A_{-\sigma}^{\alpha}$ is shown to have a smallest closed extension $A_{-}{}^{\alpha}$ which is independent of σ. When $\mathrm{Re}(\alpha) = 0$, $A^{\alpha}x$ is defined by equation 4.11 of [17-I, p. 305] for x in $D^{\sigma} \cap R^{\tau}$. If $0 < \sigma, \tau < 1$, and if $x \in D^{\sigma} \cap R^{\tau}$, $A_{\sigma\tau}^{\alpha}x = x$ if $\alpha = 0$ and if $\alpha \neq 0$,

$$A_{\sigma\tau}^{\alpha}x = -\frac{\sin(\pi\alpha)}{\pi} \left[\int_0^N t^{\alpha}(t+A)^{-1}x \, dt - \frac{N^{\alpha}}{\alpha}x - \int_N^{\infty} t^{\alpha-1}A(t+A)^{-1}x \, dt \right];$$

here N is an arbitrary positive real number; N does not influence the value of $A_{\sigma\tau}^{\alpha}x$. The right side of the above equation is analytic in α on the strip $-\tau < \mathrm{Re}(\alpha) < \sigma$ and it coincides with $A^{\alpha}{}_{-\tau}x$ and $A_{\sigma}{}^{\alpha}x$ in the subdomain $-\tau < \mathrm{Re}(\alpha) < 0$ and $0 < \mathrm{Re}(\alpha) < \sigma$, respectively; so it is possible to give another definition of fractional powers by means of the operator $A_{\sigma\tau}^{\alpha}$ even when $\mathrm{Re}(\alpha) \neq 0$. There is the important

K-4. For every complex α, $A_{\sigma\tau}^{\alpha}$ has the smallest closed extension $A_0{}^{\alpha}$ which is independent of σ and τ when $-\tau < \mathrm{Re}(\alpha) < \sigma$. If $\mathrm{Re}(\alpha) > 0$, $A_0{}^{\alpha} = A_{+}{}^{\alpha}$ on $D(A_{+}{}^{\alpha}) \cap \bar{R}(A)$ and if $\mathrm{Re}(\alpha) < 0$, $A_0{}^{\alpha} = A_{-}{}^{\alpha}$.

A result similar to K-4 holds for larger values of σ and τ; see [17-I].

If A has a bounded inverse, $R^{\tau} = X$ and $A_{-}{}^{\alpha}$ is everywhere defined and analytic in $\mathrm{Re}(\alpha) < 0$. If $x \in D^{\sigma}$, $A^{\alpha}x$ is analytic in $\mathrm{Re}(\alpha) < \sigma$. If $-(n+1) < \mathrm{Re}(\alpha) < 0$,

$$A_-^{\alpha} = \frac{-\sin(\pi\alpha)}{\pi} \frac{n!}{(\alpha+1)\cdots(\alpha+n)} \int_0^{\infty} t^{\alpha+n}(t+A)^{-n-1}\,dt$$

and

K-5. If $\mathrm{Re}(\alpha) > 0$, then $A_+^{\alpha} = A_0^{\alpha}$ is the inverse of $A_0^{-\alpha} = A_-^{-\alpha}$; the $D(A_+^{\alpha})$ is contained in the $R(A_-^{-\alpha})$. See §5 of [**17**-I].

K-6. (i) If $\mathrm{Re}(\alpha) \cdot \mathrm{Re}(\beta) > 0$, then $A_{\pm}^{\alpha}A_{\pm}^{\beta} = A_0^{\alpha}A_{\pm}^{\beta} = A_{\pm}^{\alpha+\beta}$ in the sense of the product of operators.

(ii) If α and β are any complex numbers, then $[A_0^{\alpha}A_0^{\beta}]_C = A_0^{\alpha+\beta}$ where $[T]_C$ denotes the smallest closed extension of T.

(iii) If A has a bounded inverse and if $\mathrm{Re}(\alpha) > 0$, then $A_0^{\alpha}A_0^{\beta} = A_0^{\alpha+\beta}$.

See §7 of [**17**-I].

From the assumption that $\|t(t+A)^{-1}\| \leqq M$ for $t > 0$ and the resolvent equation it follows that $(t+A)^{-1}$ exists for t in the sector $|\arg(t)| < \mathrm{Arcsin}(M^{-1})$ and that $t(t+A)^{-1}$ is bounded on each ray of this sector. Let $M(\Theta) = \sup\{\|t(t+A)^{-1}\| : |\arg(t)| = \Theta\}$, $\Theta \geqq 0$; $M(\Theta)$ is an increasing function of Θ. An operator A is said to be of type $(\omega, M(\Theta))$, $0 \leqq \omega < \pi$, if A is closed, densely defined, the resolvent set of $(-A)$ contains the sector $|\arg(t)| < \pi - \omega$, and

$$\sup\{\|t(t+A)^{-1}\| : |\arg(t)| = \Theta\} \leqq M(\Theta) < \infty$$

holds for all $0 \leqq \Theta < \pi - \omega$. An operator A is of type $(\omega, M(\Theta))$ for an $\omega < \pi/2$ if and only if $(-A)$ generates a semigroup T_t which has an analytic extension to the sector $|\arg(t)| < \pi/2 - \omega$ such that the extension is uniformly bounded on each sector $|\arg(t)| < \pi/2 - \omega - \epsilon, \epsilon > 0$; [**17**-I, §10].

K-7. If A is an operator of type $(\omega, M(\Theta))$ and $0 < \alpha\omega < \pi/2$, then $(-A_+^{\alpha})$ is the generator of the strongly continuous semigroup $\exp(-tA_+^{\alpha})$ which is analytic in the sector $|\arg(t)| \leqq \pi/2 - \alpha\omega$ and uniformly bounded on each smaller sector $|\arg(t)| \leqq \pi/2 - \alpha\omega - \epsilon$, $\epsilon > 0$; see §10 of [**17**-I].

K-8. Let A be of type $(\omega, M(\Theta))$, then $(A_+^{\alpha})^{\beta} = A_+^{\alpha\beta}$ if $0 < \alpha < \pi/\omega$ and $\mathrm{Re}(\beta) > 0$.

K-9. If $0 < \alpha < 1$ and if $T_t = \exp(-tA)$, then $T_t^{\alpha}x = \exp(-tA^{\alpha}) = \int_0^{\infty} T_s x N(\alpha, t, s)\,ds$ where $N(\alpha, t, s) = (2\pi i)^{-1} \int_{\sigma-i\infty}^{\sigma+i\infty} \exp(us - tu^{\alpha})\,du$ [**24**].

If T_t is a bounded semigroup on X, we often need to know when $A_+^{\alpha}T_t$ is a bounded operator on X; the following theorems give some information of this type.

K-10. Let A be an operator of type $(\omega, M(\Theta))$ with $\omega < \pi/2$, and let T_t be the analytic semigroup generated by $(-A)$. If $|\arg(t)| < \pi/2 - \omega$, $t \neq 0$, then $T_t x$ is in $D(A_+^{\alpha})$ for any x in X and $\mathrm{Re}(\alpha) > 0$, and we have

$$A_+^{\alpha} T_t x = (2\pi i)^{-1} \int_\Gamma (-s)^{\alpha} \exp(st)(s+A)^{-1} x\, ds,$$

where Γ is the path consisting of two rays from $\infty e^{-i\Theta}$ to 0 and from 0 to $\infty e^{i\Theta}$ with $\pi/2 < \Theta < \pi/2 + |\arg(t)|$. There is a constant N depending only on α, $\epsilon > 0$, and A such that $\|A_+^{\alpha} T_t\| \leqq N|t|^{-\mathrm{Re}(\alpha)}$, when $|\arg(t)| \leqq \pi/2 - \omega - \epsilon$.

K-11. Let T_t be a bounded semigroup and let $(-A)$ be its generator. If there is a complex number α with $\mathrm{Re}(\alpha) > 0$ such that $\|A_+^{\alpha} T_t\| \leqq N|t|^{-\mathrm{Re}(\alpha)}$, $t > 0$, with constant N, then A is of type $(\omega, M(\Theta))$ for an $\omega < \pi/2$.

K-10 and K-11 are quoted from §12 of [**17-I**].

3. *The Poisson integral on Hilbert space.* Let H be a real separable Hilbert space. For $1 < p < \infty$ let $L_p(H)$ denote the Banach space of p-power integrable functions with respect to the weak normal distribution (with variance parameter 1, centered at the origin) on H. Let $y \rightarrow T_y$ denote the regular representation of the additive group of H by isometries on $L_p(H)$. If f is a bounded tame function,

$$(T_y f)(x) = f(x-y) \exp\left[\frac{(x,y)}{p} - \frac{\|y\|^2}{2p}\right].$$

The T_y are strongly continuous and play the role of the "translation operators" on $L_p(H)$[**5**]. If μ is a finite Borel measure on H, then $T(f) = f * \mu = \int_H T_y f\, d\mu(y)$ is a bounded operator on $L_p(H)$ with norm at most $\|\mu\|$, the total variation of μ. If n_t denotes the normal distribution on H with variance parameter $t/2$, and if $B \neq 0$ is a Hilbert-Schmidt operator on H, then $n_t \circ B^{-1}$ is a Borel probability measure on H [**12**]. Let

$$H_t(f) = \int_H T_y f\, dn_t \circ B^{-1}(y),$$

$$P_y(f) = \int_0^\infty H_t(f) N_t(y)\, dt/t$$

when $N_t(y) = (\pi t)^{-1/2} y \exp(-t^{-1}y^2)$. $P_z(f)$ is the Poisson integral of f. H_t and P_z were studied in [**6**]. We shall recall some of the properties of these operators; the proofs of the properties not given here can be found in [**6**].

P-1. H_t and P_z are strongly continuous, contraction semigroups on $L_p(H)$.

P-2. There is a unique Borel probability measure p_z on H such that $P_z(f) = \int_H T_y f \, dp_z(y)$.

P-3. If $a = (a_1, \cdots, a_n)$ is a multi-index of nonnegative integers with $|a| = \sum_{i=1}^n a_i$, if A_h is the infinitesimal generator of the translation semigroup T_{tBh}, and if $A^a = A_{h_1}^{a_1} \cdots A_{h_n}^{a_n}$, then

$$A^a H_t(f) = \int_{H_B} T_{By} f \, C^a(1)(y) \, dn_t(y)$$

where $C^a = C_{h_1}^{a_1} \cdots C_{h_n}^{a_n}$ and C_{h_i} is the infinitesimal generator of T_{sh_i}, $s > 0$, acting on $L_1(H, n_t)$. Thus if $t > 0$, $A^a H_t$ is a bounded operator on $L_p(H)$ and $\|A^a H_t\|_p \leqq A(a, p) \|h_1\|^{a_1} \cdots \|h_n\|^{a_n} t^{-|a|/2}$.

P-4. P_z is infinitely differentiable with respect to z and with respect to the space variable and

$$A^a P_z(f) = \int_0^\infty A^a H_t(f) N_t(z) \, dt/t,$$

$$\left(\frac{d}{dz}\right)^n P_z(f) = \int_0^\infty H_t(f) \left(\frac{d}{dz}\right)^n N_t(z) \, dt/t.$$

P-5. If $H_t = \exp(-tA)$, then $P_y = \exp(-yT)$ where $T = 2A^{1/2}$; see [24] or K-9.

P-6. P_z extends to an analytic semigroup in $|\arg(z)| < \pi/4$. P_z is a bounded semigroup in $|\arg(z)| < \pi/4 - \epsilon$ for each $\epsilon > 0$.

Proof. $N_t(z)$ is analytic in $|\arg(z)| < \pi/2$ and the integral $P_z(f) = \int_0^\infty H_t(f) N_t(z) \, dt/t$ converges uniformly on compacts in $|\arg(z)| < \pi/4 - \epsilon$ for $\epsilon > 0$.

P-7. If $P_z = \exp(-zT)$, T is one-to-one in $L_p(H)$ and $R(T)$ is dense in $L_p(H)$.

Proof. It suffices to show that T^2 is one-to-one. If $T^2 f = 0$, $H_t 2f = f$ for all finite t. If A_h denotes the infinitesimal generator of T_{tBh}, $t > 0$, then $A_h H_t 2f = A_h f$ for all h in H. By P-3, $\|A_h f\|_p \leqq K t^{-1} \|h\| \, \|f\|_p$ for all $t > 0$; let t tend to ∞. Thus $\|A_h f\|_p = 0$ for all h in H, and $T_{tBh} f = f$ for all $t > 0$ and all h in H. If g is a tame function on H, Hörmander's result (see the proof of Theorem 1.1 of [16]) that $\|\tau_y U + U\|_p \to 2^{1/p} \|U\|_p$ for U in $L_p(E_n, dx)$ implies that $\|T_{tBh} g + g\|_p \to 2^{1/p} \|g\|_p$. Since the tame functions are dense in $L_p(H)$, an $\epsilon/3$-argument shows that $\|T_{tBh} f + f\|_p \to 2^{1/p} \|f\|_p$ as $t \to \infty$. But since $T^2 f = 0$, $2\|f\|_p = \|T_{tBh} f + f\|_p \to 2^{1/p} \|f\|_p$, and this implies that $f = 0$ and T is one-one. By Theorem 3.1 of [17-I], $R(T)$ is dense in $L_p(H)$ since $L_p(H)$ is a reflexive space and T is one-to-one.

Remark. We have to assume in what follows that B is a one-to-one

Hilbert-Schmidt operator because of the present formulation of P-3 and its influence in the proof of P-7. It is possible, however, to state P-3 in such a way that B is not required to be one-one; then P-7 follows as above and we conclude that T is one-one whenever B is not the zero operator.

4. *Interpolation.* In §IV we shall rely heavily on a special case of an interpolation theorem due to E. M. Stein ([21], [25]) to estimate the norms of the operators which were mentioned in the introduction. Let B denote a dense subset of $L_p(H)$ and let C denote a dense subset of the dual space $L_q(H)$. Let S be the strip $0 \leqq \mathrm{Re}(z) \leqq 1$, and let T_z, $z \in S$, be a family of linear operators on $L_p(H)$ which maps B into $L_p(H)$.

THEOREM. *Let T_z, $z \in S$, be a family of linear operators which maps B into $L_p(H)$ and satisfies the following conditions:*

(1) *If $f \in B$ and $g \in C$, then $\varphi(z) = \langle T_z f, g\rangle$ is continuous on S and analytic in the interior of S and*

$$\log|\varphi(x + iy)| \leqq A \exp(a|y|) \text{ for } 0 \leqq x \leqq 1 \text{ and } a < \pi;$$

(2) *$\|T_{iy}f\|_p \leqq M_1(y)\|f\|_p$ and $\|T_{1+iy}f\|_p \leqq M_2(y)\|f\|_p$ for f in B with $\log M_i(y) \leqq M_i \exp(a|y|)$, $a < \pi$, $i = 1, 2$.*

Then $\|T_t\|_p \leqq A(t)$ for $0 \leqq t \leqq 1$; $A(t)$ is bounded in t for $0 \leqq t \leqq 1$.

In place of the sets $B \subset L_p$ and $C \subset L_q$ Stein uses simple functions and assumes that the T_z map simple functions to locally integrable functions. Zygmund [25] gives an integral formula for $A(t)$ when $0 < t < 1$. If one replaces $\log M_i(y)$ by $M \exp(a|y|)$, $a < \pi$, $i = 1, 2$, in this integral and uses $M \exp(a|y|) \leqq 2M \operatorname{Cosh}(ay) = 2M \operatorname{Cos}(iay)$ and a circuit integral, it follows that $A(t)$ is a bounded function in $0 \leqq t \leqq 1$.

III. Imaginary powers of operators.

1. *Singular integrals of imaginary order.* In [18] Muckenhoupt studied a class of singular integral operators which is of fundamental importance in the study of imaginary powers of infinitesimal generators. We shall only restate some of the one dimensional results here.

Let c be a nonzero real number and set

$$(T_\epsilon f)(x) = \left[\int_{t>\epsilon} f(x - t)t^{-ic-1}\,dt - \frac{f(x)}{ic}\,\epsilon^{-ic}\right].$$

PROPOSITION 1.1. *Let $g(t)$ be a measurable function on $[0, 1]$ to a Banach space X. Let* (S) $\int_0^1 g(t)\,dt$ *denote* $\lim_{b\to 0^+} \int_0^1 bt^{b-1}g(t)\,dt$. *Then $(Tf)(x) = \lim_{\epsilon\to 0^+}(T_\epsilon f)(x)$ converges almost everywhere* *or in $L_p(-\infty, \infty)$-norm only if*

$$\begin{aligned}(\mathrm{S}) \int_0^1 f(x-t)t^{-ic-1}\,dt + \int_1^\infty f(x-t)t^{-ic-1}\,dt \\ = (\mathrm{S}) \int_0^1 (f(x-t) - f(x))t^{-ic-1}\,dt \\ - \frac{f(x)}{ic} + \int_1^\infty f(x-t)t^{-ic-1}\,dt\end{aligned}$$

exists almost everywhere or in L_p-norm.

Proposition 1.1 is a consequence of the fact that (S) is a regular summability method.

