ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 2, Spring 1972

ON $\Delta - m$ SETS, ALMOST PERIODIC FUNCTIONS AND GROUP TOPOLOGIES

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1. **Introduction.** Markov [1], in a paper in 1933, introduced a combinatorial property of subsets of the real line which he used to prove that certain stability properties of solutions of differential equations implied their almost periodicity. The only property of these sets that he used is the one that we reproduce below as Theorem 1. It turns out that this combinatorial property makes sense and is useful in arbitrary groups.

2. $\Delta - m$ **sets.**

DEFINITION 1. A set S in a group $(G, +)$ is called a $\Delta - m$ *set* (m a positive integer) in case (1) given elements of G, $t_1, t_2, \cdots, t_{m+1}$, not necessarily distinct, there exist $i \neq j$ such that $t_i - t_j \in S$; and (2) S is symmetric with respect to the identity of G, i.e., $-S = S$.

We can make several remarks. Since all the t_i's may be equal, $0 \in S$ for all $\Delta - m$ sets. Furthermore, the choice $t_1 = t$, $t_2 = 0$ implies that every $\Delta - 1$ set contains t or $-t$ for t an arbitrary element of G. Since S is symmetric, it contains both. Thus the only $\Delta - 1$ set is G itself.

We first restrict ourselves to the additive group of reals. It may be verified that $E_k = \bigcup_{n=-\infty}^{\infty} [n - 1/k, n + 1/k]$ $(k \geqq 3)$ is a $\Delta - (k - 1)$ set but not a $\Delta - (k - 2)$ set.

As these examples indicate, $\Delta - m$ sets in $\mathbf{R}$ can contain gaps; however, in some sense they cannot be too sparse. The next two results make this precise.

THEOREM 1. *Every $\Delta - m$ set in $\mathbf{R}$ is relatively dense, i.e., if S is a $\Delta - m$ set, then there exists an $L(S) > 0$ such that $S \cap [t, t + L(S)] \neq 0$ for all real t.*

PROOF. The proof is by induction on m. The statement for $\Delta - 1$ sets is clear since $\mathbf{R}$ is the only such set. Assume the theorem true for $\Delta - (m - 1)$ sets. Let S be a $\Delta - m$ set. If S is also a $\Delta - (m - 1)$

Received by the editors May 6, 1970 and, in revised form, November 14, 1970.
AMS 1970 *subject classifications.* Primary 43A60, 05B10; Secondary 22A10.
[1]The research of the first author was supported in part by the National Science Foundation under grant GP 11623, and that of the second author under GP 11605.

set then it is relatively dense by the induction hypothesis. Hence we may assume that S is not a $\Delta - (m-1)$ set. There exist $t_1, \cdots, t_m$ such that $t_i - t_j \notin S$. Let $L = 2\max|t_i| + 1$. If S is not relatively dense, there exists t such that $[t, t+L] \cap S = \emptyset$. Now there exists u such that $u + t_i \in [t, t+L]$ for all i. Consider the $m+1$ real numbers $v_i = u + t_i$, $i = 1, \cdots, m$, and $v_{m+1} = 0$. Now $\pm(v_i - v_{m+1}) = \pm v_i \notin S$ and $v_i - v_j = t_i - t_j \notin S$, $1 \leqq j \leqq m$. Thus S is not a $\Delta - m$ set. This contradiction shows that S is relatively dense.

The converse of Theorem 1 is not true. The integers are certainly relatively dense in **R**. But if n is a given positive integer, then let $t_i = \sqrt{p_i}$, $i = 1, \cdots, n$, where p_i is the ith-prime. Then $t_i - t_j$ is not an integer if $i \neq j$. This is easy to see. If $\sqrt{p_i} - \sqrt{p_j} = m$ an integer, then $p_i = m^2 + 2m\sqrt{p_j} + p_j$ implies that $\sqrt{p_j}$ is a rational number. This of course is false. Thus the integers are not a $\Delta - (n-1)$ set. We might also remark that Theorem 1 is true for any subgroup of the reals.

We now let G be an arbitrary group written additively with identity 0.

THEOREM 2. *If S is a $\Delta - m$ set, then there exist $t_1, \cdots, t_m$ in G such that $G = \bigcup_{i=1}^{m}\{t_i + S\}$.*

PROOF. Suppose no m translates of S cover G. Let $t_0 = 0$, $t_1 \notin S$. If $t_1, t_2, \cdots, t_k$ have been picked $k \leqq m-1$, then since $T = \bigcup_{n=0}^{k}\{t_n + S\} \neq G$, pick $t_{k+1} \in G - T$. By choice of the t_k, $t_i - t_j \notin S$, $i \neq j$, $0 \leqq i, j \leqq m$. This contradicts S being a $\Delta - m$ set.

We now discuss the lattice properties of $\Delta - m$ sets. Let $D \equiv \{S \subset G\colon$ there exists an m such that S is a $\Delta - m$ set$\}$. We will need a special case of Ramsey's theorem, see Ryser [2].

