ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 2, Spring 1972

DECOMPOSITIONS OF DIRECT SUMS OF CYCLIC p-GROUPS

DALTON TARWATER AND ELBERT WALKER

Throughout we will be considering only p-primary Abelian groups G which are direct sums of cyclic groups. Such groups have many basic subgroups. Recall that B is a basic subgroup of G if B is a direct sum of cyclic groups (which is automatic here), pure, and G/B is divisible. Since G is a direct sum of cyclic groups, G itself is basic in G. If $G = \sum_{i=1}^{\infty} Zx_i$, where $o(x_i) = p^{n_i}$, $n_1 < n_2 < \cdots$, then $B = \sum_{i=1}^{\infty} Z(x_i - p^{n_{i+1}-n_i}x_{i+1})$ is basic in G with $G/B \approx Z(p^{\infty})$ [**1**, Lemma 31.1, p. 103]. Let G be any direct sum of cyclics, and suppose B is basic in G with $G/B \approx Z(p^{\infty})$. Write $G = \sum_{i=1}^{\infty} G_i$, where G_i is a direct sum of cyclic groups of order p^{n_i}, $n_1 < n_2 < \cdots$, and where no $G_i = 0$. Tarwater [**6**] showed that $G = X_1 \oplus X$ with

$$B = (B \cap X_1) \oplus X, \qquad X_1 = \sum_{i=1}^{\infty} Zx_i,$$

$$B \cap X_1 = \sum_{i=1}^{\infty} Z(x_i - p^{n_{i+1}-n_i}x_{i+1}),$$

and $o(x_i) = p^{n_i}$. In particular, if C is any basic subgroup of G with $G/C \approx Z(p^{\infty})$, then there is an automorphism α of G such that $\alpha(B) = C$. More generally, he indicated in [**5**] that if G is any direct sum of cyclic groups with basic subgroups B and C such that G/B and G/C are isomorphic and have (the same) finite rank, then there is an automorphism α of G such that $\alpha(B) = C$. The idea is to show that $G = X_1 \oplus \cdots \oplus X_n = Y_1 \oplus \cdots \oplus Y_n$ with $B = (B \cap X_1) \oplus \cdots \oplus (B \cap X_n)$, $C = (C \cap Y_1) \oplus \cdots \oplus (C \cap Y_n)$, $X_i \approx Y_i$, and with $X_i/(B \cap X_i) \approx Y_i/(C \cap Y_i) \approx Z(p^{\infty})$. Now P. Hill [3] has proved that if G is any direct sum of cyclic groups and B and C are basic in G with $G/B \approx G/C$, then there is an automorphism α of G such that $\alpha(B) = C$. Hill's proof involves extending height-preserving automorphisms of subgroups, and employs a two stage transfinite induction. One would hope that the general case follows from the case when $G/B \approx G/C \approx Z(p^{\infty})$, or at least that most of the group

Received by the editors November 19, 1970 and, in revised form, December 10, 1970.

AMS 1970 *subject classifications.* Primary 20K10, 20K30; Secondary 20K25.

theory involved takes place there. This is the case, as an examination of our development will show.

We state the following theorem, whose corollary is quite important in our subsequent proofs.

THEOREM 1 (TARWATER [6], HILL [2]). *Let B and C be basic subgroups of the direct sum of cyclic groups G such that $G/B \approx G/C \approx Z(p^\infty)$. Then there is an automorphism α of G such that $\alpha(B) = C$.*

COROLLARY 1. *Let G be a direct sum of cyclic groups and let B be basic in G such that $G/B \approx Z(p^\infty)$. Suppose $G = Y_1 \oplus Y$ with Y_1 unbounded. Then $G = X_1 \oplus X$ with $B = (B \cap X_1) \oplus X$, $X_1 \approx Y_1$, $X \approx Y$.*

PROOF. Let C_1 be basic in Y_1 with $Y_1/C_1 \approx Z(p^\infty)$. Then $C = C_1 \oplus Y$ is basic in G and again $G/C \approx Z(p^\infty)$. By Theorem 1, there is an automorphism α of G with $\alpha(C) = B$. Then $X_1 = \alpha(Y_1)$ and $X = \alpha(Y)$ have the desired properties.