PROPOSITION 1.2. *If f is in $L_p(-\infty, \infty)$, $(T_\epsilon f)(x)$ converges almost everywhere and in L_p to $(Tf)(x)$ as $\epsilon \to 0^+$. T_ϵ is a uniformly bounded family of operators with $\|T_\epsilon\| \leqq Apq(|c| + 1)^2|c|^{-1}$.*

Given a bounded semigroup $K_t = \exp(-tD)$ on $L_p(H)$ we will begin by studying the analytic semigroup $(r + D)^{-\alpha}$, $\mathrm{Re}(\alpha) > 0$, $r > 0$ is fixed. When D is suitably restricted, we shall see that $(r + D)^{-ic}$ is a strongly continuous group of bounded operators on $L_p(H)$ and that $D^{-ic} = \text{S-lim}_{r\to 0^+}(r + D)^{-ic}$, the $(-ic)$th power of D is the strong limit of the $(r + D)^{-ic}$. Since we are primarily interested in imaginary powers of D_h, $T_{th} = \exp(-tD_h)$, of T, $P_z = \exp(-zT)$, and of $(r + T)^{-1}$, we shall severely restrict the semigroup K_t from the start.

Let ν be a Borel probability measure on H such that $\nu(\{0\}) = 0$ and if $\nu_t(E) = \nu(E/t)$ for $t > 0$ and Borel sets E, then $\nu_t * \nu_s = \nu_{t+s}$ for all $t, s > 0$. Set $K_t(f) = \int_H T_y f\, d\nu_t(y)$ and let $(-D)$ denote the infinitesimal generator of K_t.

Since imaginary powers were treated in detail in [9], we shall only outline the theory in the following sections.

2. *Bessel-Komatsu potentials.* For $r > 0$ and $\mathrm{Re}(\alpha) > 0$, set

$$L_r^\alpha(f) = \Gamma(\alpha)^{-1} \int_0^\infty K_t(f)t^{\alpha-1}e^{-rt}\,dt.$$

THEOREM 2.1. *L_r^α is an analytic semigroup of bounded operators on $L_p(H)$ in $|\arg(\alpha)| < \pi/2$. L_r^α is one-to-one on $L_p(H)$ if* $\mathrm{Re}(\alpha) > 0$. *$L_r^\alpha = (L_r^1)^\alpha = (r + D)^{-\alpha}$, the αth Komatsu power of L_r^1, if* $\mathrm{Re}(\alpha) > 0$. *The range of L_r^α, $R(L_r^\alpha)$, is dense in $L_p(H)$ if* $\mathrm{Re}(\alpha) > 0$. *$\|L_r^\alpha\|_p \leqq r^{-\mathrm{Re}(\alpha)}\Gamma(\mathrm{Re}(\alpha))|\Gamma(\alpha)|^{-1}$.*

Proof. To check the continuity of L_r^α, let $|\arg(\alpha)| \leqq \Theta < \pi/2$ and write

$$L_r^\alpha f - f = \Gamma(\alpha)^{-1} \int_0^\infty (K_t f - f) t^{\alpha-1} e^{-rt}\, dt + (r^{-\alpha} f - f).$$

The last term converges strongly to 0 as α tends to zero. Given $\epsilon > 0$, let $\delta > 0$ be so small that $\|K_t f - f\|_p < \epsilon$ for $0 < t \leqq \delta$. Choose $\eta > 0$ such that

$$|\Gamma(\alpha)|^{-1} \int_\delta^\infty t^{\mathrm{Re}(\alpha)-1} e^{-rt}\, dt < \epsilon r^{-\mathrm{Re}(\alpha)} \Gamma(\mathrm{Re}(\alpha)) |\Gamma(\alpha)|^{-1}$$

when $0 < \mathrm{Re}(\alpha) < \eta$. Then $\|L_r^\alpha(f) - f\|_p \leqq |r^{-\alpha} - 1|\, \|f\|_p + \epsilon r^{-\mathrm{Re}(\alpha)} \Gamma(\mathrm{Re}(\alpha)) |\Gamma(\alpha)|^{-1} (1 + 2\|f\|_p)$ if $0 < \mathrm{Re}(\alpha) < \eta$. Since $\Gamma(\mathrm{Re}(\alpha)) |\Gamma(\alpha)|^{-1} \leqq M(\Theta) < \infty$, continuity is verified.

Since $t^{\alpha-1}$ is analytic in $\mathrm{Re}(\alpha) > 0$ for $t > 0$ and since $t^{\alpha-1} \log(t) e^{-rt}$ is integrable, $L_r^\alpha(f)$ is analytic in $\mathrm{Re}(\alpha) > 0$. Set $\Gamma(\alpha)^{-1} t^{\alpha-1} e^{-rt} = g_\alpha(t)$ if $t > 0$ and $g_\alpha(t) = 0$ if $t < 0$. Then

$$L_r^\alpha L_r^\beta(f) = \int_{-\infty}^\infty K_t(f) g_\alpha * g_\beta(t)\, dt$$

and direct computation shows that $g_\alpha * g_\beta(t) = g_{\alpha+\beta}(t)$; thus $L_r^\alpha L_r^\beta = L_r^{\alpha+\beta}$.

If $L_r^\alpha(f) = 0$ for some α in $\mathrm{Re}(\alpha) > 0$, then $L_r^{\alpha+t}(f) = L_r^\alpha L_r^t(f) = 0$ for all $t > 0$. By the principle of uniqueness for analytic functions $L_r^\alpha(f) = 0$ for all $\mathrm{Re}(\alpha) > 0$. Strong continuity implies that $f = 0$ and that each L_r^α is one-one. That $L_r^\alpha = (L_r^1)^\alpha$ follows from the computation: set $L(f) = \int_0^\infty e^{-rt} K_t(f)\, dt$, then, if $0 < \mathrm{Re}(\alpha) < 1$,

$$\begin{aligned}
L^\alpha(f) &= \Gamma(\alpha)^{-1} \int_0^\infty u^{\alpha-1} e^{-ru} K_u(f)\, du \\
&= \Gamma(\alpha)^{-1} \Gamma(1-\alpha)^{-1} \int_0^\infty \int_0^\infty t^{-\alpha} e^{-ru} e^{-tu} K_u(f)\, dt\, du \\
&= \Gamma(\alpha)^{-1} \Gamma(1-\alpha)^{-1} \int_0^\infty \int_r^\infty (t-r)^{-\alpha} e^{-tu}\, dt\, K_u(f)\, du \\
&= \Gamma(\alpha)^{-1} \Gamma(1-\alpha)^{-1} \int_r^\infty (t-r)^{-\alpha} \int_0^\infty e^{-tu} K_u(f)\, du\, dt \\
&= \Gamma(\alpha)^{-1} \Gamma(1-\alpha)^{-1} \int_r^\infty (t-r)^{-\alpha} (t+D)^{-1} f\, dt \\
&= \Gamma(\alpha)^{-1} \Gamma(1-\alpha)^{-1} \int_0^\infty v^{-\alpha} (v+r+D)^{-1} f\, dv.
\end{aligned}$$

Since $(v + r + D)^{-1} = L(vL + 1)^{-1}$, set $x = v^{-1}$ to get

$$L^{\alpha}(f) = \frac{\sin(\pi\alpha)}{\pi} \int_0^{\infty} x^{\alpha-1} L(x + L)^{-1} f \, dx = (L)^{\alpha} f$$

by K-1 of §II-2. If $\mathrm{Re}(\alpha) \geqq 1$, use the semigroup property of L^{α} and K-6 to show that $L^{\alpha} = (L^1)^{\alpha}$.

$R(L_r^{\alpha})$ is dense in $L_p(H)$ since by Proposition 4.3 of [17-I] $D((r + D)^{\alpha})$ is dense in $L_p(H)$ and by K-5, $D((r + D)^{\alpha})$ is contained in $R((r + D)_-^{-\alpha})$.

Next we shall define and study the boundary value group L_r^{ic}. Let $\delta > 0$. If $c = 0$, set ${}_{\delta}L_r^{ic}(f) = f$; when $c \neq 0$ set

$${}_{\delta}L_r^{ic}(f) = \Gamma(ic)^{-1} \left[\int_{\delta}^{\infty} K_t(f) t^{ic-1} e^{-rt} \, dt + \frac{\delta^{ic}}{ic} f \right];$$

let $L_r^{ic}(f) = \lim_{\delta \to 0^+} {}_{\delta}L_r^{ic}(f)$ when this limit exists in the p-norm.

Theorem 2.2. *For $r > 0$ the ${}_{\delta}L_r^{ic}$ are uniformly bounded in $\delta > 0$, $r > 0$, and the strong limit L_r^{ic} exists as $\delta \to 0^+$. $\|{}_{\delta}L_r^{ic}\|_p \leqq Npq(|c| + 1)^2 \, |\Gamma(ic + 1)|^{-1}$ where the constant N does not depend on $\delta > 0$ or $r > 0$.*

Proof. First consider

$$(T_{\delta}^{A} f)(x) = \int_{\delta}^{\infty} f(x - y) \exp(-y/A) y^{ic-1} \, dy + \frac{\delta^{ic}}{ic} f$$

on $L_p((-\infty, \infty))$. Let $g(t) = t^{ic-1}$ if $t > 0$ and $g(t) = 0$ if $t \leqq 0$. Since $\exp(-|t|/A) = (\pi)^{-1} \int_{-\infty}^{\infty} e^{-ity} A(1 + A^2y^2)^{-1} \, dy$, set $h(A, y) = A(\pi)^{-1}(1 + A^2y^2)^{-1}$ and write

$$\begin{aligned}(T_{\delta}^{A} f)(x) &= \int_{-\infty}^{\infty} e^{-ixy} \int_{|t|>\delta} f(x - t) e^{i(x-t)y} g(t) \, dt \, h(A, y) \, dy + \frac{\delta^{ic}}{ic} f \\ &= \int_{-\infty}^{\infty} e^{-ixy} \left[\int_{|t|>\delta} f(x - t) e^{i(x-t)y} g(t) \, dt \right. \\ &\qquad \left. + \frac{\delta^{ic} e^{ixy}}{ic} f \right] h(A, y) \, dy.\end{aligned}$$

By Minkowski's integral inequality,

$$\begin{aligned}\|T^{A} f\|_p \leqq \int_{-\infty}^{\infty} h(A, y) \Bigg\| \int_{|t|>\delta} & f(x - t) e^{i(x-t)y} g(t) \, dt \\ & + \frac{\delta^{ic} e^{ixy}}{ic} f \Bigg\|_p \, dy.\end{aligned}$$

By Proposition II.1.2, the norm in the above integral is dominated by $Npq(|c| + 1)^2|c|^{-1}$. Thus $T_\delta{}^A$ is a bounded operator on $L_p(H)$ and the bound on $\|T_\delta{}^A\|_p$ does not depend on A. By Proposition III.1.2 and the bounded convergence theorem, the $T_\delta{}^A$ converge strongly to a bounded operator, T^A, on $L_p((-\infty, \infty), dx)$ as $\delta \to 0^+$.

Let

$$_\delta U_y(f) = \Gamma(ic)^{-1}\left[\int_\delta^\infty T_{ty} f t^{ic-1} e^{-rt}\, dt + \frac{\delta^{ic}}{ic} f\right],$$

and assume that f is a bounded tame function on H. Then the rotational invariance of the normal distribution can be used as in the proofs of Theorems 7 or 4 of [8] or [9] to show that as a consequence of the bound on $T_\delta{}^{\|y\|/r}$, $\|_\delta U_y f\|_p \leqq Npq(|c| + 1)^2|\Gamma(ic + 1)|^{-1}\|f\|_p$ where N does not depend on δ, r, or y. The bounded tame functions are dense in $L_p(H)$, so that the desired estimate holds. The rotational invariance of the normal distribution together with the bounded convergence theorem shows that $_\delta U_y$ converges strongly to a bounded operator U_y on $L_p(H)$ as $\delta \to 0^+$. $_\delta L_r{}^{ic}$ is the ν-integral with respect to y of the $_\delta U_y$, so that the $_\delta L_r{}^{ic}$ are bounded uniformly in $\delta > 0$ and $r > 0$. The bounded convergence theorem implies that the $_\delta L_r{}^{ic}$ converge strongly; the required estimate holds.

THEOREM 2.3. *$L_r{}^{ic}(f) = \lim\{L_r{}^{b+ic}(f) : b \to 0^+\}$ for each f in $L_p(H)$.*

PROOF. The integral $\Gamma(a)^{-1}\int_1^\infty K_t(f)t^{a-1}e^{-rt}\,dt$, $a = b + ic$, converges strongly to $_1L_r{}^{ic}$ as $b \to 0^+$. It is sufficient to consider $\int_0^1 K_t(f)t^{a-1}e^{-rt}\,dt$. This integral is

$$\int_0^1 bx^{b-1}\int_x^1 K_t(f)t^{ic-1}e^{-rt}\,dt\,dx.$$

The function bx^{b-1} gives a regular summability method on $0 \leqq x \leqq 1$. Since the integral $\int_0^1 K_t(f)t^{a-1}(e^{-rt} - 1)\,dt$ converges strongly to $\int_{0^+}^1 K_t(f)t^{ic-1}(e^{-rt} - 1)\,dt$ as $b \to 0^+$, we consider

$$\lim_{b\to 0^+}\int_0^1 bx^{b-1}\int_x^1 K_t(f)t^{ic-1}\,dt\,dx.$$

From Proposition III. 1.1, we have that this last integral exists if

$$\lim_{\epsilon\to 0^+}\int_\epsilon^1 K_t(f)t^{ic-1}\,dt + \frac{\epsilon^{ic}}{ic}f$$

exists; when the limit on ϵ exists, the limit on b exists and the two are equal. Theorem 2.2 shows that the limit on ϵ exists, so that $L_r^{ic}(f) = \lim_{b\to 0^+} L_r^{b+ic}(f)$.

COROLLARY 2.4. *If* $\mathrm{Re}(a) > 0$, $L_r^{a+ic} = L_r^a L_r^{ic}$.

PROOF. If $0 < \epsilon < \mathrm{Re}(a)$ and if $f \in L_p(H)$,

$$\begin{aligned}\|L_r^{a+ic}f - L_r^a L_r^{ic} f\|_p \\ \leqq r^{\mathrm{Re}(\epsilon-\alpha)}\Gamma(\mathrm{Re}(a-\epsilon))|\Gamma(a-\epsilon)|^{-1}\|L_r^{\epsilon+ic}f - L_r^\epsilon L_r^{ic}f\|_p\end{aligned}$$

by the boundedness assertion of Theorem 2.1. Since

$$\|L_r^{\epsilon+ic}f - L_r^\epsilon L_r^{ic}f\|_p \leqq \|L_r^{\epsilon+ic}f - L_r^{ic}f\|_p + \|L_r^{ic}f - L_r^\epsilon L_r^{ic}f\|_p,$$

let $\epsilon \to 0^+$; then Theorem 2.3 and Theorem 2.1 give the desired result.

COROLLARY 2.5. $T_r^a = L_r^{a+ic}$, $a \geqq 0$, *is a strongly continuous family of bounded operators on* $L_p(H)$ *with*

$$\|T_r^a\|_p \leqq Ar^{-a}pq(|c| + 1)^2|\Gamma(ic + 1)|^{-1}.$$

PROOF. By Corollary 2.4, $L_r^{a+ic} = L_r^a L_r^{ic}$. Since L_r^a is strongly continuous by Theorem 2.1, T_r^a is strongly continuous. The bounds for L_r^a in Theorem 2.1 and for L_r^{ic} in Theorem 2.2 give the bound on T_r^a.

COROLLARY 2.6. *For each* $r > 0$, $L_r^{ic}L_r^{id} = L_r^{i(c+d)}$.

PROOF. By Corollary 2.4, $L_r^{\epsilon+ic}L_r^{id} = L_r^{\epsilon+i(c+d)}$, and by Theorem 2.3, we may take the limit on each side of this equation as $\epsilon \to 0^+$ to get the desired equality.

COROLLARY 2.7. $\{L_r^{ic} : c \text{ real}\}$ *is a strongly continuous group of bounded operators on* $L_p(H)$ *with* L_r^{i0} = *the identity operator and* $(L_r^{ic})^{-1} = L_r^{-ic}$.

PROOF. Because of Corollary 2.6, we need only show that $\lim\{L_r^{ic}f : c \to 0\} = f$ for each f in $L_p(H)$. The bound on $\|L_r^{ic}\|_p$ is $Apq(|c| + 1)^2|\Gamma(ic + 1)|^{-1} \leqq 4\,Apq(\pi c)^{-1/2}(\sinh(\pi c))^{1/2}$ on $|c| \leqq 1$ since $|\Gamma(ic)| = (\pi)^{1/2}(c\sinh(\pi c))^{-1/2}$ as follows from the well-known identity for $\Gamma(z)\Gamma(1-z)$. Thus $\|L_r^{ic}\|$ is bounded on $-1 \leqq c \leqq 1$ and $\lim_{\epsilon\to 0^+} L_r^{\epsilon+ic}f = L_r^{ic}f$ uniformly on $-1 \leqq c \leqq 1$. Because of the strong continuity of L_r^a in $\mathrm{Re}(a) > 0$, the following equality completes the proof:

$$\begin{aligned}\lim_{c\to 0} L_r^{ic}f &= \lim_{c\to 0}\lim_{\epsilon\to 0^+} L_r^{\epsilon+ic}f = \lim_{\epsilon\to 0^+}\lim_{c\to 0} L_r^{\epsilon+ic}f \\ &= \lim_{\epsilon\to 0^+} L_r^\epsilon f = f.\end{aligned}$$

COROLLARY 2.8. *$L_r^{ic}f = (r + D)^{-ic}f = [(r + D)^{ic}]^{-1}f$ for all f in $L_p(H)$, and $(r + D)^{-ic}$ is a bounded operator on $L_p(H)$ for all real c and all $r > 0$.*

PROOF. By Theorem 2.1, $L_r^{a} = ((r + D)^{-1})^{a}$. By K-5, $((r + D)^{-1})^{a} = (r + D)^{-a}$. The desired result follows from K-4 when we note that $R^{\tau} = L_p(H)$ because of the invertibility of $(r + D)$ and the density in $L_p(H)$ of D^{σ}. A corollary of the uniform boundedness principle (p. 60 of [4]) can now be used to complete the proof.

