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WEAK PROBABILITY DISTRIBUTIONS ON REPRODUCING KERNEL HILBERT SPACES

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1. **Introduction.** In the experimental sciences, the problem of estimating a function from an approximate knowledge of a finite number of observations occurs very frequently. If the observational values were known exactly the techniques of Optimal Approximation [1] could be employed directly, but usually the given information is subject to experimental error, which may often be regarded as probabilistic in nature. One is thus led to consider a space of all the functions which might conceivably have given rise to the observations, and the possibility of constructing a probability measure on this space.

In choosing a curve of unknown algebraic form to fit experimental data there is a widespread predilection favoring "smoother" curves over those which are "not so smooth", presumably expressing the intuitive feeling that, if any given set of data could have arisen from either a smooth curve or a rough curve, then the former is "more likely" than the latter. Also in choosing such curves, one would feel that curves of equal norm should be equally likely. The first of these two considerations suggests a probability measure that favors smooth functions over unsmooth ones by assigning high probability to curves with small norms and low probability to curves with large norms. It would seem, then, that the choice of a Gaussian measure in which the functional $\exp\{-\|h\|^2\}$ plays the role of the relative likelihood of a member h of some Hilbert space is likely to be useful in numerical practice.

The second consideration carries us even further toward a Gaussian measure. To ask that our measure make curves of equal norm equally likely is to ask that it be invariant under unitary transformations.

It has been shown in [13] that in the setting we are about to consider a unitarily invariant, μ, must be of the form $\mu(\cdot) = \int_0^\infty \mu_\lambda(\cdot)\alpha(d\lambda)$ where $\{\mu_\lambda\}$ is a family of canonical Gaussian measures and α is a probability measure on $[0, \infty)$.

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Practical applications of the above ideas are explored in another paper [**11**]. The purpose of this present paper is to review and interpret enough of the theory of Gaussian measure on function spaces to provide a working basis for practical applications in Approximation Theory and Numerical Analysis like those in [**11**].

2. **The measure.** We will always assume that H is a real separable Hilbert space. For a given linearly independent set $y_1, y_2, \cdots, y_n$ in H^* and Borel set $E \subset R^{(n)}$ we define

$$\nu\{h \in H : ((y_1, h), (y_2, h), \cdots, (y_n, h)) \in E\} = \int_E \frac{|\Sigma|^{-n/2}}{(\sqrt{2\pi})^n} \exp\{-\tfrac{1}{2}x\Sigma^{-1}x^t\}\, dx$$

where $x = (x_1, x_2, \cdots, x_n) \in R^{(n)}$ and $\Sigma = [(y_i, y_j)]$ is an $n \times n$ matrix of inner products. Any such subset of H of this form is called a tame set or a cylinder set. A change of variables shows that ν is well-defined and finitely additive on the algebra of tame subsets of H. The extension of ν to the Borel subsets of H would appear to be the measure called for in §1.

3. **It is not countably additive.** The set function, ν, has one serious defect. It is not countably additive. To see this, let $y_1, y_2, \cdots, y_k, \cdots$ be an orthonormal basis for H and let $A_j = \{h \in H : -j \leqq (y_i, h) \leqq j$ for all $i \leqq \rho_j\}$ where ρ_j is chosen so that

$$\nu(A_j) = \left(\frac{1}{\sqrt{2\pi}} \int_{-j}^{j} \exp\left\{-\frac{x^2}{2}\right\} dx\right)^{\rho_j} \leqq \frac{1}{2^j}.$$

All the A_j as well as H are tame sets and $H = \bigcup_{j=2}^{\infty} A_j$, $\nu(H) = 1$. However, $\sum_{j=2}^{\infty} \nu(A_j) \leqq \frac{1}{2}$.