THEOREM 2. *Let G be a countable direct sum of cyclic groups and let B be basic in G. Then $G = \sum_{i \in I} X_i \oplus X$ with*

(a) $B = \sum_{i \in I}(B \cap X_i) \oplus X$, *and*

(b) $X_i/(B \cap X_i) \approx Z(p^\infty)$.

PROOF. We may suppose that G is unbounded. Write $G = \sum_{i=1}^{\infty} G_i$, where G_i is a direct sum of cyclic groups of order p^{n_i}, $n_1 < n_2 < \cdots$, and where no $G_i = 0$. Let $G_i[p] = S_i \oplus B \cap G_i[p]$. Let A_i be a summand of G_i such that $A_i[p] = S_i$, and let B_i be a summand of B such that $B_i[p] = B \cap G_i[p]$. All this is possible since G_i is a direct sum of cyclic groups of fixed order p^{n_i} and B is pure in G. Now $G = \sum_{i=1}^{\infty}(A_i \oplus B_i)$. Indeed, $G[p] = \sum_{i=1}^{\infty}(A_i \oplus B_i)[p]$, and every element in $\sum_{i=1}^{\infty}(A_i \oplus B_i)[p]$ has the same height in $\sum_{i=1}^{\infty}(A_i \oplus B_i)$ as it does in G since B_i is also a direct sum of cyclic groups of order p^{n_i}. Thus we have a decomposition $G = \sum_{i=1}^{\infty}(A_i \oplus B_i)$ with $E(A_i \oplus B_i) = n_i$, $1 \leqq n_1 < n_2 < \cdots$, $B \cap A_i = 0$, and $B_i \subseteq B$. We want to stipulate that infinitely many of the B_i are nonzero. Suppose not. Pick a basis of G by picking one for each A_i and B_i. Suppose $B_{k_1} = 0$. Then $A_{k_1} \neq 0$. Let a_{k_1} be a member of the basis chosen for A_{k_1}. Write $a_{k_1} = p^{n_{k_1}} \sum_{i=n_{k_1}+1}^{m_1} g_i + b_{k_1}$, with $g_i \in A_i \oplus B_i$, $E(\sum_{i=n_{k_1}+1}^{m_1} g_i) = 2n_{k_1}$, and $b_{k_1} \in B$. Pick $B_{k_2} = 0$ with $k_2 > m_1$. Then $A_{k_2} \neq 0$. Let a_{k_2} be a member of the basis chosen for A_{k_2}. Write $a_{k_2} = p^{n_{k_2}}(\sum_{i=n_{k_2}+1}^{m_2} g_i) + b_{k_2}$ with $g_i \in A_i \oplus B_i$, $E(\sum_{i=n_{k_2}+1}^{m_2} g_i) = 2n_{k_2}$, and $b_{k_2} \in B$. Continue the process. In the basis originally chosen for G, replace all the a_{k_i} by b_{k_i}. The resulting set is still a basis for G. This gives in the obvious way a new decomposition $G = \sum_{i=1}^{\infty}(A_i \oplus B_i)$ with $E(A_i \oplus B_i) = n_i$, $1 \leqq n_1 < n_2 < \cdots$, $B \cap A_i = 0$, $B_i \subseteq B$, and with infinitely many of the B_i nonzero. Since G is countable, G/B is countable. Suppose its rank is $\aleph_0$. Then $G/B = \sum_{i=1}^{\infty} X_i/B$ with $X_i/B \approx Z(p^{\infty})$. (From this case it will be obvious how to handle the case where G/B has finite rank.) Let $A = \sum_{i=1}^{\infty} A_i$. We need the following technical fact. Suppose $x_i \in X_i$. Then $x_i + B = a + B$ with $a \in A$ and $E(a) = E(a + B)$. Indeed, $x_i + B = \sum_{j=1}^{\infty}(a_j + b_j) + B$ with $a_j \in A_j$, $b_j \in B_j$, and with $E(\sum_{j=1}^{\infty}(a_j + b_j)) = E(\sum_{j=1}^{\infty}(a_j + b_j) + B)$. But dropping the b_j does not affect either of the two equalities since $b_j \in B$. In particular, there are elements in $A \cap X_i$ of arbitrary order whose orders are the same mod B. Now relabel the B_i so that B_{11}, B_{21}, B_{12}, B_{31}, B_{22}, B_{13}, B_{41}, B_{32}, B_{23}, B_{14}, $\cdots$ are the nonzero B_i in order of increasing exponents $k_{11} < k_{21} < k_{12} < k_{31} < \cdots$. Let $a_{11} \in A \cap X_1$, $E(a_{11}) = E(a_{11} + B) = k_{11}$. Let $a_{21} \in A \cap X_2$, $E(a_{21}) = E(a_{21} + B) = k_{21}$. Let $a_{12} \in A \cap X_1$, $E(a_{12}) = E(a_{12} + B) = k_{12}$, $a_{11} = p^{k_{12}-k_{11}} a_{12}$ mod B. Let $a_{31} \in A \cap X_3$, $E(a_{31}) = E(a_{31} + B) = k_{31}$. Let $a_{22} \in A \cap X_2$, $E(a_{22}) = E(a_{22} + B) = k_{22}, a_{21} = p^{k_{22}-k_{21}} a_{22}$mod B. Let $a_{13} \in A \cap X_1$, $E(a_{13}) = E(a_{13} + B) = k_{13}$, $a_{12} = p^{k_{13}-k_{12}} a_{13}$ mod B. Continue in this fashion. Pick a basis of G by picking a basis of each A_i and each B_{ij}. Let b_{ij} be a member of the basis picked for B_{ij}. Replace b_{ij} by $a_{ij} + b_{ij}$. From our construction, the resulting set is still a basis of G. Let $y_{ij} = a_{ij} + b_{ij}$, $Y_i = \sum_{j=1}^{\infty} Zy_{ij}$. Then Y_i is a summand of G, being generated by a subset of a basis of G. $Y = \sum_{i=1}^{\infty} Y_i$ is a summand of G since it too is generated by a subset of a basis of G.