COROLLARY 2.9. *L_r^{ic} is the (ic)th Komatsu power of $L_r = L_r^{1}$ for all real c.*

PROOF. By Theorem 2.1, $(L_r)^{a} = L_r^{a}$. By Theorem 8.2 of [17-I], for a dense set of f in $L_p(H)$, $(L_r)^{ic}f = \lim_{a\to 0^+}(L_r)^{a+ic}f = \lim_{a\to 0^+}L_r^{a+ic}f = L_r^{ic}f$. Since L_r^{a+ic} $(= (L_r)^{a+ic})$ is uniformly bounded in $a \geqq 0$, $(L_r)^{ic}$ is bounded because of a corollary of the uniform boundedness principle (p. 60 of [4]) so that $L_r^{ic}f = (L_r)^{ic}f$ for all f in $L_p(H)$.

3. *Imaginary powers of D.* In this section we shall assume that D is one-to-one and that $(\pm D^2)$ generates a bounded semigroup. If $K_t = P_t$, $D = T$ and $(-T^2)$ generates H_{4t}; if $K_t = T_{th}$, $D = D_h$ and $+(D_h)^2$ generates the semigroup

$$H_t^h(f) = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} T_{uh}f \exp((-u^2/4t))\, du.$$

The significance of these assumptions on D is that by Theorem 3.1 of [17-I], the range of D and the range of D^2 are dense in $L_p(H)$.

Define $D(ic)f = \lim_{r\to 0^+} L_r^{ic}(f)$ if the limit exists in the p-norm.

THEOREM 3.1. *$D(ic)$ is the strong limit of L_r^{ic} as $r \to 0^+$ and $\|D(ic)\|_p \leqq Apq(|c| + 1)^2|\Gamma(ic + 1)|^{-1}$. Furthermore, $D(ic) = D^{-ic}$, the $(-ic)$th Komatsu power of D.*

PROOF. If $r_1, r_2 > 0$,

$$L_{r_1}^{ic}(f) - L_{r_2}^{ic}(f) = \lim_{\delta\to 0^+} \Gamma(ic)^{-1}\int_{\delta}^{\infty} t^{ic-1}[e^{-r_1 t} - e^{-r_2 t}]\, K_t(f)\, dt$$

$$= \pm \lim_{\delta\to 0^+} \Gamma(ic)^{-1}\int_{\delta}^{\infty} t^{ic}\int_{r_1}^{r_2} \exp(-st)\, ds\, K_t(f)\, dt$$

$$= \pm\, ic\int_{r_1}^{r_2} L_s^{1+ic}(f)\, ds.$$

The bounded convergence theorem insures that the equality is preserved in the interchange of integrals in the last equality above. Since by Theorem 3.1 of [17-I], $L_p(H) = N(D) \oplus \bar{R}(D)$, and since $N(D)$, the null space of D, is 0 by our earlier assumption, $R(D)$ is dense in $L_p(H)$. Suppose that $f = Dg$ for some $g \in L_p(H)$; then

$$\|L_{r_1}^{ic}(Dg) - L_{r_2}^{ic}(Dg)\|_p \leqq |c| \, \|L_r^{ic}\|_p \int_{r_1}^{r_2} \|D(s+D)^{-1}g\|_p \, ds.$$

Since $\|K_t\|_p \leqq 1$, $\|D(s+D)^{-1}\|_p \leqq 2$ and $\|L_{r_1}^{ic}(Dg) - L_{r_2}^{ic}(Dg)\|_p \leqq A(c,p)|r_1 - r_2|$. Since the range of D is dense in $L_p(H)$, Theorem II.3.6 of [4] implies that $D(ic)$ is a bounded operator on $L_p(H)$.

From the definition on p. 305 of [17-I] of purely imaginary powers, $D^{-ic}f = f$ if $c = 0$ and if $c \neq 0$,

$$D^{-ic}f = \frac{\sin(\pi ic)}{\pi} \left[\int_0^N t^{-ic}(t+D)^{-1}f \, dt + \frac{N^{-ic}}{ic} f \right.$$
$$\left. - \int_N^\infty t^{-ic-1}D(t+D)^{-1}f \, dt \right]$$

and by using Corollary 2.8,

$$(r+D)^{-ic}f = \frac{\sin(\pi ic)}{\pi} \left[\int_0^N t^{-ic}(t+r+D)^{-1}f \, dt + \frac{N^{-ic}}{ic} f \right.$$
$$\left. - \int_N^\infty t^{-ic-1}(D+r)(t+r+D)^{-1}f \, dt \right]$$

where N is a positive real number. The resolvent equation implies that

$$D^{-ic}f - (r+D)^{-ic}f = rK(c) \left[\int_0^N t^{-ic}(t+D)^{-1}(t+r+D)^{-1}f \, dt \right.$$
$$- \int_N^\infty t^{-ic-1}D(t+r+D)^{-1}(t+D)^{-1}f \, dt$$
$$\left. + \int_N^\infty t^{-ic-1}(t+r+D)^{-1}f \, dt \right].$$

The second and third integrals on the right are bounded operators on $L_p(H)$. If $f = D^2 g$, the first integral on the right converges. Since $(\pm D^2)$ generates a bounded semigroup and D^2 is one-one, the range of D^2 is dense in $L_p(H)$ by Theorem 3.1 of [17-I]. Thus

Theorem II.3.6 of [4] implies that $(r + D)^{-ic}$ converges strongly to D^{-ic} as $r \to 0^+$ and $D(ic) = D^{-ic}$; thus

$$\|D^{-ic}\|_p \leqq Apq(|c| + 1)^2 |\Gamma(ic + 1)|^{-1}.$$

COROLLARY 3.2. *D^{ic} is a strongly continuous group of bounded operators on $L_p(H)$.*

PROOF. $D^{ic}D^{id} = D^{i(c+d)}$ by K-6(ii). The continuity of D^{ic} follows from the fact that the imaginary powers are continuous on a dense subset of $L_p(H)$ and from the uniform boundedness of the operators D^{ic} in $-1 \leqq c \leqq 1$.

COROLLARY 3.3. *If the domain $D(D^a)$ is equipped with the graph norm, $D(D^a) = D(D^b)$ when* $\mathrm{Re}(a) = \mathrm{Re}(b)$.

PROOF. D^{ic} is bounded, so $D^a = D^{b+ic} = D^b D^{ic}$ has the graph norm on $D(D^a)$ equivalent to the graph norm on $D(D^b)$.

THEOREM 3.4. *D^{-ic} is given by Muckenhoupt's singular integral*

$$D^{-ic}f = \lim_{\delta \to 0^+} \left[\Gamma(ic)^{-1} \int_\delta^\infty K_t(f) t^{ic-1}\, dt + \frac{\delta^{ic}}{\Gamma(ic+1)} f \right].$$

PROOF. Set $D(ic)$ equal to the integral operator in the statement of the theorem. By arguing as in the proof of Theorem 2.2, one sees that $D(ic)$ is a bounded operator on $L_p(H)$. For $f \in R(D)$,

$$\begin{aligned} &D(ic)f - (r + D)^{-ic}f \\ &\quad = \pm\, \Gamma(ic)^{-1} \lim_{\delta \to 0} \int_\delta^\infty K_t(f) t^{ic-1} \int_0^r te^{-st}\, ds\, dt \\ &\quad = \pm\, ic \int_0^r L_s{}^{ic+1}(f)\, ds. \end{aligned}$$

The dominated convergence theorem insures the last equality. Since $L_s{}^{ic+1} = L_s{}^{ic} L_s{}^1$ and since $\|L_s{}^{ic}\|$ does not depend on s, it suffices to assume that $f = Dg$ for some g in $L_p(H)$ as the range of D is dense in $L_p(H)$.

Then $\|D(ic)f - (r + D)^{-ic}f\|_p \leqq rK$ which tends to zero as $r \to 0^+$. Since the set of f in $L_p(H)$ of the form $f = Dg$ is dense in $L_p(H)$ by Theorem 3.1 of [17-I] and since both $D(ic)$ and D^{-ic} are continuous, D^{-ic} has the desired form.

THEOREM 3.5. *If $f \in R(D) \cap D(D)$, the infinitesimal generator of the semigroup D^{ic}, $c > 0$, applied to f is*

$$V(f) = -i\left[\int_{0^+}^{1} (K_t(f) - f)\,dt/t + \int_{1}^{\infty} K_t(f)\,dt/t + Cf\right]$$

where C is Euler's constant. $R(D) \cap D(D)$ is dense in $L_p(H)$.

PROOF. If $f \in R(D)$, $t^{-1}\int_0^t K_s(f)\,ds$ is in $R(D) \cap D(D)$ and as $t \to 0^+$, the integral converges to f. Since $R(D)$ is dense in $L_p(H)$, $D(D) \cap R(D)$ is dense in $L_p(H)$.

Let $f \in D(D) \cap R(D)$. Then by Theorem 3.4,

$$\begin{aligned} c^{-1}(D^{ic}f - f) &= \lim_{\delta \to 0^+} -i\Big[\Gamma(1-ic)^{-1}\int_{\delta}^{1} (K_tf - f)t^{-ic-1}\,dt \\ &\qquad + \Gamma(1-ic)^{-1}\int_1^{\infty} K_tft^{-ic-1}\,dt \\ &\qquad + f(\Gamma(1-ic)^{-1} - 1)(-ic)^{-1}\Big] \\ &= -i\Gamma(1-ic)^{-1}\Big[\int_{0^+}^{1} (K_tf - f)t^{-ic-1}\,dt + \int_1^{\infty} K_tft^{-ic-1}\,dt \\ &\qquad + f(1 - \Gamma(1-ic))(-ic)^{-1}\Big]. \end{aligned}$$

Because $f \in D(D)$, the first integral converges absolutely, and as $c \to 0^+$, the first integral converges to the appropriate integral. Since $f \in R(D)$, $f = Dg$ for some g in $L_p(H)$. Restrict c to $-1 \leqq c \leqq 1$; after integration by parts

$$\int_1^{\infty} K_t(f)t^{-ic-1}\,dt = K_1(g) + (-ic - 1)\int_1^{\infty} K_t(g)t^{-ic-2}\,dt.$$

This last integral converges absolutely and one can apply the dominated convergence theorem to take the limit as $c \to 0^+$. Since $C = -\Gamma'(1)$, $V(f)$ has the desired form.

IV. **Some analytic semigroups.** In this section we shall use the boundedness of certain purely imaginary powers to estimate the norms of operators and to study some analytic semigroups which arise in the study of singular integrals. We shall begin with a discussion of the indefinite integral.

1. *Powers of the integral.* Let $\mathrm{Re}(\alpha) > 0$ and set

$$(I^{\alpha}f)(x) = \Gamma(\alpha)^{-1}\int_0^x (x-y)^{\alpha-1}f(y)\,dy$$

for f in $L_p(0, \infty)$. In [14] Hardy and Littlewood showed that if $0 < \mathrm{Re}(\alpha) < p^{-1}$ and $r = p(1 - p\,\mathrm{Re}(\alpha))^{-1}$, then $\|I^{\alpha}f\|_r \leqq A(\alpha, p)\|f\|_p$. When restricted to $L_p(0, 1)$, I^{α} is an analytic semigroup in $|\arg(\alpha)| < \pi/2$. We shall define and study I^{ic} for c real. Let $\epsilon > 0$, c be real, and f be in $L_p(0, \infty)$ and set $I^{ic}f = f$ if $c = 0$ and if $c \neq 0$, set

$$I_\epsilon^{ic}f(x) = \Gamma(ic)^{-1} \left[\int_\epsilon^x f(x-y)y^{ic-1}\,dy + \frac{\epsilon^{ic}}{ic} f(x) \right].$$

Set $I^{ic}f(x) = \lim_{\epsilon \to 0^+} I_\epsilon^{ic}f(x)$ if this limit exists.

THEOREM 1.1. *I_ϵ^{ic} is a bounded family of continuous linear operators on $L_p(0, \infty)$ for $\epsilon > 0$ and each fixed c; $\|I_\epsilon^{ic}\|_p \leqq Apq(|c| + 1)^2 \cdot |\Gamma(ic + 1)|^{-1}$. For f in $L_p(0, \infty)$, as $\epsilon \to 0^+$, $I_\epsilon^{ic}f(x)$ converges almost everywhere and in $L_p(0, \infty)$. I^{ic} is a strongly continuous group of bounded operators on $L_p(0, \infty)$.*

PROOF. Notice that $I_\epsilon^{ic}f(x) = 0$ if $x < 0$. Set $F(x) = f(x)$ if $x > 0$ and $F(x) = 0$ if $x < 0$. Then $F \in L_p(-\infty, \infty)$ and $\|f\|_p = \|F\|_p$ and

$$I_\epsilon^{ic}f(x) = T_\epsilon^{c}F(x) = \Gamma(ic)^{-1} \left[\int_\epsilon^\infty F(x-y)y^{ic-1}\,dy + \frac{\epsilon^{ic}}{ic} F(x) \right].$$

By Proposition III.1.2, $\|T_\epsilon^c F\|_p \leqq Apq(|c| + 1)^2 |\Gamma(ic+1)|^{-1} \|F\|_p$; I_ϵ^{ic} is uniformly bounded in $\epsilon > 0$ on $L_p(0, \infty)$. By Theorem 6 of [18], $T_\epsilon^c F(x)$ converges almost everywhere and in $L_p(-\infty, \infty)$ to $T^c F(x)$ as $\epsilon \to 0^+$; then, of course, $\|T^c\|_p \leqq Apq(|c| + 1)^2 \cdot |\Gamma(ic + 1)|^{-1}$ and I^{ic} has the desired form and bound.

An integration by parts shows that the Laplace transform of $I^{ic}f$ is $t^{-ic}(Lf)(t)$ and this shows that $I^{ic}I^{id} = I^{i(c+d)}$ for all real c and d. To show that I^{ic} is strongly continuous we need only show that $\lim_{c \to 0} I^{ic}f = f$ for a fundamental set in L_p since $\|I^{ic}\|_p$ is bounded on $|c| \leqq 1$. Characteristic functions of the intervals $[0, a]$, $a > 0$, generate the step functions so that we need only show that $I^{ic}f_a$ converges to f_a in L_p when f_a is the characteristic function $[0, a]$. Direct computation shows that $I^{ic}f_a(x) = \Gamma(ic + 1)^{-1}x^{ic}$ on $0 < x < a$ and $I^{ic}f_a(x) = \Gamma(ic + 1)^{-1}(x^{ic} - (x - a)^{ic})$ if $x > a$. By the bounded convergence theorem,

$$\lim_{c \to 0} \int_0^a |x^{ic} - 1|^p\,dx = 0,$$

$$\lim_{c \to 0} \int_a^{2a} |x^{ic} - (x-a)^{ic}|^p\,dx = 0.$$

Since $\quad |x^{ic} - (x-a)^{ic}| = |1 - (1 - a/x)^{ic}| = |c|\,|\int_0^{a/x}(1-t)^{ic-1}\,dt|$

$\leqq |c|(a/x)(1 - a/x)^{-1} = |c|a(x - a)^{-1}$, and since $(x-a)^{-1}$ is in $L_p(2a, \infty)$, the dominated convergence theorem implies that

$$\lim_{c \to 0} \int_{2a}^{\infty} |x^{ic} - (x-a)^{ic}|^p \, dx = 0.$$

I^{ic} is a strongly continuous group of bounded operators on $L_p(0, \infty)$.

Now restrict to $L_p(0, 1)$.

COROLLARY 1.2. *I^α is an analytic semigroup on $L_p(0, 1)$ in $|\arg(\alpha)| < \pi/2$, and I^α is a strongly continuous semigroup in* $\mathrm{Re}(\alpha) \geqq 0$.

PROOF. The analyticity is proved in [15]. I^{ic} is a strongly continuous group by Theorem 1.1 and by taking Laplace transforms, one shows that $I^{ic}I^\alpha = I^{ic+\alpha}$ for $\mathrm{Re}(\alpha) > 0$. Strong continuity in $\mathrm{Re}(\alpha) \geqq 0$ follows from the strong continuity of I^{ic} and I^t, $t > 0$.

THEOREM 1.3. *$I^{ic} = (d/dx)^{-ic}$ for all real numbers c.*

PROOF. Set $D = d/dx$, the negative of the infinitesimal generator of $(T_t f)(x) = f(x - t)$. For $f \in L_p(0, \infty)$, extend f to $(-\infty, \infty)$ by setting $f(x) = 0$ if $x < 0$. Then

$$(r + D)^{-ic} f(x) = \lim_{\epsilon \to 0^+} \Gamma(ic)^{-1} \left[\int_\epsilon^\infty f(x-y) e^{-ry} y^{ic-1} \, dy + \frac{\epsilon^{ic}}{ic} f(x) \right]$$

$$= \lim_{\epsilon \to 0^+} \Gamma(ic)^{-1} \left[\int_\epsilon^x f(x-y) e^{-ry} y^{ic-1} \, dy + \frac{\epsilon^{ic}}{ic} f(x) \right]$$

has $(r + D)^{-ic} f(x) = 0$ if $x < 0$. Thus $(r + D)^{-ic}$ and D^{-ic} map $L_p(0, \infty)$ to $L_p(0, \infty)$. For f in $R(D)$,

$$[(r + D)^{-ic} - I^{ic}] f(x) = \lim_{\delta \to 0^+} \Gamma(ic)^{-1} \int_\delta^x f(x-y)(e^{-ry} - 1) y^{ic-1} \, dy$$

$$= -\Gamma(ic)^{-1} \lim_{\delta \to 0^+} \int_\delta^x f(x-y) \int_0^r e^{-sy} \, ds \, y^{ic} \, dy$$

$$= -ic \int_0^r (s + D)^{-1-ic} f(x) \, ds.$$

Since $f \in R(D)$, $f = Dg$ and $\|(r + D)^{-ic} f - I^{ic} f\|_p \leqq rA(c, p)\|g\|_p$, so that $(r + D)^{-ic}$ converges strongly to I^{ic}. Thus $D^{-ic} = I^{ic}$.

