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PRESERVATION OF COPRODUCTS BY $\text{Hom}_R(M, -)$

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The functor $\text{Hom}_R(M, -)$ from the category of left R-modules into the category of abelian groups always preserves products but preserves coproducts only in special cases. An obvious sufficient condition for the preservation of coproducts is that M be finitely generated. In several significant special cases (for example, when M is projective or R is left Noetherian) finite generation is also necessary. H. Bass has stated [**1**, p. 54] that finite generation is not in general necessary for the preservation of coproducts and he has given a necessary and sufficient condition which we state in slightly altered form: $\text{Hom}_R(M, -)$ *preserves coproducts if and only if M is not the union of any nest of proper submodules of the form* $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$ *(i a positive integer)*. In this note we present a simple example of a non-finitely generated module M for which $\text{Hom}_R(M, -)$ preserves coproducts and we discuss the effect of some additional hypotheses on coproduct preservation.

We make four assumptions that hold throughout this note: R is a ring with identity. All modules are unitary left R-modules. A map is an R-homomorphism. N is the set of positive integers.

THEOREM. *There exists a Boolean ring $\boldsymbol{R}$ which has cardinal $\aleph_1$ and contains a maximal ideal $\boldsymbol{M}$ which is neither finitely nor countably generated but for which* $\text{Hom}_{\boldsymbol{R}}(\boldsymbol{M}, -)$ *preserves coproducts.*

PROOF. For each ordinal number β let S_β be the set of all ordinals α such that $\alpha < \beta$. Let Ω be the least ordinal of uncountable cardinal. The validity of our example will be seen to stem from the following fact: A subset X of S_Ω is cofinal (i.e., for every $\alpha \in S_\Omega$ there is a $\beta \in X$ such that $\alpha < \beta$) if and only if it is uncountable.

Let $\boldsymbol{R}$ be the subring of the ring of all subsets of S_Ω that is generated by the set of all 'segments' $\{S_\alpha \mid \alpha \leqq \Omega\}$. Then $\boldsymbol{R}$ is a Boolean ring with identity S_Ω and has cardinal $\aleph_1$. Let $\boldsymbol{M}$ be the ideal of $\boldsymbol{R}$ generated by the set of all 'short' segments $\{S_\alpha \mid \alpha < \Omega\}$. Then $\boldsymbol{M}$ is proper and maximal. Let $A_i \in \boldsymbol{M}$ $(i \in N)$. For each i in N we have an $\alpha(i) < \Omega$ such that $A_i \subseteq S_{\alpha(i)}$. Since $\{\alpha(i) \mid i \in N\}$ is countable ($=$ not cofinal),

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there is a $\beta < \Omega$ such that $\alpha(i) < \beta$ for all $i \in N$. Then $S_\beta \in \boldsymbol{M}$ but not in the ideal generated by $\{A_i \mid i \in N\}$. Thus $\boldsymbol{M}$ is neither finitely nor countably generated.

Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$ $(i \in N)$ be a nest of ideals of $\boldsymbol{R}$ that has $\boldsymbol{M}$ as its union. For each $\alpha < \Omega$, $S_\alpha \in \boldsymbol{M}$ and consequently $S_\alpha \in A_i$ for some $i \in N$. We define a function $f: S_\Omega \to N$ by letting $f(\alpha)$ be the least positive integer for which $S_\alpha \in A_{f(\alpha)}$. There is an $n \in N$ such that $f^{-1}(n)$ is uncountable ($=$ cofinal). Thus for every $\alpha \in S_\Omega$ there is a $\beta \in f^{-1}(n)$ such that $\alpha < \beta$ and we have $S_\alpha \subset S_\beta \in A_{f(\beta)} = A_n$. Then $S_\alpha \in A_n$ for all $\alpha < \Omega$ and $A_n = \boldsymbol{M}$. We conclude that $\mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{M}, -)$ preserves coproducts.

Assume for the remainder of this note that M is a left R-module for which $\mathrm{Hom}_R(M, -)$ *preserves coproducts.* We will discuss three conditions under which M must be finitely generated.

If M is the coproduct of countably generated modules then M is finitely generated: Suppose $M = \coprod \{A_i \mid i \in I\}$ where each A_i $(i \in I)$ is nonzero and countably generated. From the coproduct preservation property we conclude that I is finite. Then M is countably generated. Let $\{g_i \mid i \in N\}$ be a set of generators for M. For each $i \in N$ let A_i be the submodule of M generated by $\{g_j \mid j \leqq i\}$. Since $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$ $(i \in N)$ and $\bigcup\{A_i \mid i \in N\} = M$, we conclude that $M = A_n$ for some $n \in N$. Then M is generated by $\{g_i \mid i \leqq n\}$.

From this observation it follows that the example $\boldsymbol{M}$ of the theorem is minimal in two senses: There is no example (for any ring R) of a nonfinitely generated R-module M, for which $\mathrm{Hom}_R(M, -)$ preserves coproducts, that has either smaller cardinal or a generating set of smaller cardinal.

If M is projective it is finitely generated: Every projective module is the coproduct of countably generated modules by a theorem of Kaplansky [2]. Our first observation then implies that M is finitely generated. For a more direct proof see [1, p. 53].

By our theorem above, the hypothesis 'M is projective' cannot be weakened to 'M is a submodule of a (finitely generated) free module', nor can it be weakened to 'M is flat' since Boolean rings are regular and all modules over regular rings are flat [3, p. 134].

If R is left Noetherian then M is finitely generated: Suppose M is not finitely generated. Then there is a strictly ascending nest $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ $(i \in N)$ of submodules of M. Let $K = \bigcup\{A_i \mid i \in N\}$. For each $i \in N$ let $K/A_i \subseteq Q_i$ be an embedding of K/A_i in an injective module. We have a map $h: K \to$

$\coprod\{Q_i \mid i \in N\}$ given by $h(k) = (k + A_1, k + A_2, \cdots, k + A_i, \cdots)$ for each $k \in K$. Since R is left Noetherian, a coproduct of injective R-modules is injective. Thus h can be extended to a map $h_1 : M \to \coprod\{Q_i \mid i \in N\}$ and for any such h_1 we have $p_i h_1(M) \supseteq p_i h(M) = K/A_i \neq 0$ for every $i \in N$. Since the existence of such an h_1 would violate our coproduct preservation hypothesis, we conclude that M is finitely generated.

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