If $S_i = \sum_{j=1}^{\infty} Z(y_{ij} - p^{k_{ij+1}-k_{ij}} y_{ij+1})$, then $S_i = B \cap Y_i$. Suppose $b = \sum_{j=1}^{n} m_j y_{ij} \in B \cap Y_i$. We use induction on n. If $n = 1, 0 = b \in S_i$ since $B \cap A_i = 0$. Suppose any shorter linear combination of the y_{ij} that is in B is in S_i. Now $b - m_1 y_{i1} + p^{k_{i2}-k_{i1}} m_1 y_{i2} = (p^{k_{i2}-k_{i1}} m_1 + m_2) y_{i2} + \sum_{j=3}^{n} m_j y_{ij} \in B$, and by induction is in S_i. Since $-m_1 y_{i1} + p^{k_{i2}-k_{i1}} m_1 y_{i2} \in S_i$, it follows that $b \in S_i$ and hence that $B \cap Y_i = S_i$. $B \cap Y_i$ is pure in Y_i and $Y_i/(B \cap Y_i) \approx Z(p^{\infty})$. Since $Y_i \subseteq X_i$ and the sum $\sum_{i=1}^{\infty} X_i/B$ is direct, it follows that $B \cap Y = \sum_{i=1}^{\infty}(B \cap Y_i)$. Write $G/(Y \cap B) = Y/(Y \cap B) \oplus K/(Y \cap B)$. Since Y is a summand of G, $G/Y \approx K/(Y \cap B)$ is a direct sum of cyclics, whence $K = W \oplus (Y \cap B)$. Thus $G = Y \oplus W$, and $B = (B \cap Y) \oplus (B \cap W)$. But $Y + B = G$ by construction, since $Y_i + B = X_i$ for all i. Therefore $G = Y \oplus (B \cap W)$, whence $W = B \cap W$. We have proved Theorem 2.

It should be noted that the above result follows from Hill's theorem [2]. The next two theorems reduce everything to the countable case.

THEOREM 3. *Let G be a direct sum of cyclic groups, and let B be basic in G. Then* $G = \sum_{i \in I} X_i$ *with* $B = \sum_{i \in I}(B \cap X_i)$ *and* $|X_i| \leqq \aleph_0$.

PROOF. Let Λ be a basis of G. Consider the set of all independent families $\{X_i\}_{i \in I}$ of subgroups of G such that
(a) $|X_i| \leqq \aleph_0$ for all $i \in I$,
(b) $\sum_{i \in I} X_i$ is generated by a subset of Λ,
(c) $B \cap (\sum_{i \in I} X_i) = \sum_{i \in I}(B \cap X_i)$, and
(d) $B \cap X_i$ is basic in X_i for all $i \in I$.