We shall continue to let $D = d/dx$ in the next theorem.

THEOREM 1.4. *Let $\omega > \pi/2$, let $p \geqq r$, and set*

$$\mu(f, r)(x) = \left(\int_0^\infty |\exp(-\omega|c|)D^{ic}f(x)|^r \, dc \right)^{1/r}.$$

Then $\|\mu(f, r)\|_p \leqq A(p, r, \omega)\|f\|_p$.

PROOF. By Theorems 1.1 and 1.3, $\|D^{ic}f\|_p \leqq A(p, \eta) \exp(\eta|c|)\|f\|_p$ where $\eta > \pi/2$ and $\eta - \pi/2$ is arbitrarily small. Minkowski's integral inequality implies that $\|\mu(f, r)\|_p \leqq A(p, r, \omega)\|f\|_p$ if $p \geqq r$.

In the next section we shall study another semigroup which consists of smoothing operators.

2. *Bessel potentials.* In §III let K_t be the semigroup P_t, the Poisson integral. Then for $r > 0$, $L_r^\alpha = J_r^\alpha$ is the semigroup of Bessel potentials. The following theorem restates the main properties of L_r^α for J_r^α.

THEOREM 2.1. *J_r^α is an analytic semigroup of bounded operators in $|\arg(\alpha)| < \pi/2$ and J_r^α is a strongly continuous semigroup in* $\mathrm{Re}(\alpha) \geqq 0$. *For* $\mathrm{Re}(\alpha) \geqq 0$, *$J_r^\alpha$ is the αth Komatsu power of J_r^1, J_r^α is one-to-one, and the range of J_r^α is dense in $L_p(H)$.*

DEFINITION. Let $L_p^\alpha(H)$ be $R(J_1^\alpha)$ with the norm $\|g\|_{p,\alpha} = \|f\|_p$ when $g = J^\alpha f = J_1^\alpha f$.

COROLLARY 2.2. *If* $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta)$, $L_p^\alpha(H) = L_p^{\mathrm{Re}(\beta)}(H)$ *with equivalent norms.*

This corollary follows from the fact that J^{ic} is a bounded operator with a bounded inverse. We shall see soon that $L_p^\alpha(H)$ could have been defined with any of the J_r^α and an equivalent space of functions would have resulted. We want to show that $L_p^\alpha(H) = D(T^\alpha)$ with equivalent norms when $D(T^\alpha)$ is equipped with the graph norm; $(-T)$ is the infinitesimal generator of the Poisson integral. To do this we shall study the semigroup $(r + T)^\alpha J_u^\alpha$ when $r \geqq 0$ and $u > 0$.

THEOREM 2.3. *$(r + T)^\alpha J_u^\alpha$ is an analytic semigroup of bounded operators on $L_p(H)$ in $|\arg(\alpha)| < \pi/2$ and a strongly continuous semigroup of bounded operators on $L_p(H)$ in* $\mathrm{Re}(\alpha) \geqq 0$ *for* $r \geqq 0$ *and* $u > 0$. *If* $\mathrm{Re}(\alpha) \leqq n$,

$$\|(r + T)^\alpha J_u^\alpha\|_p \leqq A(p) \exp(2\pi|\mathrm{Im}(\alpha)|) \, [u^{-1}(u + |r - u|)]^n.$$

PROOF. $(r + T)J_u^1 = I - (r - u)J_u^1$ so that $\|(r + T)^n J_u^n\|_p \leqq (u^{-1}(u + |r - u|))^n$. By Theorem III.2.2 and Corollary III.2.8,

$$\begin{aligned} \|(r + T)^{ic}J_u^{ic}\|_p &\leqq A^2p^2q^2(|c| + 1)^4|\Gamma(ic + 1)|^{-2} \\ &\leqq A(p) \exp(2\pi|\mathrm{Im}(\alpha)|). \end{aligned}$$

By Proposition 6.3 of [17-I], $(r + T)^\alpha J_u^\alpha = [(r + T)(u + T)^{-1}]^\alpha$ for

$\operatorname{Re}(\alpha) > 0$, so that by Proposition 8.2 of [**17**-I] and by K-6, $(r + T)^{\alpha} J_u^{\alpha}$ is an analytic semigroup in $|\arg(\alpha)| < \pi/2$. Strong continuity on $\operatorname{Re}(\alpha) \geqq 0$ follows from the strong continuity of $(r + T)^{ic} J_u^{ic}$ and the strong continuity of $(r + T)(u + T^{-1})^t$ in $t \geqq 0$. By Stein's interpolation theorem,

$$\|(r + T)^{\alpha} J_u^{\alpha}\|_p \leqq A(p) \exp(2\pi |\operatorname{Im}(\alpha)|)(u^{-1}(u + |r - u|))^n$$

when $\operatorname{Re}(\alpha) \leqq n$.

Remark. An exact computation can be used to show that $(r + T)^{\alpha} J_u^{\alpha}$ is given by convolution with a finite Borel measure if $\operatorname{Re}(\alpha) > 0$. See [7] where this is done for $r = 0$ and $u = 1$.

Since J^{ic} is bounded and invertible and since T^{ic} is bounded and invertible, $L_p^{\alpha}(H) = L_p^{\operatorname{Re}(\alpha)}(H)$ and $D(T^{\alpha}) = D(T^{\operatorname{Re}(\alpha)})$. Thus to prove that $L_p^{\alpha}(H) = D(T^{\alpha})$, it suffices to verify this equivalence for real α.

Theorem 2.4. *$L_p^{\alpha}(H) = D(T^{\alpha})$ with equivalent norms for* $\operatorname{Re}(\alpha) \geqq 0$.

Proof. It will be sufficient to prove equivalence when α is real. For $f \in L_p^{\alpha}(H)$, $\|f\|_{p,\alpha} = \|(1 + T)^{\alpha} f\|_p$. Thus

$$\begin{aligned}
\|f\|_p + \|T^{\alpha} f\|_p &\leqq \|f\|_p + \|T^{\alpha} J^{\alpha}(1 + T)^{\alpha} f\|_p \\
&\leqq \|J^{\alpha}(1 + T)^{\alpha} f\|_p + \|T^{\alpha} J^{\alpha}(1 + T)^{\alpha} f\|_p \\
&\leqq (1 + \|T^{\alpha} J^{\alpha}\|_p)\|(1 + T)^{\alpha} f\|_p \\
&\leqq A(\alpha, p)\|f\|_{p,\alpha}
\end{aligned}$$

by Theorem 2.3, and $L_p^{\alpha}(H) \subset D(T^{\alpha})$.

Let $f \in D(T^{\alpha})$ and suppose first that $0 \leqq \alpha \leqq 1$. Then $(1 + T)^{\alpha} f = T^{\alpha} f + Bf$ where B is a bounded operator of $L_p(H)$. For

$$\begin{aligned}
(1 + T)^{\alpha} f &= \frac{\sin(\pi\alpha)}{\pi} \int_0^{\infty} t^{\alpha-1}(T + 1)(t + 1 + T)^{-1} f \, dt \\
&= \frac{\sin(\pi\alpha)}{\pi} \int_0^{\infty} t^{\alpha-1}(t + 1 + T)^{-1} f \, dt \\
&\quad + \frac{\sin(\pi\alpha)}{\pi} \int_0^{\infty} t^{\alpha-1} T(t + 1 + T)^{-1} f \, dt.
\end{aligned}$$

Since $\|(t + 1 + T)^{-1}\| \leqq A(t + 1)^{-1}$, the first integral on the right represents a bounded operator on $L_p(H)$. By the resolvent equation $(t + 1 + T)^{-1} - (t + T)^{-1} = -(t + 1 + T)^{-1}(t + T)^{-1}$. Thus

$$\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}T(t+1+T)^{-1}f\,dt$$

$$= T^\alpha f - \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}T(t+T)^{-1}(t+1+T)^{-1}f\,dt.$$

Since $\|T(t+T)^{-1}\| \leqq A$ and $\|(t+1+T)^{-1}\| \leqq A(t+1)^{-1}$, the last integral on the right represents a bounded operator on $L_p(H)$. Thus if $0 \leqq \alpha < 1$, $\|f\|_{p,\alpha} = \|(1+T)^\alpha f\|_p \leqq \|Bf\|_p + \|T^\alpha f\|_p \leqq A(p,\alpha)(\|f\|_p + \|T^\alpha f\|_p)$. For $\alpha = n + \partial$, $0 \leqq \partial < 1$, n a positive integer, $(1+T)^\alpha = (1+T)^n(T^\partial + B) = \sum_{k=0}^n A_k T^k + \sum_{k=0}^n B_k T^{k+\partial}$ where the A_k and the B_k are bounded operators on $L_p(H)$. By Theorem 6.5 of [17-I], $D(T^\alpha) \subset D(T^\beta)$ continuously for $\beta \leqq \alpha$, so that

$$\|(1+T)^\alpha f\|_p \leqq A(p,\alpha)(\|f\|_p + \|T^\alpha f\|_p) \quad \text{and} \quad D(T^\alpha) \subset L_p^\alpha(H).$$

3. *Directional derivatives.* Let $h \in H$ and let A_h denote the infinitesimal generator of the translation semigroup T_{tBh}. We shall study the semigroup $(-A_h)^\alpha J^\alpha$ for $\operatorname{Re}(\alpha) \geqq 0$ when J^α is the Bessel potential of order α; $J^\alpha = J_1{}^\alpha$.

PROPOSITION 3.1. *$(-A_h)^{ic}J^{ic}$ is a strongly continuous group of bounded operators on $L_p(H)$. $\|(-A_h)^{ic}J^{ic}\|_p \leqq A(p)\exp(2\pi|c|)$.*

PROOF. $(-A_h)^{ic}$ and J^{ic} are strongly continuous groups of bounded operators by Corollary III.3.2 and Corollary III.2.7. Since Theorems III.2.2 and III.3.1 give estimates for the norm of each of these operators of the form $Apq(|c|+1)^2|\Gamma(1+ic)|^{-1}$, and since $|\Gamma(ic)| = (\pi)^{1/2}(c\sinh(\pi c))^{-1/2}$, we have the desired estimate for $(-A_h)^{ic}J^{ic}$.

PROPOSITION 3.2. *A_hJ is a bounded operator on $L_p(H)$.*

PROOF. The proof requires the following useful lemma.

LEMMA 3.3. *If f is in $L_p(H)$, $1 < p < \infty$, and if $\varphi(t)$ is the Fourier transform of a bounded, even, Borel measure μ on the real line, then*

$$\Phi_A(f) = \lim_{\epsilon\to 0}\int_{|t|>\epsilon} T_{ty}f\varphi(tA^{-1})\,dt/t$$

satisfies $\|\Phi_A(f)\|_p \leqq N(p)\|\mu\|\,\|f\|_p$ where the constant $N(p)$ is independent of A and y in H.

PROOF. First set

$$(T_Af)(x) = \lim_{\epsilon\to 0}\int_{|t|>\epsilon} f(x-t)\varphi(tA^{-1})\,dt/t$$

for f in $L_p[(-\infty,\infty), dx]$. Then for smooth f with compact support

$$(T_A f)(x) = \int_{-\infty}^{\infty} e^{iux} \lim_{\epsilon\to 0} \int_{|y|>\epsilon} f(x-y)e^{iu(y-x)} \frac{dy}{y}\, d\mu_A(u)$$

where $\mu_A(E) = \mu(AE)$ for Borel sets E in the real line. By the M. Riesz theorem on the Hilbert transform, $\|T_A f\|_p \leqq N(p)\|\mu\|\,\|f\|_p$, since $\|\mu\| = \|\mu_A\|$ for all $|A| > 0$; $N(p)$ depends only on p.

Let f be a bounded continuous tame function on H which is based in the finite dimensional subspace E of H; dimension of $E = k$. Since the normal distribution on H is rotationally invariant, let K be the span of E and y and let $e_1, \cdots, e_{k+1}$ be an orthonormal basis for K with $e_1 = \omega = y\|y\|^{-1}$. Then

$$\left\| \int_{\delta<|t|<\rho} T_{ty} f\varphi(tA^{-1})\, dt/t \right\|_p^p$$
$$= \int_K \left| \int_{\delta\|y\|\leqq|t|<\rho\|y\|} g(x-t\omega)D_p(x,t\omega) \cdot\varphi(t\|y\|^{-1}A^{-1})\, dt/t \right|^p dn(x)$$

where g is the restriction of f to K and where $D_p(x, t\omega) = \exp[(x, t\omega)/p - t^2/2p]$. If we write the integral over K as an iterated integral and write the first integral as

$$M \int_{-\infty}^{\infty} \left| \int_{\delta\|y\|\leqq|t|<\rho\|y\|} g(x_1 - t, x_2, \cdots, x_{k+1}) \cdot \exp\left[-\frac{(x_1-t)^2}{2p} \right] \varphi(t\|y\|^{-1}A^{-1})\, dt/t \right|^p dx_1,$$

it follows from the discussion of T_A over $(-\infty, \infty)$ in the first paragraph of this proof and from the dominated convergence theorem that $\|\Phi_A(f)\|_p \leqq N(p)\|\mu\|\,\|f\|_p$, the desired conclusion.

PROOF OF PROPOSITION 3.2. By P-3 of §II.3,

$$A_h H_t 2(f) = t^{-1} \int_{H_B} T_{tBy} f C_h(1)(y)\, dn_1(y),$$

where C_h is the infinitesimal generator of T_{sh} acting on $L_1(H, n_1)$. Since A_h is a closed operator,

$$A_h J(f) = \int_{0^+}^{\infty} \int_{H_B} T_{tBy} f C_h(1)(y)\, dn_1(y)\varphi(t)\, dt/t$$

where

$$\varphi(t) = t^{-2} \int_0^{\infty} z \exp(-z^2 t^{-2}) e^{-z}\, dz, \qquad t > 0.$$

Since $C_h(1)(y)$ is a homogeneous polynomial of degree 1 in y, this last integral may be written as

$$A_h J(f) = \frac{1}{2} \lim_{\epsilon \to 0} \int_{|t|>\epsilon} \int_{H_B} T_{tBy} f C_h(1)(y)\, dn_1(y) \varphi(t)\, dt/t$$

when $\varphi(t) = \varphi(-t)$ for negative t. By Minkowski's integral inequality,

$$\|A_h J f\|_p \leqq M \int_{H_B} \|\Phi_{\|By\|}(f)\|_p\, |C_h(1)(y)|\, dn_1(y) \leqq N(p)\|\mu\|\, \|f\|_p$$

by Lemma 3.3, if $\varphi(t)$ is the Fourier transform of a finite even Borel measure μ on the real line. Note that $\varphi(t)$ is even by definition and that, on $t > 0$,

$$\varphi(t) = \int_0^\infty z \exp[-z^2 - tz]\, dz$$

so that $\varphi'(t) \leqq 0$ and $\varphi''(t) > 0$. Thus Polya's criterion [3, p. 169] guarantees that $\varphi(t)$ is the Fourier transform of a finite even Borel measure on the line.