4. **Enlarging the space.** As a result of §3 one is forced to choose between the following two alternatives. The requirements of §1 can be dropped in favor of some nonnegative set function which is countably additive on the algebra of tame sets of H and hence can be extended to a measure on H. Or, we can introduce a well developed integration theory that will allow us to have the conditions of §1 satisfied. It is this second choice that we will discuss in the following paragraphs. We attempt at this point a heuristic discussion. A seminorm, $\|\cdot\|_1$, on H is sought with the property that if B is the completion of H under $\|\cdot\|_1$ then ν can be defined on the cylinder sets of B and ν is countably additive on this algebra. Such seminorms do exist and are called measurable seminorms. They will always be

weaker than the original Hilbert space norm (see [7, Corollary 5.4]). Given such a $\|\cdot\|_1$, $\int F(h)\nu\{dh\}$ can then be defined for certain classes of functions on H in terms of an extension of F to B and an integral over B. We now give a formal definition of a measurable seminorm due to Gross [7].

DEFINITION. A seminorm, $\|\cdot\|_1$, on H is called a measurable seminorm if for every $\epsilon > 0$ there exists a finite-dimensional projection, P_0, such that for every finite-dimensional, P, orthogonal to P_0 we have $\nu\{h \in H : \|Ph\|_1 > \epsilon\} < \epsilon$. If T is a positive definite finite trace class operator on H then $\|x\|_1 = (Tx, x)^{1/2}$ is always a measurable norm on H. In the case when H is a reproducing kernel Hilbert space of more interest is the question of when the sup norm is a measurable norm. This is a question that will be discussed in §7.

5. **Gross' Theorem.** We first introduce a cylinder set measure, μ, on B. B^* is dense in H^* and $\|\cdot\|_1$ is weaker than the Hilbert space norm so that, for each $y \in B^*$, $y/H \in H^*$. Thus we can define μ by

$$\mu\{b \in B : (y_1(b), \cdots, y_n(b)) \in E\}$$
$$= \nu\{h \in H : ((y_1/H)(h), \cdots, (y_n/H)(h)) \in E\}.$$

We can now state the

THEOREM. *If H is a real separable Hilbert space and B is the completion of H under $\|\cdot\|_1$ where $\|\cdot\|_1$ is a measurable norm on H, then μ is countably additive on the cylinder sets of B.*

This result together with its proof appears as Theorem 1 in [8].

6. **Integration.** We now turn to the problem of integrating functions on H with respect to the cylinder set measure ν. Our discussion follows the theory developed by L. Gross in [7], [8], but differs in the sense that we have tried to be more explicit in the construction of the weak canonical normal distribution on H. We hasten to add that we knowingly have sacrificed some generality and perhaps a certain neatness in this attempt, but hopefully this will be offset by ease in application to the problems in §1.

DEFINITION. A function f on H is said to be a tame function based on P if there exists a finite-dimensional projection P such that $f(Ph) = f(h)$ for all $h \in H$, and f is measurable with respect to the algebra of tame sets.

It follows that if f is a tame function based on P and Q is a finite-dimensional projection such that $QH \supseteq PH$ then f is also based on Q. Furthermore, one can show that if f is a tame function based on P and

$$\Phi_P(u_1, \cdots, u_n) = f(u_1 e_1 + \cdots + u_n e_n) \tag{6.1}$$

where $e_1, \cdots, e_n$ is an orthonormal basis for $P(H)$, then Φ_P is Borel measurable on $R^{(n)}$ and

$$(2\pi)^{-n/2} \int_{R^{(n)}} \Phi_P(u) \exp\{-\|u\|^2/2\}\, du \tag{6.2}$$

is independent of the base P of f. The main ideas used to prove these facts are similar to those employed in proving that ν is well-defined on the tame sets.

In view of the previous remarks we now are able to define the integral of a tame function.

Definition. If f is a tame function based on P, then f is integrable if

$$\int_{R^{(n)}} |\Phi_P(u)| \exp\{-\|u\|^2/2\}\, du < \infty. \tag{6.3}$$

We then define the integral $\int_H f(h)\, \nu(dh)$ as (6.2).