Partial order these families in the obvious way. Zorn's lemma clearly applies. Let $\{X_i\}_{i \in I}$ be a maximal family. Let $X = \sum_{i \in I} X_i$, and $Y = \sum_{i \in I}(B \cap X_i)$. Write $G/Y = X/Y \oplus K/Y$ with $K \supseteq B$. Conditions (c) and (d) enable us to do this. Since X is a summand of G, $G/X \approx K/Y$ is a direct sum of cyclic groups, so $K = C \oplus Y$ and $G = X \oplus C$. Also $B = Y \oplus (B \cap C)$ since $K \supseteq B$. If $C = 0$, we are done. So suppose not. Write $G = X \oplus H$, where H is generated by the elements of Λ not in X. Let K_1 be any nonzero countable subgroup of C. Since $B \cap C = D$ is basic in C, C/D is divisible, and $(K_1 + D)/D$ is in a countable divisible subgroup C_1/D of C/D. Then $C_1 = S_1 + D$ with $|S_1| \leqq \aleph_0$ and $K_1 \subseteq S_1$. Now $(S_1 + D)/D \approx S_1/(S_1 \cap D)$ is divisible. Put $S_1 \cap D$ in a countable pure subgroup P of D. We have $(S_1 + P)/((S_1 + P) \cap D) = (S_1 + P)/P \approx S_1/(S_1 \cap P)$ is divisible and P is pure in $S_1 + P$, being pure in D and hence in G. Set $S_1 + P = T_1$. Thus we have $T_1/(T_1 \cap D)$ is divisible, $T_1 \cap D$ is pure in T_1, whence $T_1 \cap D$ is basic in T_1. Furthermore, $|T_1| \leqq \aleph_0$, and $K_1 \subseteq T_1 \subseteq C$. For each $t \in T_1$, write $t = x + h, x \in X, h \in H$. Write $h = \sum_{a_\lambda \in \Lambda} n_\lambda a_\lambda$, and write $a_\lambda = x' + c'$, $x' \in X$, $c' \in C$. Adjoin all such c' to T_1, getting a countable subgroup K_2 of C. From K_2 get a countable subgroup T_2 of S_2 in the same way T_1 was gotten from K_1. Let $T = \bigcup_{i=1}^{\infty} T_i$. Now $\sum_{i \in I} X_i \oplus T$ is generated by a subset of Λ by construction. Hence $\sum_{i \in I} X_i \oplus T$ is a summand of G. Since $T \subseteq C$, T is a summand of C. Thus since $B \cap (X \oplus C) = (B \cap X) \oplus (B \cap C)$, we have $B \cap (X \oplus T) = (B \cap X) \oplus (B \cap T)$. Also $B \cap T = \bigcup_{i=1}^{\infty}(B \cap T_i)$ is basic in T. Since T is countable, it follows that $\{X_i\}_{i \in I}$ is not maximal after all. This concludes the proof.

THEOREM 4. *Let G be a direct sum of cyclic groups, and let B and C be basic subgroups of G such that* $G/B \approx G/C$. *Then* $G = \sum_{i \in I} X_i = \sum_{i \in I} Y_i$ *with*
(a) $X_i \approx Y_i$ *and countable for all* $i \in I$;
(b) $B = \sum_{i \in I}(B \cap X_i)$; $C = \sum_{i \in I}(C \cap Y_i)$; *and*
(c) $X_i/(B \cap X_i) \approx Y_i/(C \cap Y_i)$ *for all* $i \in I$.

PROOF. By Theorem 3, there are decompositions $G = \sum_{i\in I} X_i = \sum_{i\in I} Y_i$ with each X_i and Y_i countable and with condition (b) holding. The proof is completed by two applications of the following theorem which we state for the reader's convenience.

THEOREM (RICHMAN AND WALKER [4]). *Let m be an infinite cardinal number. Let f be a function from the set X to the cardinal numbers such that*

$$f(x) = \sum_{i\in I} f_i(x) = \sum_{i\in I} g_i(x) \quad \text{for all } x \in X,$$

where $\sum_{x\in X} f_i(x) \leqq m \geqq \sum_{x\in X} g_i(x)$ for all $i \in I$. Then there exists a partition of I into subsets S_α of cardinal $\leqq m$ such that

$$\sum_{i\in S_\alpha} f_i(x) = \sum_{i\in S_\alpha} g_i(x).$$

We come now to the main theorem.