THEOREM 3.4. *$(-A_h)^\alpha J^\alpha$ is an analytic semigroup of bounded operators on $L_p(H)$ in $|\arg(\alpha)| < \pi/2$ and $(-A_h)^\alpha J^\alpha$ is a strongly continuous semigroup in* $\operatorname{Re}(\alpha) \geqq 0$. *For* $\operatorname{Re}(\alpha) \leqq n$,

$$\|(-A_h)^\alpha J^\alpha\|_p \leqq A(p)\|h\|^{\operatorname{Re}(\alpha)} \exp(2\pi|\operatorname{Im}(\alpha)|)(1 + \|A_\omega J\|_p)^n, \qquad \|\omega\| = 1.$$

PROOF. If $f \in D(A_h{}^N)$ for $N > \operatorname{Re}(\alpha)$, $(-A_h)^\alpha J^\alpha f$ is analytic in $0 < \operatorname{Re}(\alpha) < N$ by Theorem 8.2 of [17-I]. Since $D(A_h{}^N)$ is dense in $L_p(H)$, Stein's interpolation theorem applies if we set $U_z = (-A_h)^{n+z} J^{n+z}$ when n is a nonnegative integer and $0 \leqq \operatorname{Re}(z) \leqq 1$. Then the boundedness of $(-A_h)^{ic} J^{ic}$ and the boundedness of $(-A_h)^n J^n = (-A_h J)^n$ imply that $\|U_t\|_p \leqq A(t, p) \leqq A(n, p)$. Since $(-A_h)^\alpha = (-\|h\| A_\omega)^\alpha = \|h\|^\alpha (-A_\omega)^\alpha$, $\|\omega\| = 1$, and since $(-A_h)^\alpha J^\alpha = (-A_h)^t J^t (-A_h)^{ic} J^{ic}$ when $\alpha = t + ic$, we have boundedness for $(-A_h)^\alpha J^\alpha$ if $\operatorname{Re}(\alpha) \geqq 0$. Set $S_\alpha = (-A_h)^\alpha J^\alpha$. Then if $f \in L_p(H)$,

$$\begin{aligned} \|S_t f - f\|_p &\leqq \|S_t f - S_t J^\epsilon f\|_p + \|S_t J^\epsilon f - J^\epsilon f\|_p + \|J^\epsilon f - f\|_p \\ &\leqq A(p)\|J^\epsilon f - f\|_p + \|S_t J^\epsilon f - J^\epsilon f\|_p \end{aligned}$$

since $\|S_t\|$ is bounded on $0 < t < 1$. Let $\delta > 0$ and take ϵ sufficiently small that $\|J^\epsilon f - f\|_p < \delta$. Let $t < \epsilon$; then $J^\epsilon f \in D((-A_h)^t)$ and $\lim_{t\to 0^+} \|S_t J^\epsilon f - J^\epsilon f\|_p = 0$. Thus S_t is strongly continuous in $t \geqq 0$. Let N be a positive integer and let $0 < \operatorname{Re}(\alpha) < N$. If

$f \in D(A_h{}^N)$, $S_\alpha f$ is analytic in $\mathrm{Re}(\alpha) < N$ by Theorem 8.2 of [17-I]. For any $f \in L_p(H)$, let $\{f_n\} \subset D(A_h{}^N)$ converge to f in $L_p(H)$. Since $\|(-A_h)^\alpha J^\alpha\|_p$ is bounded in $\mathrm{Re}(\alpha) \leqq N$, $|\mathrm{Im}(\alpha)| \leqq A < \infty$, $S_\alpha f$ is a uniform limit of analytic functions so that $S_\alpha f$ is analytic in $\mathrm{Re}(\alpha) > 0$. If $|\arg(\alpha)| \leqq \Theta < \pi/2$, $\alpha = t + ic$,

$$\|S_\alpha f - f\|_p \leqq \|S_\alpha f - S_t f\|_p + \|S_t f - f\|_p$$
$$\leqq A(p, t)\|S_{ic} f - f\|_p + \|S_t f - f\|_p.$$

Since both S_t and S_{ic} are both strongly continuous semigroups, if $\alpha \to 0$ in $|\arg(\alpha)| \leqq \Theta < \pi/2$, $S_\alpha f$ tends to f. S_α is an analytic semigroup in $|\arg(\alpha)| < \pi/2$ and S_α is strongly continuous in $\mathrm{Re}(\alpha) \geqq 0$. Stein's interpolation theorem gives the desired estimate for $\|S_\alpha\|_p$.

Corollary 3.5. *Let $r > 0$. $(-A_h)^\alpha J_r{}^\alpha$ is an analytic semigroup of bounded operators on $L_p(H)$ in $|\arg(\alpha)| < \pi/2$ and $(-A_h)^\alpha J_r{}^\alpha$ is a strongly continuous semigroup in* $\mathrm{Re}(\alpha) \geqq 0$. *For* $\mathrm{Re}(\alpha) \leqq N$,

$$\|(-A_h)^\alpha J_r{}^\alpha\|_p \leqq A(N, p)\exp(2\pi|\mathrm{Im}(\alpha)|)(r^{-1}(r + |r - 1|))^N.$$

Proof. Since $(-A_h)^\alpha J_r{}^\alpha = (-A_h)^\alpha J^\alpha[(1 + T)^\alpha J_r{}^\alpha]$, and since $(-A_h)^\alpha J^\alpha$ and $(1 + T)^\alpha J_r{}^\alpha$ have the properties of analyticity and continuity, $(-A_h)^\alpha J_r{}^\alpha$ has these properties. The estimate follows from Theorems 2.3 and 3.4.

Corollary 3.6. *There is a constant $A_1(t, p)$ such that*

$$\|(\partial/\partial t)[(-A_h)^t J^t(f)]\|_p \leqq t^{-1}A_1(p, t)(1 + |\log\|h\||)\|h\|^t\|f\|_p;$$

$A_1(p, t)$ is bounded for $0 \leqq t \leqq N < \infty$. Furthermore,

$$(\partial/\partial t)[(-A_h)^{t+ic}J^{t+ic}(f)] = -i(\partial/\partial c)[(-A_h)^{t+ic}J^{t+ic}(f)].$$

Proof. The last derivative formula is a consequence of the analyticity of $(-A_h)^\alpha J^\alpha$ in $\mathrm{Re}(\alpha) > 0$. Set $S_t = (-A_\eta)^t J^t$ for $\eta = h\|h\|^{-1}$ and let $\omega_0 = \liminf_{t\to 0}(\log\|S_t\|/t)$. Then for $\omega > \omega_0$, $\|S_t\| \leqq M\exp(\omega t)$, $M = M(\omega)$, and $e^{-\omega t}S_t = U_t$ is a bounded, strongly continuous semigroup in $t \geqq 0$ which extends to an analytic semigroup in $|\arg(\alpha)| < \pi/2$. By K-11 of §II-2, $\|(\partial/\partial t)U_t\| \leqq N/t$ where N is a finite constant. Thus $\|(\partial/\partial t)S_t\| \leqq (M_1/t)\exp(\omega_1 t)$ for $\omega_1 > \omega > \omega_0$. Since $(-A_h)^t J^t = \|h\|^t S_t$, the desired estimate of $\|(\partial/\partial t)[(-A_h)^t J^t]\|$ holds.

4. *Riesz operators.* If $(-T)$ denotes the infinitesimal generators of the Poisson integral and if A_h denotes the infinitesimal generator of the translation semigroup T_{tBh}, $t > 0$, then $(-A_h)T^{-1}$ is the Riesz operator for the direction h. We shall consider the semigroup $(-A_h)^\alpha T^{-\alpha}$ for $\mathrm{Re}(\alpha) \geqq 0$. Since P_z is an analytic semigroup in $|\arg(z)| < \pi/4$, $T^{-\alpha}f = \lim_{N\to\infty}\Gamma(\alpha)^{-1}\int_0^N P_t(f)t^{\alpha-1}\,dt$ for each f in

$D(T^{-\alpha})$ by Theorem 6.3 of [17-III].

THEOREM 4.1. *$(-A_h)^\alpha T^{-\alpha}$ is an analytic semigroup of bounded operators on $L_p(H)$ for $|\arg(\alpha)| < \pi/2$. $(-A_h)^\alpha T^\alpha$ is strongly continuous in* $\mathrm{Re}(\alpha) \geqq 0$. *For* $\mathrm{Re}(\alpha) \leqq n$,

$$\|(-A_h)^\alpha T^{-\alpha}\|_p \leqq A(p)\|h\|^{\mathrm{Re}(\alpha)}\exp(2\pi|\mathrm{Im}(\alpha)|)(1 + \|A_\omega T^{-1}\|_p)^n$$

when $\omega = -h\|h\|^{-1}$.

PROOF. $(-A_h)^{ic}T^{-ic}$ is a strongly continuous group of bounded operators on $L_p(H)$ since both $(-A_h)^{ic}$ and T^{-ic} have this property by Corollary III.3.2. By Theorem III.3.1, $\|(-A_h)^{ic}T^{-ic}\|_p \leqq A(p)\exp(2\pi|\mathrm{Im}(\alpha)|)$.

Let $f \in D(A_h) \cap R(T)$, then

$$A_hT^{-1}f = \int_0^\infty A_hP_y(f)\,dy = \int_0^\infty \int_{0^+}^\infty A_hH_t2(f)N_t2(y)\,\frac{dt}{t}\,dy.$$

$$U_h(f) = \int_{0^+}^\infty A_hH_t2(f)\,dt$$

$$= \frac{1}{2}\int_{\tilde{H}} P\int_{-\infty}^\infty T_{tBy}f\,\frac{dt}{t}\,C_h(1)(y)\,dN(y)$$

by P-3 of §II-3. Lemma 3.3 and Minkowski's integral inequality show that $\|U_h\|_p \leqq A(p)\|h\|$. Since the operators $\int_{\epsilon\leqq|t|\leqq R} T_{tBy}f\,dt/t$ are uniformly bounded in ϵ and R, we may interchange integrals above and write $A_hT^{-1}f = KU_h(f)$. Thus $\|A_hT^{-1}\|_p \leqq A(p)\|h\|$. Now apply Stein's interpolation theorem to the family of operators $U_z = (-A_h)^{n+z}T^{-(n+z)}$; the boundedness of the operators $(-A_h)^\alpha T^{-\alpha}$ follows.

To see that $(-A_h)^\alpha T^{-\alpha}(f)$ is analytic in $\mathrm{Re}(\alpha) > 0$, let $f \in R(T)$, $f = Tg$, and consider

$$(-A_h)^\alpha T^{-\alpha}(f) - (-A_h)^\alpha J_r^{\alpha}(f)$$

$$= \Gamma(\alpha)^{-1}\int_{0^+}^\infty (-A_h)^\alpha P_t(f)t^{\alpha-1}(1 - e^{-rt})\,dt$$

$$= \Gamma(\alpha)^{-1}\int_{0^+}^\infty (-A_h)^\alpha P_t(f)t^\alpha \int_0^r e^{-ut}\,du\,dt$$

$$= \alpha\int_0^r (-A_h)^\alpha J_u^{\alpha+1}(f)\,du$$

$$= \int_0^r \alpha(-A_h)^\alpha T^{-\alpha}T^{\alpha+1}J_u^{\alpha+1}(g)\,du.$$

Since $\|(-A_h)^\alpha T^{-\alpha}\|_p \leqq A(p, \alpha)\|h\|^{\mathrm{Re}(\alpha)} \leqq A < \infty$ for $0 \leqq \mathrm{Re}(\alpha) \leqq N$ and $|\mathrm{Im}(\alpha)| \leqq \Gamma < \infty$, and since $\|T^{\alpha+1} J_u^{\alpha+1}\|_p \leqq A(p, \alpha) < A$ (independent of u) for $0 \leqq \mathrm{Re}(\alpha) \leqq N$ and $|\mathrm{Im}(\alpha)| \leqq \Gamma < \infty$,

$$\|(-A_h)^\alpha T^{-\alpha} f - (-A_h)^\alpha J_r^\alpha(f)\|_p \leqq rA\|g\|_p$$

for α in a compact subset of the right half-plane. Thus $(-A_h)^\alpha J_r^\alpha f$ converges uniformly on compact subsets of $\mathrm{Re}(\alpha) > 0$ to $(-A_h)^\alpha T^{-\alpha} f$ as $r \to 0^+$ and $(-A_h)^\alpha T^{-\alpha} f$ is analytic in $\mathrm{Re}(\alpha) > 0$ if $f \in R(T)$. Since $R(T)$ is dense in $L_p(H)$ and since $\|(-A_h)^\alpha J_r^\alpha\|_p \leqq \|(-A_h)^\alpha T^{-\alpha}\|_p \|T^\alpha J_r^\alpha\|_p \leqq A < \infty$ if α is in a compact subset of the right half-plane, for $r > 0$, an $\epsilon/3$-argument shows that $(-A_h)^\alpha T^{-\alpha}(f)$ is analytic in $\mathrm{Re}(\alpha) > 0$ for all f in $L_p(H)$. To see that $S_t = (-A_h)^t T^{-t}$ is strongly continuous in $t \geqq 0$, let $f \in R(T)$ and write

$$\|S_t f - f\|_p \leqq \|S_t f - (-A_h)^t J_r^t f\|_p + \|(-A_h)^t J_r^t f - f\|_p;$$

let $\epsilon > 0$ and take r sufficiently small that $\|S_t f - (-A_h)^t J_r^t f\|_p < \epsilon$ for $0 \leqq t \leqq 1$; $\|(-A_h)^t J_r^t f - f\|_p$ tends to zero as $t \to 0$. Since $R(T)$ is dense in $L_p(H)$ and since S_t is bounded on $0 \leqq t \leqq 1$, S_t is strongly continuous. Since S_t and S_{ic} are strongly continuous, S_α has the required continuity properties.

5. *A characterization of* $L_p^\alpha(H)$. We know that $L_p^\alpha(H)$ is equivalent to $D(T^\alpha)$ and that if $f \in L_p^\alpha(H)$, then $(-A_h)^\alpha f$ is in $L_p(H)$ for all $h \in H$ with $\|(-A_h)^\alpha f\|_p \leqq A(p, \alpha)\|h\|^{\mathrm{Re}(\alpha)}\|g\|_p$ when $f = J^\alpha g$. We shall prove a converse of this last fact.

Let B be the one-one Hilbert-Schmidt operator in the definition of the Poisson integral and let $G: H^* \to$ Borel measurable functions on H_B denote the Wiener space representative for the normal distribution on H. Let p' be a Borel probability measure on H_B such that $P_t f = \int_{H_B} T_{tBy} f \, dp'(y)$. In [12] it is shown that such a measure p' exists.

THEOREM 5.1. *Let $f \in L_p(H)$ and suppose that $\varphi(y) = (-A_y)^\alpha f$ is a Borel measurable function from H_B to $L_p(H)$ such that $\int_{H_B} (-A_y)^\alpha f \, dp'(y)$ is in $L_p(H)$. Then f is in $L_p^\alpha(H)$.*

PROOF. By Theorem 4.4 of [17-II], if $0 < \mathrm{Re}(\alpha) \leqq m$ and if $f \in D((-A_y)^\alpha)$,

$$(-A_y)^\alpha f = K(\alpha, m)^{-1} \int_{0^+}^\infty (I - T_{tBy})^m f t^{-\alpha-1} \, dt$$

where $K(\alpha, m) = \int_{0^+}^\infty (1 - e^{-t})^m t^{-\alpha-1} \, dt$. Since $(-A_y)^\alpha f$ is p'-measurable and $\int_{H_B} (-A_y)^\alpha f \, dp'(y) \in L_p(H)$,

$$\begin{aligned}T^\alpha f &= K(\alpha, m)^{-1}\int_{0^+}^\infty (I - P_t)^m f t^{-\alpha-1}\, dt \\ &= K(\alpha, m)^{-1}\int_{0^+}^\infty \int_{H_B} (I - T_{tBy})^m f\, dp'(y) t^{-\alpha-1}\, dt \\ &= \int_{H_B} (-A_y)^\alpha f\, dp'(y)\end{aligned}$$

and $f \in D(T^\alpha)$ which is equivalent to $L_p{}^\alpha(H)$.

V. **Singular integrals.** In this section we shall use the analytic semigroups of §IV to study the singular integrals of Calderon-Zygmund, Muckenhoupt, and Wheeden. Let T_{tBy} denote the translation semigroup for the direction By and let A_y denote its infinitesimal generator. Let n be a nonnegative integer and set

$$\begin{aligned}R_n(f, y, t) &= (n!)^{-1} \int_0^t (t - u)^n T_{uBy} A_y{}^{n+1} f\, du \\ &= \Gamma(n)^{-1} \int_0^t (t - u)^{n-1}[T_{uBy}A_y{}^n f - A_y{}^n f]\, du \qquad \text{if } n \geqq 1, \\ &= T_{tBy} f - f \qquad \text{if } n = 0.\end{aligned}$$

Let μ be a Borel measure (possibly unbounded) on H such that $d\mu_\alpha(y) = \|y\|^\alpha\, d\mu(y)$ is finite and has zero mass at $y = 0$ and set

$$G^\alpha(f) = \int_{0^+}^\infty \int_H R_n(f, y, t)\, d\mu(y) t^{-\alpha-1}\, dt$$

when $0 \leqq n \leqq \operatorname{Re}(\alpha) < n + 1$. The operators G^α include the classes of singular integral operators mentioned above when μ is suitably restricted. We shall state three theorems regarding these operators. The proofs will be given after all of the theorems have been stated.