Another way to view the integral of a tame function f on H is by extending f to B, the completion of H in a measurable norm $\|\cdot\|_1$. To do this we first define the extension to B of the functional $x \to (x, h)$ where $h \in H$ is fixed and $x \in H$. Since $\|\cdot\|_1$ is weaker than the Hilbert space norm $\|\cdot\|$ and H is dense in B we have B^* dense in H^* which we identify with H. Thus there exists a complete orthonormal sequence $\{\alpha_k\} \subseteq B^* \subseteq H^*$ and we define

$$(x, h)^{\sim} = \lim_n \sum_{k=1}^{n} (x, \alpha_k)(h, \alpha_k) \qquad (x \in B). \tag{6.4}$$

Now $\{(x, \alpha_k)\}$ is a sequence of independent Gaussian functions with mean zero and variance one with respect to the measure μ on B and $\sum_{k=1}^{\infty} (h, \alpha_k)^2 = \|h\|^2$; thus it follows that $\lim_n \sum_{k=1}^{n} (x, \alpha_k)(h, \alpha_k)$ exists for almost all $x \in B$ and that $(x, h)^{\sim}$ is Gaussian with mean zero and variance $\|h\|^2$. It also is easy to show that $(x, h)^{\sim}$ is independent of the defining sequence $\{\alpha_k\} \subseteq B^*$ and that, for $h_1, \cdots, h_k \in H$, $(x, h_1)^{\sim}, \cdots, (x, h_k)^{\sim}$ have a joint Gaussian distribution with mean zero and covariance matrix $\Sigma = [(h_i, h_j)]$. Furthermore, if $h \in B^*$ it can be shown that $(x, h) = (x, h)^{\sim}$ with μ-measure one on B.

We now extend the function $x \to (x, h)$ defined on H to μ-almost all of B by the formula $x \to (x, h)^{\sim}$ and call it a stochastic inner product.

If f is a tame function based on P as in (6.1) we then extend f to B by the equation

$$\tilde{f}(x) = f[(x, e_1)^{\sim} e_1 + \cdots + (x, e_n)^{\sim} e_n] \\ = \Phi_P[(x, e_1)^{\sim}, \cdots, (x, e_n)^{\sim}]. \tag{6.5}$$

Since Φ_P is a Borel function, it follows that $\tilde{f}$ is measurable on B and we have

$$\int_B \tilde{f}(x)\mu\{dx\} = \frac{1}{(\sqrt{2\pi})^n} \int_{R^{(n)}} \Phi_P(u) \exp\{-\|u\|^2/2\}\, du \tag{6.6}$$

so $\int_B \tilde{f}(x)\,\mu(dx) = \int_H f(h)\,\nu(dh)$ provided any of the integrals exist. Again, since f may be based on various projections P and hence determines various Borel functions Φ_P, we mention that the extension of f to B via any Φ_P yields the same measurable function (with μ probability one). Further, the right-hand side of (6.6) is independent of the Φ_P used to represent f, so $\int_B \tilde{f}\, d\mu$ is well-defined provided the right-hand side of (6.6) exists.

In summary then, we can integrate a tame function f on H by forming its extension to B which we will denote by $\tilde{f}$, and then integrating $\tilde{f}$ with respect to the measure μ on B induced by ν. Further, the map $\Gamma(f) = \tilde{f}$ is a linear multiplicative map on the tame functions to measurable functionals (random variables) on the probability space (B, μ). To integrate other functions on H we attempt to extend Γ.

If f is a function on H we define

$$\Gamma(f) = \tilde{f} = \lim_n (\widetilde{f \circ P_n})$$

provided the limit exists in μ-measure for every sequence of finite dimensional projections increasing strongly to the identity and the limit function is independent of the sequence $\{P_n\}$. Here, of course, the limit exists on a Banach space B obtained from a measurable norm $\|\cdot\|_1$ on H, and μ is the measure induced on B by ν.