THEOREM 5. *Let G be a direct sum of cyclic groups and let B and C be basic subgroups of G such that $G/B \approx G/C$. Then there is an automorphism α of G such that $\alpha(B) = C$.*

PROOF. By Theorem 4, we may suppose that G is countable. (We will assume that the rank of G/B is $\aleph_0$. How to treat the finite case will become clear from this case.) Using Theorem 2, we write

$$G = \sum_{i=1}^{\infty} X_i \oplus X = \sum_{i=1}^{\infty} Y_i \oplus Y$$

with

$$B = \sum_{i=1}^{\infty} (B \cap X_i) \oplus X, \qquad C = \sum_{i=1}^{\infty} (C \cap Y_i) \oplus Y,$$

$$X_i/(B \cap X_i) \approx Y_i/(C \cap Y_i) \approx Z(p^\infty) \quad \text{for all } i.$$

Now we are going to use Corollary 1 several times. We want to arrange for X_1 and Y_1 to have isomorphic unbounded summands; that is, that $X_1 = K \oplus L$, and $Y_1 = M \oplus N$ with $K \approx M$ and K unbounded. The crucial thing is to maintain $X_1/(B \cap X_1) \approx Y_1/(C \cap Y_1) \approx Z(p^\infty)$. If some X_i and Y_j have isomorphic unbounded summands, just renumber. Otherwise, for $i > 1$, write $X_i = K_i \oplus L_i$ so that L_i and Y_1 have no isomorphic summands, L_i is unbounded, and $K_i \subseteq B$. Similarly, write $X = K \oplus L$. Now $H = X_1 \oplus \sum_{i=2}^{\infty} K_i \oplus K$ and Y_1 clearly have isomorphic unbounded summands.

So write $H = S_1 \oplus S$, $Y_1 = T_1 \oplus T$ with $S_1 \approx T_1$ unbounded summands, $S \subseteq B$, and $T \subseteq C$. Further, choose S_1 and T_1 so that they are direct sums of cyclic groups of distinct orders; that is so that each Ulm invariant of S_1 and of T_1 is 0 or 1. Now write $S_1 = \sum_{i=1}^{\infty} D_i$, $T_1 = \sum_{i=1}^{\infty} E_i$, $D_i \approx E_i$ and unbounded, and $\sum_{i=2}^{\infty} D_i \subseteq B$, $\sum_{i=2}^{\infty} E_i \subseteq C$. We now have

$$G = D_1 \oplus \sum_{i=2}^{\infty} (D_i \oplus L_i) \oplus (L \oplus S)$$

$$= E_1 \oplus \sum_{i=2}^{\infty} (E_i \oplus Y_i) \oplus (Y \oplus T).$$

Since $(D_i + L_i)/(B \cap (D_i \oplus L_i)) \approx Z(p^{\infty})$ and L_i is unbounded, we can write, for $i \geqq 2$, $D_i \oplus L_i = V_i \oplus V_{1i}$ with $V_i \approx D_i$ and $V_{1i} \subseteq B$. Similarly, write $E_i \oplus Y_i = W_i \oplus W_{1i}$ with $W_i \approx E_i$ and $W_{1i} \subseteq C$. Finally, set $D_1 = V_1$, $E_1 = W_1$, $V = \sum_{i=2}^{\infty} V_{1i} \oplus L \oplus S$, $W = \sum_{i=2}^{\infty} W_{1i} \oplus Y \oplus T$. We then have

$$G = \sum_{i=1}^{\infty} V_i \oplus V = \sum_{i=1}^{\infty} W_i \oplus W,$$

$$B = \sum_{i=1}^{\infty} (B \cap V_i) \oplus V, \qquad C = \sum_{i=1}^{\infty} (C \cap W_i) \oplus W,$$

$$V_i/(B \cap V_i) \approx W_i/(C \cap W_i) \approx Z(p^{\infty}) \quad \text{and} \quad V_i \approx W_i \quad \text{for all } i,$$

and our condition on the Ulm invariants of S_1 and T_1 guarantees that $V \approx W$. Theorem 1 completes the proof.

There are several immediate corollaries to Theorem 5 that we can draw.