THEOREM 1. *Let μ be a Borel measure on H such that $\int_H \|y\|^{\operatorname{Re}(\alpha)}\, d|\mu|(y) < \infty$. If $n < \operatorname{Re}(\alpha) < n + 1$, n a nonnegative integer, then*

$$G^\alpha J^\alpha(f) = \frac{-\pi}{\Gamma(\alpha + 1)\sin(\pi\alpha)} \int_H (-A_y)^\alpha J^\alpha(f)\, d\mu(y)$$

when J^α is the Bessel potential of order α. $G^\alpha J^\alpha$ is a bounded operator on $L_p(H)$ with norm

$$\|G^\alpha J^\alpha\|_p \leqq A(p, \alpha) \int_H \|y\|^{\operatorname{Re}(\alpha)}\, d|\mu|(y).$$

THEOREM 2. *Let $\operatorname{Re}(\alpha) = n$, a nonnegative integer, and let*

$\mathrm{Im}(\alpha) = c \neq 0$. *Suppose that* $\int_H \|y\|^n \, d|\mu|(y) < \infty$ *and that* $\int_H p_n(y) \, d\mu(y) = 0$ *for all Borel measurable functions* $p_n(y)$ *with* $|p_n(y)| \leqq M\|y\|^n$ *which are homogeneous of degree n. Then*

$$G^\alpha J^\alpha(f) = \frac{-\pi}{\Gamma(\alpha + 1)\sin(\pi\alpha)} \int_H (-A_y)^\alpha J^\alpha(f) \, d\mu(y)$$

when J^α *is the Bessel potential of order* α. $G^\alpha J^\alpha$ *is a bounded operator on* $L_p(H)$ *with norm*

$$\|G^\alpha J^\alpha\|_p \leqq A(p, \alpha) \int_H \|y\|^n \, d|\mu|(y).$$

THEOREM 3. *Let* $\mathrm{Re}(\alpha) = n$, *a positive integer, and let* $\mathrm{Im}(\alpha) = 0$. *Suppose that* $\int_H (\|y\|^{n+\epsilon} + \|y\|^{n-\epsilon}) \, d|\mu|(y) < \infty$ *for all sufficiently small* $\epsilon > 0$ *and suppose that* $\int_H p_n(y) \, d\mu(y) = 0$ *for all Borel measurable functions* $p_n(y)$ *with* $|p_n(y)| \leqq M\|y\|^n$ *which are homogeneous of degree n. Then*

$$G^n J^n(f) = \frac{(-1)^{n+1}}{n!} \int_H \frac{\partial}{\partial\alpha}(-A_y)^\alpha J^\alpha(f) \,\Big|_{\alpha=n} d\mu(y)$$

when $J^n(J^\alpha)$ *is the Bessel potential of order* $n(\alpha)$. $G^n J^n$ *is a bounded operator on* $L_p(H)$ *with norm*

$$\|G^n J^n\|_p \leqq A(p, n) \int_H (1 + |\log\|y\||)\|y\|^n \, d|\mu|(y).$$

PROOF OF THEOREM 1. Set $\alpha = n + \beta$ where $0 < \mathrm{Re}(\beta) < 1$ and $R_n(f, y, t) = T_{tBy}f - f$ if $n = 0$ and

$$R_n(f, y, t) = \Gamma(n)^{-1} \int_0^t (t - u)^{n-1}(T_{uBy}A_y{}^n f - A_y{}^n f) \, du \quad \text{if } n \geqq 1.$$

Since $A^n_{ty}f = t^n A_y{}^n f$, $R_n(f, y, t) = R_n(f, ty, 1)$ and

$$\int_{0^+}^\infty [T_{tuBy}A_y{}^n J^\alpha f - A_y{}^n J^\alpha f] \, t^{-\beta-1} \, dt = (-1)^n \Gamma(-\beta) u^\beta (-A_y)^\alpha J^\alpha(f)$$

by K-3. Since $\Gamma(n)^{-1}\int_0^1 (1 - u)^{n-1}u^\beta \, du = \Gamma(\beta + 1)\Gamma(\alpha + 1)^{-1}$, we have that

$$G^\alpha J^\alpha(f) = \frac{(-1)^n \Gamma(-\beta)\Gamma(\beta + 1)}{\Gamma(\alpha + 1)} \int_H (-A_y)^\alpha J^\alpha(f) \, d\mu(y)$$

$$= \frac{-\pi}{\Gamma(\alpha + 1)\sin(\pi\alpha)} \int_H (-A_y)^\alpha J^\alpha(f) \, d\mu(y).$$

The estimate for the norm of $G^\alpha J^\alpha$ follows from Minkowski's integral inequality and Theorem IV.3.4.

PROOF OF THEOREM 2. Since $\int_H p_n(y)\,d\mu(y) = 0$ for measurable functions which are homogeneous of degree n and have $|p_n(y)| \leqq M\|y\|^n$, $\int_H A_y{}^n J^\alpha f\,d\mu(y) = 0$ for all f in $L_p(H)$. Thus

$$\int_H R_n(J^\alpha f, y, t)\,d\mu(y) = \int_H R_{n-1}(J^\alpha f, y, t)\,d\mu(y).$$

Write

$$R_{n-1}(J^\alpha f, y, t) = \Gamma(n)^{-1}\int_0^t (t-u)^{n-1} T_{uBy} A_y{}^n J^\alpha(f)\,du.$$

Then $R_{n-1}(J^\alpha f, y, t) = R_{n-1}(J^\alpha f, ty, 1)$ and since $t^{-n}A^n_{ty} = A_y{}^n$, we have that

$$\begin{aligned} G^\alpha J^\alpha(f) &= \int_H \int_{0^+}^\infty t^{-n} R_{n-1}(J^\alpha f, ty, 1) t^{-ic-1}\,dt\,d\mu(y) \\ &= \int_0^1 \frac{(1-v)^{n-1}v^{ic}}{\Gamma(n)} \int_H \Big[\lim_{\epsilon\to 0^+}\int_{\epsilon v}^\infty T_{uBy}A_y{}^n J^\alpha f u^{-ic-1}\,du \\ &\qquad\qquad - \frac{(\epsilon v)^{-ic}}{ic} A_y{}^n J^\alpha f\Big]\,d\mu(y)\,dv \\ &= \frac{(-1)^n\Gamma(-ic)\Gamma(1+ic)}{\Gamma(\alpha+1)}\int_H (-A_y)^\alpha J^\alpha f\,d\mu(y) \\ &= \frac{-\pi}{\Gamma(\alpha+1)\sin(\pi\alpha)}\int_H (-A_y)^\alpha J^\alpha(f)\,d\mu(y) \end{aligned}$$

by Theorem III.3.4. The estimate for the norm of $G^\alpha J^\alpha$ follows from Minkowski's integral inequality and Theorem IV.3.4.

PROOF OF THEOREM 3. First consider the constant

$$\begin{aligned} &-\pi\Gamma(\alpha+1)^{-1}(\sin(\pi\alpha))^{-1} \\ &\qquad = \Gamma(\alpha+1)^{-1}\left[\frac{\pi(n-\alpha)}{\sin(\pi n)-\sin(\pi\alpha)}\right](n-\alpha)^{-1}; \end{aligned}$$

as α tends to n, $\pi(n-\alpha)(\sin(\pi n)-\sin(\pi\alpha))^{-1}$ converges to $(\cos(\pi n))^{-1} = (-1)^n$. Since $\int_H(-A_y)^n J^n(f)\,d\mu(y) = 0$,

$$G^\alpha J^\alpha(f) = \frac{(-1)^{n+1}}{\Gamma(\alpha+1)} M'(\alpha)\int_H \frac{[(-A_y)^\alpha J^\alpha f - (-A_y)^n J^n f]}{(\alpha - n)}\,d\mu(y)$$

for α near n where $M'(\alpha)$ tends to 1 as α tends to n. Let $\alpha = t$ be real so that

$$(-A_y)^t J^t(f) - (-A_y)^n J^n(f) = \int_n^t (\partial/\partial u)[(-A_y)^u J^u(f)]\, du.$$

By Corollary IV.3.6,

$$\left\| \frac{\partial}{\partial u} [(-A_y)^u J^u(f)] \right\|_p \leqq t^{-1} A_1(p, t)(1 + |\log\|y\||)\|y\|^t \|f\|_p;$$

$A_1(p, t)$ is bounded on finite t-intervals. Thus

$$\|(-A_y)^t J^t f - (-A_y)^n J^n f\| \leqq \int_n^t (1 + |\log\|y\||)\|y\|^u \frac{du}{u} A(p)\|f\|_p.$$

Since $(1 + |\log\|y\||)\|y\|^u \leqq A(\epsilon)(\|y\|^{n+\epsilon} + \|y\|^{n-\epsilon})$ for small $\epsilon > 0$ and because the last function is $|\mu|$-integrable,

$$\lim_{\alpha \to n} G^\alpha J^\alpha(f) = \frac{(-1)^{n+1}}{\Gamma(n+1)} \int_H \frac{\partial}{\partial t} (-A_y)^t J^t(f) \,|_{t=n}\, d\mu(y) = K^n(f)$$

is a bounded operator on $L_p(H)$ with

$$\|K^n\|_p \leqq A(n, p) \int_H (1 + |\log\|y\||)\|y\|^n \, d|\mu|(y).$$

By Corollary IV.3.6,

$$\frac{\partial}{\partial t} [(A_y)^{t+ic} J^{t+ic}(f)] = -i \frac{\partial}{\partial c} [(-A_y)^{t+ic} J^{t+ic}(f)]$$

$$= -i \left[V_y(-A_y)^{t+ic} J^{t+ic}(f) + (-A_y)^{t+ic} \frac{\partial}{\partial c} J^{t+ic}(f) \right].$$

Since J^α is an analytic semigroup, $-i(\partial/\partial c)J^{t+ic}(f) = (\partial/\partial\alpha)J^\alpha(f)$ is a bounded operator on $L_p(H)$ for $\mathrm{Re}(\alpha) > 0$ and $\int_H (-A_y)^n (\partial/\partial\alpha)J^\alpha(f)|_{\alpha=n}\, d\mu(y) = 0$. Thus

$$K^n(f) = \frac{(-1)^{n+1}(-i)}{\Gamma(n+1)} \int_H V_y(-A_y)^n J^n(f)\, d\mu(y)$$

when V_y is the infinitesimal generator of the semigroup $(-A_y)^{ic}$, $c > 0$. By Theorem III.3.5,

$$V_y(g) = -i \left[\int_{0^+}^1 (T_{tBy} g - g)\, dt/t + \int_1^\infty T_{tBy} g\, dt/t + Cg \right]$$

when $g \in R(A_y) \cap D(A_y)$ and C is Euler's constant. Set $g_y = (-A_y)^n J^n(f)$ so that $g_y \in R(A_y)$ and assume that f is in $L_p^{\,1}(H)$, $f \in D(A_y)$, so that $g_y \in D(A_y)$. Then

$$K^n(f) = \frac{(-1)^n}{n!} \int_H \left[\int_{0^+}^1 (T_{tBy} g_y - g_y)\, dt/t \right.$$

$$\left. + \int_1^\infty T_{tBy} g_y dt/t + C g_y \right] d\mu(y)$$

$$= \frac{(-1)^n}{n!} \int_H \int_{0^+}^\infty T_{tBy}(-A_y)^n J^n(f) \frac{dt}{t}\, d\mu(y)$$

since $\int_H (-A_y)^n J^n(f)\, d\mu(y) = 0$. Thus $K^n = G^n J^n$. This completes the proof of Theorem 3.

When $0 < \alpha < 2$, G^α is the hypersingular integral operator studied by Wheeden in [23]. If $\mathrm{Re}(\alpha) = 0$, $\mathrm{Im}(\alpha) \neq 0$, G^α is the singular integral operator studied by Muckenhoupt in [18]. When $\alpha = 0$ and $\int_H d\mu(y) = 0$, G^α is a Calderon-Zygmund operator; the present treatment says nothing about the boundedness of Calderon-Zygmund operators. See [1] and [5].

VI. **Littlewood-Paley theory.** In this section we shall use imaginary powers of the directional derivatives and imaginary powers of T to estimate the p-norm of the Littlewood-Paley g-function and the p-norms of the maximal functions associated with the Poisson integral. The Littlewood-Paley g-function is

$$g_\alpha^r(f) = \left(\int_0^\infty |y^\alpha T^\alpha P_y(f)|^r\, dy/y \right)^{1/r},$$

$1 \leqq r < \infty$, $1 \leqq \mathrm{Re}(\alpha) < \infty$ and the most interesting maximal functions are

$$M_\alpha(f) = \sup_{y>0} |y^\alpha T^\alpha P_y(f)|,$$

$0 \leqq \mathrm{Re}(\alpha) < \infty$. Neither of these functions is linear, but, by Minkowski's inequality, each of them is sublinear. M_α is just g_α^∞ for $\mathrm{Re}(\alpha) \geqq 1$. We shall use certain linear operators to approximate M_α and g_α^r and to aid in the estimation of $\|g_\alpha^r(f)\|_p$ and $\|M_\alpha(f)\|_p$. As before, p is fixed in $1 < p < \infty$.

1. *Certain linear operators.* Let k be a positive integer, let P_y denote the Poisson integral, set $(-d/dx)^k P_y = P_y^{(k)} = T^k P_y$, and if $\mathrm{Re}(\alpha) \geqq 0$, set $P_y^{(\alpha)} = T^\alpha P_y$. The following lemma will be used repeatedly.

LEMMA 1.1. *There is a polynomial $P_k(u)$ of degree k such that* $y^k(d/dy)^k(y \exp[-t^{-1}y^2]) = yP_k(t^{-1}y^2)\exp[-t^{-1}y^2]$.

PROOF. Since $(d/dy)(y\exp[-t^{-1}y^2]) = (1-2t^{-1}y^2)\exp[-t^{-1}y^2]$, the statement is true for $k=1$. Assume that it is true for $k-1$ and write $y^{k-1}(d/dy)^{k-1}(y\exp[-t^{-1}y^2]) = yP_{k-1}(t^{-1}y^2)\exp[-t^{-1}y^2]$. Then

$$y^k(d/dy)^k(y\exp[-t^{-1}y^2]) = y^k(d/dy)(y^{2-k}P_{k-1}(t^{-1}y^2)\exp[-t^{-1}y^2]),$$

and direct computation shows that this expression has the desired form.

Let φ be a function from $(0,\infty) = R^+$ to measurable functions on H such that $\varphi \in L_1(R^+, dy/y) \cap L_2(R^+, dy/y)$. Let $G: H^* \to$ measurable functions on $(\tilde{H}, N)$ be a representative of the normal distribution on H and set

$$\|\varphi\|_{r\infty} = \operatorname*{ess\,sup}_{x\in\tilde{H}} \left(\int_0^\infty |\varphi(y)|^r\, dy/y\right)^{1/r}$$

for $1 \leqq r < \infty$. Set

$$T_\varphi{}^\alpha(f) = T_\varphi(f) = \int_{0^+}^\infty y^\alpha P_y{}^{(\alpha)}(f)\varphi(y)\, dy/y$$

when $\mathrm{Re}(\alpha) \geqq 1$. The linear operators $T_\varphi{}^\alpha$ are closely related to the g-function and to the maximal functions.

LEMMA 1.2. *Let $\|\varphi\|_{r\infty} < \infty$ for $r = 1, 2$. If k is a positive integer,* $T_\varphi{}^k(f)$

$$= K\int_H \int_{-\infty}^\infty \Gamma(k-iv)D_y{}^{iv}(f)\Gamma\left(\frac{1-iv}{2}\right)\hat{\Phi}(v)\, dv\, dn_1 \circ B^{-1}(y)$$

when $(-D_y)$ is the infinitesimal generator of the translation semi-group T_{ty}, $t>0$, and $\hat{\Phi}$ is the Mellin transform of φ.

PROOF. Assume that $\varphi \in L_1(R^+, dy/y) \cap L_2(R^+, dy/y)$. Then

$$T_\varphi{}^k(f) = T_\varphi(f) = \int_{0^+}^\infty y^k P_y{}^{(k)}(f)\varphi(y)\, dy/y$$

$$= K\int_{0^+}^\infty H_t 2(f)\psi(t)\, dt/t$$

where

$$\psi(t) = \int_{0^+}^\infty N_k(y/t)\varphi(y)\, dy/y \quad \text{when } N_k(y/t) = y^k(-\partial/\partial y)^k N_t 2(y).$$

By Lemma 1.1, $N_k(y/t) = (y/t)P_k(y^2t^{-2})\exp(-y^2t^{-2})$ when P_k is a polynomial of degree k. Set $t = e^u$ and $y = e^Y$, then $\psi(e^u) = \nu(u)$ is the convolution of two functions each of which is in $L_2(-\infty, \infty)$; thus $\nu(u)$ is the Fourier transform of a function $L(v)$ in $L_1(-\infty, \infty)$. Replace $t = e^u$ to write $\psi(t) = \int_{-\infty}^{\infty} t^{iv}L(v)\,dv$. Set $M_k(X) = N_k(e^X)$ and $\Phi(X) = \varphi(e^X)$. Then $L(v) = K\hat{M}_k(v)\hat{\Phi}(-v)$ where $\hat{}$ denotes the Fourier transform; $\hat{\Phi}(u)$ is also the Mellin transform of φ at u.

We shall calculate $\hat{M}_k(v)$ explicitly, show that $u^{-1}\hat{M}_k(u)$ is a rapidly decreasing smooth function in the sense of Laurent Schwartz, and show that

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} t^{iv}v^{-1}L(v)\,dv = 0$$

with the limit existing boundedly in $x \in \tilde{H}$. Set $\hat{M}_k(Y) = \int_{-\infty}^{\infty} e^{iYX}M_k(X)\,dX$ and replace $x = e^X$ so that $\hat{M}_k(Y) = \int_0^{\infty} x^{iY}N_k(x)\,dx/x$. Since $N_k(x) = Kx^k(d/dx)^k(x\exp(-x^2))$, k integrations by parts show that

$$\hat{M}_k(Y) = K\prod_{n=0}^{k-1}(n+iY)\int_0^{\infty} x^{iY}\exp(-x^2)\,dx.$$

Set $t = x^2$ to conclude that

$$\hat{M}_k(Y) = K\Gamma((iY+1)/2)\prod_{n=0}^{k-1}(n+iY).$$

Since

$$\int_0^{\infty} x^{iY}\exp(-x^2)\,dx = \int_{-\infty}^{\infty} e^{iYX}\exp(X)\exp(-e^{2X})\,dX,$$

$Y^{-1}\hat{M}_k(Y)$ is a rapidly decreasing smooth function since $\hat{M}_k(Y)$ contains a factor of Y and since $\exp(X)\exp(-e^{2X})$ is a rapidly decreasing smooth function. Thus $Y^{-1}\hat{M}_k(y)$ is in $L_p(R, dY)$ for all $1 \leqq p \leqq \infty$. By Hölder's inequality and the Hausdorff-Young theorem, $\int_{-\infty}^{\infty} |v^{-1}L(v)|\,dv$ is in $L^{\infty}(H)$ since $\|\varphi\|_{r\infty} \leqq M$ for some r in $1 \leqq r \leqq 2$. Set $t = e^{-\rho}$, so that $\int_{-\infty}^{\infty} t^{iv}v^{-1}L(v)\,dv$ is the Fourier transform of an L_1-function evaluated at $(-\rho)$. By the Riemann-Lebesgue lemma, the integral converges to zero as ρ tends to $+\infty$. This shows that $\lim_{t\to 0^+}\int_{-\infty}^{\infty} t^{iv}v^{-1}L(v)\,dv = 0$ and the limit exists boundedly in $x \in \tilde{H}$.