Definition. A function f on H is integrable if $\tilde{f}$ exists on a Banach space B and $\tilde{f}$ is μ-integrable on B. We then define $\int_H f d\nu$ as $\int_B \tilde{f}\, d\mu$.

Further, we remark that $(\widetilde{f \circ P_n})$ converges in probability on B iff $\widetilde{f \circ P_n}$ converges in probability on B_0 where B_0 is the completion of H under another measurable norm $\|\cdot\|_0$ and that $\int_B \tilde{f}\, d\mu = \int_{B_0} \tilde{f}\, d\mu$. Hence the integral of f is well-defined.

The next result indicates how to integrate certain smooth functions on H.

Proposition 1. *$\|\cdot\|_1$ is a measurable norm on H,*
B is the completion of H under $\|\cdot\|_1$,
$\nu(\cdot)$ is the canonical normal distribution on H,

$\mu(\cdot)$ is the countably additive extension of $\nu(\cdot)$ to the Borel subsets of B.

Suppose: 1 — *$f(x)$ is defined on H,*

2 — *$f(\cdot)$ has a unique continuous extension to B which we again denote by f,*[3]

3 — *$|f(x)| \leqq A \exp\{C\|x\|_1^{2-\delta}\}$ for $A, C, \delta > 0$ for $x \in H$.*

Then, f is integrable on H. Further, let $\|x\|_1 = (Tx, x)^{1/2}$ where T is a compact symmetric positive operator on H with eigenvectors $\{\varphi_1, \varphi_2, \cdots\}$ and eigenvalues $\{\lambda_k\}$ such that $\sum_k \lambda_k < \infty$. Let $P_N(x) = \sum_{k=1}^N (x, \varphi_k)\varphi_k$. Then

$$f(x) = \tilde{f}(x) = \lim_N (\widetilde{f \circ P_N})(x) = \lim_N f \circ \tilde{P}_N(x) \tag{6.7}$$

for almost all x in B and

$$\begin{aligned} \int_B \tilde{f}(x)\, \mu(dx) &= \lim_N \int_B (\widetilde{f \circ P_N})(x)\, \mu(dx) \\ &= \lim_N \int_H f \circ P_N(h)\, \nu(dh). \end{aligned} \tag{6.8}$$

Proof. Since f has a unique continuous extension to B and for any increasing sequence of projections converging to the identity $\lim \|\tilde{P}_N(x) - x\|_1 = 0$ with μ-measure one [10] it follows that $\tilde{f}(x) = \lim_N (\widetilde{f \circ P_N})(x) = \lim_N f \circ \tilde{P}_N(x) = f(x)$ with μ-measure one. Now $|f(x)| \leqq A \exp\{C\|x\|_1^{2-\delta}\}$ for A, C, $\delta > 0$ and $x \in H$ implies the same inequality holds for $x \in B$ since H is dense in B and f is continuous on B. Using the recent results of Shepp and Landau [14] and Fernique [3] we know the exponential function bounding $f = \tilde{f}$ is integrable on B and hence $\int_B f\, d\mu = \int_B \tilde{f}\, d\mu$ exists. Thus f integrable on H.

To verify (6.7) we simply argue as above since we know $f = \tilde{f}$. Since $(T\tilde{P}_N x, \tilde{P}_N x) = \sum_{k=1}^N \lambda_k [(x, \varphi_k)^{\sim}]^2$ is increasing in N we have $\|\tilde{P}_N x\|_1$ increasing to $\|x\|_1$ with μ-measure one and hence by (3) we have $|(\widetilde{f \circ P_N})(x)| \leqq A \exp\{C\|x\|_1^{2-\delta}\}$ for $N = 1, 2, \cdots$ and μ-measure one. Thus by (6.7) and the dominated convergence theorem (6.8) holds. As already noted, measurable norms are always weaker than the Hilbert space norm. Therefore functions on H with a unique continuous extension to B must be "very smooth" on H. In particular we get from Corollary 4 of [8] that positive functions with this continuity property cannot have a 0 integral (see the example of §9). Another class of functions on H that can be integrated