Corollary 2. *Let G be a direct sum of cyclic groups, let m be a cardinal number $\geqq 1$, and let $\mathcal{B}$ be the set of all basic subgroups B of G such that the rank of $G/B = m$. Suppose $\mathcal{B} \neq \emptyset$. Let $\mathcal{S}(\mathcal{B})$ be the symmetric group on $\mathcal{B}$, let $A(G)$ be the automorphism group of G, and let $\phi : A(G) \to \mathcal{S}(\mathcal{B})$ be the canonical map. Then* $\operatorname{Im}(\phi)$ *is transitive and* $\operatorname{Ker}\phi$ *is the center of $A(G)$, which is the group of p-adic units.*

Proof. Theorem 5 says that $\operatorname{Im}\phi$ is transitive. Suppose $f \in \operatorname{Ker}\phi$. Then f maps every basic subgroup $B \in \mathcal{B}$ onto itself. Let $G = Zx \oplus H$, and suppose $f(x) \notin Zx$, $f \in \operatorname{Ker}\phi$. Then $f(x) = nx + h$,

$h \in H$, $h \neq 0$. Let B be a basic subgroup of H such that H/B has rank m, and such that $h \notin B$. This is easy to arrange. Then $Zx \oplus B \in \mathcal{B}$, but $f(x) = nx + h \notin Zx \oplus B$. Therefore, if $f \in \operatorname{Ker} \phi$, then f maps every cyclic summand of G into itself. This forces f to be in the center of the endomorphism ring of G [1, proof of 56.3, p. 217]. But the center of the endomorphism ring of G is the ring of p-adic integers [1, 56.3, p. 217]. Thus $\operatorname{Ker} \phi$ is the group U of p-adic units. That the center of $A(G)$ is exactly U follows in our case from an argument entirely analogous to the proof of 56.3, p. 217 in [1]. This completes the proof.

Let B be basic in G, G a direct sum of cyclic groups. Let $\bar{G}$ be the torsion completion of G. Then B and G are basic in $\bar{G}$, and there is an automorphism α of $\bar{G}$ such that $\alpha(B) = G$. This is well known. However, one can achieve the same thing not by passing to $\bar{G}$, but merely by going to a bigger direct sum of cyclic groups in which G is basic.

Corollary 3. *Let G be a direct sum of cyclic groups. Then there is a direct sum of cyclic groups H with G basic in H such that if B is any basic subgroup of G, there is an automorphism α of H such that $\alpha(B) = G$.*

Proof. Let C be a lower basic subgroup of G [1, p. 105]. That is, C is basic in G with the rank of G/C as large as possible. There is an isomorphism $\alpha : G \to C$. Thus there is a direct sum of cyclic groups H with G a lower basic subgroup in H. Let B be any basic subgroup of G. Then $H/B \approx H/G$. Theorem 5 completes the proof.

Corollary 4. *Let B be a direct sum of cyclic groups, let $\bar{B}$ be its torsion completion, let G and H be pure in $\bar{B}$, G, $H \supseteq B$, let G and H be direct sums of cyclic groups, and let $G/B \approx H/B$. Then there is an automorphism α of $\bar{B}$ such that $\alpha(G) = H$ and $\alpha(B) = B$.*

Proof. By Theorem 5, there is an isomorphism $\alpha : G \to H$ with $\alpha(B) = B$. But α extends to an automorphism of $\bar{B}$.

References

1. L. Fuchs, *Abelian groups*, Akad. Kiadó, Budapest, 1958; republished by Internat. Series of Monographs on Pure and Appl. Math., Pergamon Press, New York, 1960. MR 21 #5672; MR 22 #2644.

2. Paul Hill, *Automorphisms of countable primary abelian groups*, Proc. Amer. Math. Soc. 25 (1970), 135–140. MR 41 #334.

3. ——, *The covering theorem for upper basic subgroups*, Michigan Math. J. 18 (1971), 187–192.

4. Fred Richman and Elbert Walker, *Extending Ulm's theorem without group theory*, Proc. Amer. Math. Soc. **21** (1969), 194–196. MR 39 #1553.

5. J. Dalton Tarwater, *Orbits of basic subgroups*, Notices Amer. Math. Soc. **17** (1970), 149. Abstract #672-233.

6. ———, *Orbits of basic subgroups of primary abelian groups under automorphisms*, Texas J. Sci. **23** (1971), 431–432.

TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409

NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88001

RICE UNIVERSITY, HOUSTON, TEXAS 77001