Now let $y \in H$ and set

$$U_y(f) = \int_{0^+}^{\infty} T_{ty} f \psi(t)\, dt/t$$

$$= \int_{-\infty}^{\infty} \int_{0^+}^{\infty} T_{ty} f t^{iv} \frac{dt}{t} L(v)\, dv$$

$$= \int_{-\infty}^{\infty} \lim_{\epsilon \to 0^+} \left[\int_{\epsilon}^{\infty} T_{ty} f t^{iv-1}\, dt + \frac{\epsilon^{iv}}{iv} f - \frac{\epsilon^{iv}}{iv} f \right] L(v)\, dv.$$

By an argument similar to that used in the proof of Theorem III.2.2, one shows that the inner integral defines a bounded operator on $L_p(H)$with norm at most $Apq[(|v| + 1)^2 |v|^{-1} + |v|^{-1}]$ and that $U_\epsilon f = \int_\epsilon^\infty T_{ty} f t^{iv-1}\, dt + (\epsilon^{iv}/iv)\, f$ converges almost everywhere and in $L_p(H)$ to $\Gamma(iv) D_y{}^{-iv} f$ as a sequence $\epsilon_n \searrow 0$. Thus for almost every $x \in \tilde{H}$, $U_\epsilon f(x)$ is a bounded function of ϵ_n. Since $L(v)(x)$ is in $L_1(R, dv)$ for almost every $x \in \tilde{H}$, the dominated convergence theorem implies that

$$U_y(f) = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0^+} \left[\int_{\epsilon}^{\infty} T_{ty} f\, t^{iv-1}\, dt + \frac{\epsilon^{iv}}{iv} f \right] L(v)\, dv$$

$$- \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{\epsilon^{iv}}{iv} L(v)\, dv f$$

$$= \int_{-\infty}^{\infty} \Gamma(iv) D_y{}^{-iv}(f) L(v)\, dv$$

$$= K \int_{-\infty}^{\infty} \Gamma(-iv) D_y{}^{iv}(f) \hat{M}_k(-v) \hat{\Phi}(v)\, dv$$

$$= K \int_{-\infty}^{\infty} \Gamma(k - iv) D_y{}^{iv}(f) \Gamma\left(\frac{1 - iv}{2}\right) \hat{\Phi}(v)\, dv.$$

Since $T_\varphi{}^k(f) = \int_H U_y(f)\, dn_1 \circ B^{-1}(y)$, we have the desired identity.

We need one last basic lemma to estimate the $T_\varphi{}^\alpha$.

Lemma 1.3. *Let h be a nonzero element of H and let $\omega > \pi/2$. Then*

$$\mu(f, r, h)(x) = \left(\int_{-\infty}^{\infty} |\exp(-\omega|c|) D_h{}^{ic} f(x)|^r dc \right)^{1/r}$$

satisfies $\|\mu(f, r, h)\|_p \leqq A(p, r, \omega) \|f\|_p$ for $r < p < \infty$.

Proof. Since $\|D_h{}^{ic} f\|_p \leqq A(p) \exp(\eta|c|) \|f\|_p$ for $\eta > \pi/2$ and

$\eta - \pi/2$ arbitrarily small, Minkowski's integral inequality implies that the desired estimate for μ holds.

THEOREM 1.4. *If* $\mathrm{Re}(\alpha) \geqq 1$, *if* $1 \leqq r \leqq 2$, *and if* $p \geqq r$, *then* $\|T_\varphi{}^\alpha f\|_p \leqq A(p, r, \alpha)\|f\|_p\|\varphi\|_{r\infty}$.

PROOF. Since $|\Gamma(ic)| = \pi^{1/2}(|c|\sinh \pi|c|)^{-1/2}$, for $\delta > 0$ and small, $|\Gamma(k + ic)| \leqq A(\delta)\exp(-(\pi/2 - \delta)|c|)$; since

$$\pi^{1/2}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z + 1/2), \quad |\Gamma((1 + ic)/2)| \leqq A \exp(-\pi|c|/4).$$

Thus if

$$U_y(f) = \int_{-\infty}^{\infty} \Gamma(k - iv)D_y{}^{iv}(f)\Gamma\left(\frac{1 - iv}{2}\right)\hat{\Phi}(v)\, dv,$$

then $|U_y(f)| \leqq \mu(f, r, y)\|\hat{\Phi}\|_{r'} \leqq \mu(f, r, y)\|\varphi\|_{r\infty}$ when $r^{-1} + r'^{-1} = 1$ by Hölder's inequality and the Hausdorff-Young Theorem. Since $T_\varphi{}^k(f) = K \int_H U_y(f)\, dn_1 \circ B^{-1}(y)$, $\|T_\varphi{}^k(f)\|_p \leqq A(p, r, k)\|f\|_p\|\varphi\|_{r\infty}$ by Minkowski's integral inequality and Lemma 1.3.

If $\alpha = k + iu$ when k is a positive integer and u is a real number, then if $\varphi_1(y) = y^{iu}\varphi(y)$, $T_\varphi{}^\alpha(f) = T^k_{\varphi_1}(T^{iu}f)$ and $\|T_\varphi{}^\alpha(f)\|_p \leqq A(p, r, k)\exp(\pi|u|)\|f\|_p\|\varphi\|_{r\infty}$. Since $T^\alpha P_y(f)$ is analytic in $k < \mathrm{Re}(\alpha) < k + 1$ and continuous in $k \leqq \mathrm{Re}(\alpha) \leqq k + 1$ for a dense set of f in $L_p(H)$ by Theorem 8.2 of [17-I], Stein's interpolation theorem applies to the $T_\varphi{}^\alpha$ and $\|T_\varphi{}^t f\|_p \leqq A(p, t)\|f\|_p\|\varphi\|_{r\infty}$ for $k \leqq t \leqq k + 1$. Since if $\alpha = t + iu$, $T_\varphi{}^\alpha(f) = T^t_{\varphi_1}(T^{iu}f)$, we have that the desired inequality holds for all $\mathrm{Re}(\alpha) \geqq 1$.

2. *The maximal theorems.* In this section we shall use the operators $T_\varphi{}^\alpha$ to estimate the p-norm of the maximal function $M_\alpha(f)$, $\mathrm{Re}(\alpha) \geqq 1$, and we shall investigate some of the implications of the inequalities for $M_\alpha(f)$. If $\mathrm{Re}(\alpha) \geqq 0$, set

$$M_\alpha(f) = \sup_{y>0} |y^\alpha T^\alpha P_y(f)|$$

and if $\mathrm{Re}(\alpha) < 0$, set

$$M_\alpha(f) = \sup_{y>0} \left| y^\alpha\Gamma(-\alpha)^{-1}\int_0^y (y - x)^{-\alpha-1}P_y(f)\, dy \right| .$$

In general, denote $M_\alpha f = \sup_{y>0}|y^\alpha(-d/dy)^\alpha P_y(f)|$.

THEOREM 2.1. *If* $\mathrm{Re}(\alpha) \geqq 1$ *and if* $1 < p < \infty$, $\|M_\alpha(f)\|_p \leqq A(p, \alpha)\|f\|_p$.

PROOF. For $\operatorname{Re}(\alpha) \geqq 1$, $\|M_\alpha f\|_p = \sup\{\|T_\varphi{}^\alpha f\|_p : \|\varphi\|_{1\infty} \leqq 1\}$. By Theorem 1.4, $\|T_\varphi{}^\alpha\|_p \leqq A(p, \alpha)\|\varphi\|_{1\infty}$, so that $\|M_\alpha(f)\|_p \leqq A(p, \alpha)\|f\|_p$.

We shall now extend the result of Theorem 2.1 to $-\infty < \operatorname{Re}(\alpha) < 1$; the following propositions contribute to the general result.

PROPOSITION 2.2. *Let $y \to T_y$ be the regular representation of the additive group of H acting on $L_p(H)$ and let $h \in H$. Then*

$$(Nf)(x) = \sup_{t>0} t^{-1} \left| \int_0^t (T_{sh}f)(x)\, ds \right|$$

satisfies $\|f\|_p \leqq \|Nf\|_p \leqq A(p)\|f\|_p$ where $A(p)$ does not depend on H.

PROOF.

$$t^{-1} \int_0^t T_{sh}f\, ds = (t\|h\|)^{-1} \int_0^{t\|h\|} T_{s\omega}f\, ds \quad \text{where } \omega = \|h\|^{-1}h,$$

so that Nf is independent of $\|h\|$ if $h \neq 0$. If $h = 0$, $Nf = f$. Suppose first that f is a bounded tame function based on K_1 and let K denote the span of K_1 and h. Then Nf is tame and based on K, and

$$\|Nf\|_p^p = \int_K \left| \sup_{t>0} t^{-1} \int_0^t f(x - t\omega) \cdot \exp\left[\frac{(t\omega, x)}{p} - \frac{t^2}{2p}\right] dt \right|^p dn(x).$$

Let $F(x) = f(x)\exp[-\|x\|^2/2p]$, so that

$$\begin{aligned} \|Nf\|_p^p &= (2\pi)^{-k/2} \int_K \left|\sup_{t>0} t^{-1} \int_0^t F(x - t\omega)\, dt\right|^p dx \\ &= (2\pi)^{-k/2} \int_K |F^*(x)|^p\, dx \end{aligned}$$

where $k = \dim(K)$ and $F^*(x) = \sup_{t>0} t^{-1}|\int_0^t F(x - t\omega)\, dt|$. By the Dunford-Schwartz Ergodic Theorem [4], $\|F^*\|_p \leqq A(p)\|F\|_p = A(p)\|f\|_p$; so that $\|Nf\|_p \leqq A(p)\|f\|_p$; $A(p) = 2q^{1/p}$ does not depend on ω. Let $f \geqq 0$ be in $L_p(H)$ and let $f_n \geqq 0$ be a sequence of bounded tame functions which converge to f in $L_p(H)$. Since $|Nf_n - Nf_m| \leqq N(|f_n - f_m|)$, the sequence Nf_n is Cauchy in $L_p(H)$; let $G(x)$ be the limit of the Nf_n in $L_p(H)$. By taking a subsequence if necessary, we may suppose that the $Nf_n(x)$ converge almost everywhere to $G(x)$ and that the f_n converge almost everywhere to $f(x)$. Then $t^{-1}\int_0^t (T_{s\omega}f_n)(x)\,ds \leqq Nf_n(x)$ almost everywhere for all n, and $t^{-1}\int_0^t (T_{s\omega}f)(x)\,ds \leqq G(x)$ almost everywhere. Thus $(Nf)(x) \leqq G(x)$ and $\|Nf\|_p \leqq \|G\|_p \leqq A(p)\|f\|_p$. Since $t^{-1}|\int_0^t (T_{s\omega}f)(x)\,ds| \leqq t^{-1}\int_0^t T_{s\omega}|f|(x)\,ds$, the right-hand side of the desired inequality is verified. For bounded tame functions, $\lim_{t\to 0^+} t^{-1}\int_0^t (T_{s\omega}f)(x)\,ds = f(x)$ almost everywhere, so that the left-hand inequality holds also; in fact, $|f(x)| \leqq (Nf)(x)$ almost everywhere.

Corollary 2.3. *Let $P_z(f)$ be the Poisson integral of f and set*

$$(N_1 f)(x) = \sup_{t>0} t^{-1} \left| \int_0^t P_z(f)(x)\,dz \right| .$$

Then $\|f\|_p \leqq \|N_1 f\|_p \leqq A(p)\|f\|_p$.

Proof. Write $P_z(f) = \int_H (T_y f)\,dp_z(y) = \int_H T_{zy}f\,dp_1(y)$. Then $(N_1 f)(x) \leqq \int_H (Nf)(x)\,dp_1(y)$. By Minkowski's integral inequality and by Proposition 2.2, $\|N_1 f\|_p \leqq A(p)\|f\|_p$. Since $\lim_{t\to 0} t^{-1} \cdot \int_0^t P_z(f)\,dz = f$ in L_p, there is a sequence t_n tending to zero such that $t_n{}^{-1}\int_0^{t_n} P_z f\,dz$ converges to f almost everywhere and the left-hand inequality holds.

Corollary 2.4. *Set*

$$(N_2 f)(x) = \sup_{t>0} t^{-1} \left| \int_0^t H_s 2(f)(x)\,ds \right| .$$

Then $\|f\|_p \leqq \|N_2 f\|_p \leqq A(p)\|f\|_p$.

Proof. Write $H_s 2(f) = \int_H T_{sy}f\,dn_1 \circ B^{-1}(y)$, so that $N_2 f(x) \leqq \int_H Nf(x)\,dn_1 \circ B^{-1}(y)$, and the desired right-hand inequality holds. The left-hand inequality holds as in the proof of Corollary 2.3.

Proposition 2.5. *Let* $(M_0 f)(x) = \sup_{y>0}|P_y f(x)|$, *then* $\|f\|_p \leqq \|M_0 f\|_p \leqq A(p)\|f\|_p$.

Proof. Since $P_z f$ tends to f in L_p as $z \to 0$, any sequence z_n which tends to zero has a subsequence, also called z_n, such that $P_{z_n} f$ converges almost everywhere to f. Thus $|f(x)| \leqq M_0 f(x)$, and the left-hand side of the inequality holds.

To prove the right-hand inequality, write

$$P_z(f) = \int_0^\infty H_t f N_t(z)\,dt/t = 2\int_0^\infty H_t 2(f) N_t 2(z)\,dt/t.$$

Set $N_t2(z) = (\pi)^{-1/2}(z/t)\exp[-(z/t)^2] = N(z/t)$, and integrate by parts to get

$$P_zf = -2\int_0^\infty \left[t^{-1}\int_0^t H_s2(f)\,ds\right]\left[t\frac{d}{dt}t^{-1}N(z/t)\right]dt.$$

Thus

$$|P_zf(x)| \leqq 2N_2f(x)\int_0^\infty t\left|\frac{d}{dt}(t^{-1}N(z/t))\right|dt.$$

The integral on the right of this inequality is finite and independent of z, so $|P_zf(x)| \leqq AN_2f(x)$ and $\|M_0f\|_p \leqq A(p)\|f\|_p$.

Corollary 2.6. *If* $\mathrm{Re}(\alpha) < 0$, *set*

$$y^\alpha\left(-\frac{d}{dy}\right)^\alpha P_y(f) = \Gamma(-\alpha)^{-1}y^\alpha\int_0^y (y-t)^{-\alpha-1}P_t(f)\,dt.$$

Then $(M_\alpha f)(x) = \sup_{x>0}|y^\alpha(-d/dy)^\alpha P_y(f)(x)|$ *satisfies* $\|f\|_p \leqq K_\alpha\|M_\alpha f\|_p \leqq A(\alpha,p)\|f\|_p$.

Proof. A subsequence argument shows that $|f(x)| \leqq K(\alpha)M_\alpha f(x)$, so that the left-hand side of the inequality holds. On the right side

$$\left|y^\alpha\left(-\frac{d}{dy}\right)^\alpha P_yf(x)\right| \leqq A(\alpha)\int_0^1 (1-u)^{-1-\mathrm{Re}(\alpha)}P_{uy}(|f|)\,du$$
$$\leqq A_1(\alpha)M_0f(x);$$

and $\|M_\alpha f\|_p \leqq A(\alpha,p)\|f\|_p$.

Corollary 2.7. *If* $\mathrm{Re}(\alpha) = 0$, $\alpha = i\gamma$, *and if*

$$(M_{i\gamma}f)(x) = \sup_{y>0}\left|y^{i\gamma}\left(-\frac{d}{dy}\right)^{i\gamma}P_yf(x)\right|,$$

then $\|f\|_p \leqq A_1(\gamma,p)\|M_{i\gamma}f\|_p \leqq A_2(\gamma,p)\|f\|_p$.

Proof. For γ real, $(-d/dy)^{i\gamma}P_y(f) = P_y(T^{i\gamma}f)$. Thus $M_{i\gamma}f(x) = M_0(T^{i\gamma}f)(x)$, and since $T^{i\gamma}$ is a bounded and invertible operator, the desired inequality follows from Proposition 2.5.

Theorem 2.8. *If* $-\infty < \mathrm{Re}(\alpha) \leqq 1$, $\|M_\alpha f\|_p \leqq A(p,\alpha)\|f\|_p$.

Proof. It only remains to prove the theorem for $0 < \mathrm{Re}(\alpha) < 1$. Set $T_\varphi{}^\alpha(f) = \int_{0^+}^\infty y^\alpha P_y{}^{(\alpha)}(f)\varphi(y)\,dy/y$ for $0 \leqq \mathrm{Re}(\alpha) \leqq 1$ when $\|\varphi\|_{1\infty} < \infty$. Then $\|M_\alpha f\|_p = \sup\{\|T_\varphi{}^\alpha f\|_p : \|\varphi\|_{1\infty} \leqq 1\}$. By Theorem 2.1, $\|T_\varphi{}^{1+iu}(f)\|_p \leqq A(p,u)\|f\|_p\|\varphi\|_{1\infty}$ and by Corollary 2.7, $\|T_{\varphi}{}^{iu}f\|_p \leqq A(p, u)\|f\|_p\|\varphi\|_{1\infty}$. Since $T^\alpha P_y(f)$ is analytic in $0 < \mathrm{Re}(\alpha) < 1$ and continuous in $0 \leqq \mathrm{Re}(\alpha) \leqq 1$ for a dense set of f in $L_p(H)$ by Theorem 8.2 of [17-I], Stein's interpolation theorem applies and $\|T_{\varphi}{}^t\|_p \leqq A(t, p)\|\varphi\|_{1\infty}$ for $0 \leqq t \leqq 1$. If $\varphi_1 = y^{iu}\varphi(y)$, $T_{\varphi}{}^{\alpha}(f) = T^t_{\varphi_1}(T^{iu}f)$ for $\alpha = t + iu$, and $\|M_\alpha f\|_p \leqq A(p, \alpha)\|f\|_p$ for $0 \leqq \mathrm{Re}(\alpha) \leqq 1$.