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[3]For example, suppose f is uniformly continuous in $\|\cdot\|_1$ on bounded subsets of H.

with the help of the computational formula of Proposition 1 is the class of tame functions. In fact, if f is a tame function we know how to define $\tilde{f} = \Gamma(f)$ but it is not at all clear that for every increasing sequence of finite-dimensional projections converging to the identity

$$\lim_n \widetilde{(f \circ P_n)} = \tilde{f}$$

where the limit is taken in μ-measure and $\tilde{f} = \Gamma(f)$ is defined as in (6.5). That this is the case follows from a result of Friedrichs and Shapiro [4], and if f is assumed to be integrable then they prove $\lim_n \int_B |\widetilde{(f \circ P_n)} - \tilde{f}|\, d\mu = 0$. In particular, (6.8) holds when f is an integrable tame function and $\{P_N\}$ is any sequence of finite-dimensional projections increasing to the identity.

REMARK. It is also easy to see that if f is as in Proposition 1 and $|f(x)| \leqq A$ for some $A > 0$ and all $x \in H$ then (6.8) holds for f with respect to any sequence of finite-dimensional projections $\{P_N\}$ increasing to the identity.

7. **The role of the reproducing kernel.** Let H be a reproducing kernel Hilbert space of real valued functions on some separable metric space, D. We assume that the reproducing kernel, R, is continuous on $D \times D$ so that H is a separable Hilbert space of continuous functions on D. If $\|h\|_1 = \sup_{x \in D} |h(x)|$ is a measurable norm on H and this sup norm is used to construct B, then the point evaluation linear functionals on B will be continuous and hence will be elements of B^*. If $X_t \in B^*$ is a continuous point evaluation at t functional on B then $\{X_t(b), t \in D\}$ is a Gaussian stochastic process on B with $E(X_t) = \int_B X_t(b)\, \mu\{db\}$ and $E(X_tX_s) = \int_B X_t(b)X_s(b)\mu\{db\} = (X_t/H, X_s/H) = R(t, s)$. Hence we have

PROPOSITION 2. *Let H, R, $\|\cdot\|_1$, and X_t be as in the above then $\{X_t(b), t \in D\}$ is a Gaussian stochastic process on $(B, \mathcal{B}, \mu)$ with 0 mean and covariance function $R(s, t)$. Here B is the completion of H with respect to the sup norm, $\mathcal{B}$ is the σ-algebra of Borel subsets of B, and μ is the extended measure of §5. The near converse to Proposition 2 is due to Kallianpur* [9].

KALLIANPUR'S THEOREM. *Let D be compact and $C = C(D)$ be the linear space of continuous functions on D. If μ is a countably additive Gaussian measure on $\mathcal{A}(C)$ with 0 mean and continuous covariance function, $R(s, t)$, and if $H(R)$ is the reproducing kernel Hilbert space with kernel R, then $\|h\|_1 = \sup_{d \in D_1} |h(d)|$ is a measurable norm on $H(R)$. Here $\mathcal{A}(C)$ is the σ-algebra generated by the Borel cylinder* *sets on C and the cylinder set measure on $H(R)$ is the canonical normal measure of* §2.

Given a continuous positive definite function, R, on $D \times D$, it is, unfortunately, not always possible to construct a mean 0 Gaussian measure, μ, with covariance function R with the property that the support of μ is a subset of $C(D)$ (see [5]). Equivalently, there exist reproducing kernel Hilbert spaces with continuous kernels having the property that the sup norm is not a measurable norm. Here, again, the cylinder set measure is that of §2. Proposition 3, to follow, and Proposition 1 make it clear that one would like conditions on the reproducing kernel, R, which would insure that the sup norm be measurable. Sufficient conditions on R for the sample path continuity of the associated Gaussian process, with the help of Kallianpur's Theorem, become sufficient conditions on R for the sup norm to be a measurable norm on $H(R)$. Hence if D is a compact subset of $R^{(1)}$ and if R is continuous, then the sup norm on $H(R)$ is a measurable norm if any one of the following are satisfied:

(7.1) There exist constants $c > 0$ and $a > 1$ such that for all sufficiently small h, $\|R(t+h, \cdot) - R(t, \cdot)\|^2_{H(R)} = R(t+h, t+h) - 2R(t, t+h) + R(t,t) \leqq c/|\ln|h||^a$ [2];

(7.2) $\int_D (\ln(1/u))^{1/2}\, dP(u) < \infty$ where

$$P(u) = \max_{|s-t| \leqq u} \{\|R(s, \cdot) - R(t, \cdot)\|_{H(R)}\}$$

$$= \max_{|s-t| \leqq u} \{[R(s,s) - 2R(s,t) + R(t,t)]^{1/2}\}$$

[6]; or

(7.3) $R(s,t) = R(t-s)$ is a function only of the difference, $|t-s|$, and there exists $\epsilon > 0$ and an increasing function, ψ, satisfying

$$\int_0^\epsilon \frac{\psi(x)\, dx}{x(\ln(1/x))^{1/2}} < \infty$$

such that for $0 \leqq |x| \leqq \epsilon$ we have $\|R(x, \cdot) - R(0, \cdot)\|_{H(R)} = (2[R(0) - R(x)])^{1/2} \leqq \psi(x)$, [12].

Dudley's condition, (7.1), is probably the easiest to verify in specific cases. (7.2) is equivalent to a condition of Fernique. (7.3) is due to Fernique. Proofs for (7.2) and (7.3) can be found in the indicated references.

8. A Fubini-type theorem.

PROPOSITION 3. *Let H be a Hilbert space of functions (not necessarily an RKHS) defined on $D \subset R^{(1)}$. For each $h \in H$ let $f(\cdot, h)$*

be a measurable function on D. Let $\| \cdot \|_1$ *be a measurable norm on* H *and* B *be the completion of* H *under* $\| \cdot \|_1$. *Assume that for each* $x \in D$, $f(x, \cdot)$ *has with respect to* $\| \cdot \|_1$ *a unique continuous extension to* B *which we again denote by* $f(x, \cdot)$; *assume that the extended function,* $f(\cdot, \cdot)$ *is measurable and integrable in the usual sense on* $D \times B$; *finally assume that* $g(h) = \int_D f(x, h)\, dx$ *has, with respect to* $\| \cdot \|_1$, *a unique continuous extension to* B *given by* $\int_D f(x, b)\, dx$. *Then* $g(h)$ *is integrable in the sense of* §6 *and* $\int_H g(h)\, \nu\{dh\} = \int_B \int_D f(x, b)\, dx\, \mu\{db\} = \int_D \int_B f(x, b)\, \mu\{db\}\, dx$.

Proof. The integrability of g and its evaluation follow from Proposition 1. The interchange of integrals is nothing more than an application of the classical Fubini Theorem.

It should be noted that if D is a bounded subset of $R^{(1)}$ and if f is uniformly continuous on $(D, |\cdot|) \times (H, \|\cdot\|_1)$ then all of the above extension and measurability conditions are satisfied.

9. **Example.** Let $F(h) = \exp\{-(h, h)\}$. Let $\{P_n\}$ be an increasing sequence of projections converging to I with the dimension of the range of P_n equal to n. Then $\int_H F(P_n h)\, \nu\{dh\} = 3^{-n/2}$. This approaches 0 as n approaches infinity. However F is not the 0 function. Therefore there exists no measurable norm on H with the property that F possesses a unique continuous extension to B where B is the completion of H under that norm.

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