Because of Theorem IV.4.1, we can define a maximal function for the directional derivatives. If $h \in H$ and $\mathrm{Re}(\alpha) \geqq 0$, set

$$M_\alpha{}^h(f) = \sup_{y>0} |y^\alpha(-A_h)^\alpha P_y(f)|.$$

But $M_\alpha{}^h(f) = M_\alpha((-A_h)^\alpha T^{-\alpha}f)$ and $(-A_h)^\alpha T^{-\alpha}$ is a bounded operator on $L_p(H)$ for $\mathrm{Re}(\alpha) \geqq 0$. $(-A_h)^\alpha T^{-\alpha}$ is invertible if $\mathrm{Re}(\alpha) = 0$. Thus we have proved

Corollary 2.9. *If* $\mathrm{Re}(\alpha) \geqq 0$, $\|M_\alpha{}^h(f)\|_p \leqq A(p, \alpha)\|h\|^{\mathrm{Re}(\alpha)}\|f\|_p$ *and if* $\mathrm{Re}(\alpha) = 0$, $\|f\|_p \leqq A_0(p, \alpha)\|M_\alpha{}^h(f)\|_p$.

3. *Applications of the maximal theorems.* In this section we shall investigate a few of the implications of the general maximal inequality. More applications of the maximal theorems will be given below in §5.

Theorem 3.1. *$P_y(f)$ converges to f almost everywhere as y tends to zero through positive values.*

Proof. $P_z(f)$ is analytic in $|\arg(z)| < \pi/4$ and the power series representation for $P_z(f)$ about z_0 can be thought of as converging almost everywhere since it converges in $L_p(H)$ and a subsequence of the sequence of partial sums converges almost everywhere. Regard the series as converging everywhere. Therefore if $z_0 > 0$, $\lim_{z\to z_0} P_z(f) = P_{z_0}(f)$ almost everywhere. Then

$$\begin{aligned}
&\limsup_{y\to 0^+} |P_y(f)(x) - f(x)| \\
&\qquad \leqq \limsup_{y\to 0^+} |P_y(f - P_t(f))(x)| \\
&\qquad\quad + \limsup_{y\to 0^+} |P_yP_t(f)(x) - P_t(f)(x)| + |P_t(f)(x) - f(x)| \\
&\qquad \leqq \sup_{y>0} |P_y(f - P_t(f))(x)| + |P_t(f)(x) - f(x)| \\
&\qquad = M_0(f - P_tf) + |P_tf - f|.
\end{aligned}$$

Therefore, by Theorem 2.8, $\|\limsup_{y\to 0^+} |P_yf - f|\,\|_p \leqq \|M_0(f - P_tf)\|_p + \|P_tf - f\|_p \leqq A(p)\|f - P_tf\|_p$ for all $t > 0$. By letting $t \to 0^+$, we get the desired result.

If k is a positive integer, and we write

$$y^k\left(\frac{d}{dy}\right)^k P_y(f) = \int_0^\infty [H_t(f) - f]\, y^k\left(\frac{d}{dy}\right)^k N_t(y)\, dt/t = \int_0^\epsilon + \int_\epsilon^\infty$$

it follows from the strong continuity of H_t, that $\|\int_0^\epsilon\|_p \leqq A(k)\epsilon$ and it follows from the properties of $y^k(d/dy)^k N_t(y) = P_k(t^{-1}y^2)N_t(y)$ that $\lim_{y\to 0^+} \|\int_\epsilon^\infty\|_p = 0$, so that $\lim_{y\to 0^+} \|y^k(d/dy)^k P_y(f)\|_p = 0$. In addition, there is

THEOREM 3.2. *Let* $\mathrm{Re}(\alpha) > 0$. *As y tends to zero through positive values, $y^\alpha P_y^{(\alpha)}(f)$ converges to zero almost everywhere.*

PROOF. Let k be a positive integer and assume first that $\mathrm{Re}(\alpha) < k$ and that $f \in D(T^k)$. Then $y^\alpha P_y^{(\alpha)}(f) = y^\alpha P_y(T^\alpha f)$, and by Theorem 3.1, $\lim_{y\to 0^+} P_y(T^\alpha f) = T^\alpha f$ almost everywhere. Thus $\lim_{y\to 0^+} y^\alpha P_y(T^\alpha f) = 0$ almost everywhere. For any f in $L_p(H)$, use the density of $D(T^k)$ in $L_p(H)$ to choose a sequence $\{f_n\}$ in $D(T^k)$ which converges in $L_p(H)$ to f. Then

$$\begin{aligned} F &= \limsup_{y\to 0^+} |y^\alpha P_y^{(\alpha)}(f)| \\ &\leqq \limsup_{y\to 0^+} |y^\alpha P_y^{(\alpha)}(f - f_n)| + \limsup_{y\to 0^+} |y^\alpha P_y^{(\alpha)}(f_n)| \\ &\leqq \sup_{y>0} |y^\alpha P_y^{(\alpha)}(f - f_n)| = M_\alpha(f - f_n). \end{aligned}$$

By Theorems 2.1 and 2.8, $\|F\|_p \leqq \|M_\alpha(f - f_n)\|_p \leqq A(p, \alpha)\|f - f_n\|_p$. By letting $n \to \infty$, we get the desired result.

A similar result holds for the directional derivatives $(-A_h)^\alpha$ since $(-A_h)^\alpha P_y(f) = P_y^{(\alpha)}((-A_h)^\alpha T^{-\alpha} f)$ and $(-A_h)^\alpha T^{-\alpha}$ is a bounded operator on $L_p(H)$ by Theorem IV.4.1.

COROLLARY 3.3. *If* $\mathrm{Re}(\alpha) > 0$, $\lim_{y\to 0^+} y^\alpha(-A_h)^\alpha P_y(f) = 0$ *almost everywhere.*

COROLLARY 3.4. *If* $\mathrm{Re}(\alpha) < 0$, $\lim_{y\to 0^+} y^\alpha(-d/dy)^\alpha P_y(f) = \Gamma(1-\alpha)^{-1} f$ *almost everywhere.*

PROOF.

$$\begin{aligned} &y^\alpha\left(-\frac{d}{dx}\right)^\alpha P_y(f) - \frac{f}{\Gamma(1-\alpha)} \\ &\qquad = \Gamma(-\alpha)^{-1}\int_0^1 (1-u)^{-\alpha-1}(P_{uy}f - f)\, du. \end{aligned}$$

By the maximal theorem for P_y, the dominated convergence theorem applies for almost every x and by Theorem 3.1, $\lim_{y\to 0^+} y^{\alpha}(-d/dy)^{\alpha}P_y(f) = \Gamma(1-\alpha)^{-1}f$ almost everywhere.

THEOREM 3.5. *If* $-\infty < \operatorname{Re}(\alpha) < \infty$, $y^{\alpha}(-d/dy)^{\alpha}P_y(f)$ *converges almost everywhere and in* $L_p(H)$ *to* 0 *as* y *tends to* $+\infty$.

PROOF. If $\operatorname{Re}(\alpha) \geqq 0$, $\|y^{\alpha}P_y^{(\alpha)}f\|_p \leqq A(p,\alpha)\|f\|_p$ and $|y^{\alpha}P_y^{(\alpha)}f| \leqq |y^{\alpha}T^{\alpha}P_{y/2}P_{y/2}(f)| \leqq A(\alpha)P_{y/2}(M_{\alpha}(f)) \leqq A_1(\alpha)M_0(M_{\alpha}f)$. Thus it is sufficient to prove that P_yf converges almost everywhere to zero in order to prove the statements of the theorem. If $\operatorname{Re}(\alpha) < 0$,

$$|y^{\alpha}(-d/dy)^{\alpha}P_y(f)| < |\Gamma(-\alpha)|^{-1}\int_0^1 (1-u)^{-\operatorname{Re}(\alpha)-1}|P_{uy}(f)|\,du.$$

By the maximal theorem, the dominated convergence theorem applies and it is sufficient to prove that $P_{uy}(f)$ converges to zero almost everywhere for each $u > 0$ in order to verify the statements of the theorem.

Let $f^* = \limsup_{y\to+\infty}|P_y(f)|$ and assume first that $f \in R(T)$ so that $f = -Tg$ for some g in $L_p(H)$. Then

$$P_y(f) = \frac{\partial}{\partial y}P_y(g) = \int_0^{\infty} H_t(g)\frac{\partial}{\partial y}N_t(y)\,dt/t$$

and

$$\left|\frac{\partial}{\partial y}N_t(y)\right| = |(\pi t)^{-1/2}(1-2y^2/t)\exp(-y^2/t)| \leqq Ay^{-1}N_t(my)$$

where A and m are positive constants. Thus $|P_y(f)| \leqq Ay^{-1}P_{my}(|g|) \leqq Ay^{-1}M_0(|g|)$ converges to zero almost everywhere as $y \to \infty$. Let $\epsilon > 0$, let $f \in L_p(H)$, and let $f_1 \in R(T)$ with $\|f - f_1\|_p < \epsilon$. Then $f^* \leqq f_1^* + (f-f_1)^* \leqq M_0(f-f_1)$. By Theorem 2.8, $\|f^*\|_p \leqq A(p)\|f-f_1\|_p < \epsilon A(p)$. Thus $\lim_{y\to\infty} P_y(f) = 0$ almost everywhere and the theorem holds.

4. *Littlewood-Paley inequality.* For $f \in L_p(H)$, set

$$g_{\alpha}^{r}(f) = \left(\int_0^{\infty} |y^{\alpha}T^{\alpha}P_y(f)|^r\,dy/y\right)^{1/r}$$

for $\operatorname{Re}(\alpha) \geqq 1$ and $r < \infty$. g_{α}^{r} is the Littlewood-Paley g-function. If $2 \leqq r < \infty$ and $r^{-1} + s^{-1} = 1$, $\|g_{\alpha}^{r}(f)\|_p = \sup\{\|T_{\varphi}^{\alpha}(f)\|_p : \|\varphi\|_{s\infty} \leqq 1\}$.

PROPOSITION 4.1. *If* $r \geqq 2$ *and* $t - s > 1/r'$, $1/r + 1/r' = 1$, $g_s^{r}(f) \leqq A(r,s,t)g_t^{r}(f)$.

PROOF. By Theorem 6.3 of [17-III],

$$P_z^{(s)}(f) = \Gamma(t-s)^{-1} \int_0^\infty u^{t-s-1} P_{u+z}^{(t)}(f)\, du$$

$$= \Gamma(t-s)^{-1} \int_z^\infty (y-z)^{t-s-1} P_y^{(t)}(f)\, dy.$$

Thus

$$|P_z^{(s)}(f)| = \Gamma(t-s)^{-1} \left| \int_z^\infty (y-z)^{t-s-1} y^s P_y^{(t)}(f)\, dy/y^s \right|$$

$$\leqq z^{-s+1/r'} A(r,t,s) \left(\int_z^\infty |(y-z)^{t-s-1} y^s P_y^{(t)}(f)|^r\, dy \right)^{1/r}.$$

Thus

$$\frac{|z^s P_z^{(s)}(f)|^r}{z} \leqq A(s,t,r) z^{r-2} \left(\int_z^\infty |(y-z)^{t-s-1} y^s P_y^{(t)}(f)|^r\, dy/y \right)$$

$$\leqq A(s,t,r) \left(\int_z^\infty y^{r-1} |(y-z)^{t-s-1} y^s P_y^{(t)}(f)|^r\, dy/y \right).$$

By Fubini's theorem, if $t - s > 1/r'$,

$$\int_0^\infty |z^s p_z^{(s)}(f)|^r\, dz/z$$

$$\leqq A(s,t,r) \int_0^\infty \left(\int_0^y (y-z)^{rt-rs-r}\, dz \right) y^{r+rs-1} |P_y^{(t)}(f)|^r\, dy/y$$

$$\leqq A(s,t,r) \int_0^\infty |y^t P_y^{(t)}(f)|^r\, dy/y.$$

The next theorem gives the Littlewood-Paley inequality.

Theorem 4.2. *If $1 < r' \leqq p < \infty$ and $2 \leqq r < \infty$, $\|g_\alpha^r(f)\|_p \leqq A(p,\alpha)\|f\|_p$. If $1 \leqq s \leqq 2$ and if $1 < p \leqq s$, then $\|f\|_p \leqq A_1(p,k)\|g_k^s(f)\|_p$ when k is a positive integer.*

Proof. The first inequality follows from Theorem IV.1.4 and the fact that $\|g_\alpha^r(f)\|_p = \sup\{\|T_\varphi{}^\alpha f\|_p : \|\varphi\|_{r'\infty} \leqq 1\}$, $1/r + 1/r' = 1$. To prove the second inequality we need a lemma whose proof will be provided after the theorem's proof has been completed.

Lemma 4.3. *Let $f_1 \in L_p(H)$ and $f_2 \in L_q(H)$, let $P_t(f_1)$ be the Poisson integral of f_1 in $L_p(H)$ and let $Q_t(f_2)$ be the Poisson integral of f_2 in $L_q(H)$. Then $\langle f_1, f_2\rangle = K(k)\int_{0^+}^\infty t^{2k-1}\langle P_t^{(k)}(f_1), Q_t^{(k)}(f_2)\rangle dt$ where $\langle f, g\rangle$ denotes the dual pairing between L_p and L_q.*

By Lemma 4.3, $|\langle f_1, f_2\rangle| \leqq |K(k)|\, \|g_k{}^s(f_1)\|_p \|g_k{}^{s'}(f_2)\|_q$. By the first inequality in Theorem 4.2, $\|g_k{}^{s'}(f)\|_q \leqq A(q, s')\|f\|_q$ for $2 \leqq s' \leqq \infty$. Thus $\|f_1\|_p \leqq A_1(p, s)\|g_k{}^s(f_1)\|_p$.

Proof of Lemma 4.3. Q_t is the semigroup dual to P_t and the infinitesimal generator of Q_t is the adjoint of the infinitesimal generator of P_t. Therefore

$$\int_{\tilde{H}} \int_\delta^\rho t^{2k-1} \left(\frac{d}{dt}\right)^{2k} (P_t(f_1)Q_t(f_2))\, dt\, dN$$

$$= K(k) \int_{\tilde{H}} \int_\delta^\rho y^{2k-1} P_y{}^{(k)}(f_1) Q_y{}^{(k)}(f_2)\, dy\, dN$$

when $0 < \delta < \rho < \infty$. Integrate the left-hand side by parts to obtain

$$\int_\delta^\rho t^{2k-1} \left(\frac{d}{dt}\right)^{2k} (P_t(f_1)Q_t(f_2))\, dt$$

$$= \sum_{n=0}^{2k-1} A_n \left[t^n \left(\frac{d}{dt}\right)^n (P_t(f_1)Q_t(f_2)) \right]_\delta^\rho$$

where the A_n are certain real constants. Repeated applications of Theorems 3.1, 3.2, and 3.5 show that the sum on the right converges almost everywhere and in $L_1(H)$ to Kf_1f_2 as $\delta \searrow 0$ and $\rho \nearrow \infty$. This proves the lemma.

Corollary 4.4. *For $h \in H$, set*

$$h_\alpha{}^r(f) = \left(\int_0^\infty |y^\alpha (-A_h)^\alpha P_y(f)|^r\, dy/y \right)^{1/r}.$$

If $1 < r^1 \leqq p < \infty$ and $2 \leqq r < \infty$, then $\|h_\alpha{}^r(f)\|_p \leqq A(p, r, \alpha)\|f\|_p$.

Proof. $(-A_h)^\alpha P_y(f) = (-A_h)^\alpha T^{-\alpha} P_y{}^{(\alpha)}(f)$ and by Theorem IV.4.1, $\|(-A_h)^\alpha T^{-\alpha}\|_p \leqq A(p, \alpha)\|h\|^{\mathrm{Re}(\alpha)}$. Thus by Theorem 4.2, $\|h_\alpha{}^r(f)\|_p \leqq \|g_\alpha{}^r((-A_h)^\alpha T^{-\alpha} f)\|_p \leqq A(p, r, \alpha)\|h\|^{\mathrm{Re}(\alpha)}\|f\|_p$ for the same p and r as in Theorem 4.2.

Theorem 4.5. *If k is a positive integer*

$$g_k{}^2(f) = A\left(\int_{-\infty}^\infty |\Gamma(k - is) T^{is}(f)|^2\, ds \right)^{1/2}.$$

Proof. The Mellin transform is an isometry from $L_2(R^+, dy/y)$ to $L_2(R, ds)$. The Mellin transform of $y^k P_y{}^{(k)}(f)$ is

$$K\int_{-\infty}^{\infty} y^k P_y^{(k)}(f)y^{-is-1}\,dy$$

and k integrations by parts and use of Theorem III.3.4 shows that this integral is $K\Gamma(k - is)T^{is}(f)$ for all real numbers s. Thus the desired identity holds.

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