RAMSEY'S THEOREM. *If m is a given integer, there exists $N(m)$ such that for every set E with cardinal $(E) \geqq N(m)$ and every division of the 2-element subsets of E into two classes, there exists an m-element subset F of E such that all of the 2-element subsets of F are in the same class.*

THEOREM 3. *The collection D is a filter of sets containing* 0.

PROOF. Each member of D is nonempty and contains 0. The collection D is closed under supersets. Thus it remains to show that D is closed under intersections. Let S be a $\Delta - m$ set and T a $\Delta - n$ set, say with $m \geqq n$. Now T is also a $\Delta - m$ set so we may assume $m = n$. Let $E = \{t_1, t_2, \cdots, t_{N(m+1)}\}$ be an arbitrary collection of elements of G with $N(m+1)$ as in Ramsey's theorem. Divide the

2-element subsets of E into two classes, E_1 and E_2, by $\{t_i, t_j\} \in E_1$ if and only if $t_i - t_j \in S$. By Ramsey's theorem, there exists an $(m+1)$-element subset F of E such that every 2-element subset of F is either in E_1 or E_2. Since S is a $\Delta - m$ set, this cannot be in E_2. Hence all 2-element subsets of F are in E_1; that is, there exist subscripts $j_1, \cdots, j_{m+1}$ such that $t_{j_k} - t_{j_n} \in S$ for all $1 \leqq k, n \leqq m+1$. Since T is a $\Delta - m$ set, one of the differences is also in T. That is $t_i - t_j \in S \cap T$ for some $i \neq j$. So $S \cap T$ is a $\Delta - N(m+1)$ set.

For the group of reals, it is easy to see that

$$F_{k,j} = \bigcup_{n=-\infty}^{\infty} (jn - 1/k, jn + 1/k)$$

for j and k positive integers is in D and $\bigcap_{k,j>0} F_{k,j} = \{0\}$. Thus for $G = \mathbf{R}$ the collection D is a filter of sets with 0 the only common element.

3. **Group topologies.** The properties of the collection D are reminiscent of the properties of the neighborhood system of zero in a topological group. The only property missing is the property that given $S \in D$, there exists $T \in D$ such that $t_1 - t_2 \in S$ for all $t_1, t_2 \in T$. If some subcollection of D which is a filter had this difference property, then this subcollection forms a base for a totally bounded topological group according to Theorem 2.

Recall that a topological group G is totally bounded if for every neighborhood U of 0, there exist elements $a_1, a_2, \cdots, a_n$ such that $G = \bigcup_{i=1}^{n} \{a_i + U\}$. (It is easily verified that left and right totally bounded are equivalent.) The converse of the above statement is true. The neighborhoods of the identity in a totally bounded topological group form a filter of $\Delta - m$ sets. It is easy to see they form a filter. The rest is contained in

THEOREM 4. *If S is a symmetric neighborhood of* 0 *in a totally bounded group G, then S is a $\Delta - m$ set.*

PROOF. Let T be a symmetric neighborhood of 0 such that $T + T \subset S$. Let $G = \bigcup_{i=1}^{n} \{T + t_i\}$. Then if $s_1, s_2, \cdots, s_{n+1}$ are given elements of G, then there exist $i \neq j$ and k such that $s_i, s_j \in \{T + t_k\}$. Then $s_i - t_k = u \in T$ and $s_j - t_k \in T$. Since T is symmetric, $t_k - s_j = v \in T$. Then $s_i - s_j = u + v \in T + T \subset S$, and S is a $\Delta - n$ set.

That it is possible to get families of $\Delta - m$ sets which are filters and satisfy the difference property can be seen by considering almost periodic functions. Indeed, the notion of $\Delta - m$ set is intimately connected with almost periodic functions. Recall that on an arbitrary group G, a complex valued function f is von Neumann left almost periodic if and only if, for $\epsilon > 0$, there is a finite set of group elements $a_1, \cdots, a_n$ such that for each $t \in G$, there is an $i(t)$ such that $|f(t + x) - f(a_{i(t)} + x)| < \epsilon$ for all $x \in G$. Right almost periodicity is defined analogously. It is well know that left and right almost periodicity are equivalent (Turing [9]).

THEOREM 5. *A function f is von Neumann almost periodic on a group G if and only if, for every $\epsilon > 0$, $\{x : |f(x + t) - f(t)| < \epsilon$ for all $t \in G\}$ is a $\Delta - m$ set.*

PROOF. If f is almost periodic, let $a_1, \cdots, a_n$ be chosen as above for $\epsilon/2$. If $t_1, \cdots, t_{n+1}$ are given then there are numbers $i, j, k, i \neq j$, such that

$$|f(t_i + x) - f(a_k + x)| < \epsilon/2$$

and

$$|f(t_j + x) - f(a_k + x)| < \epsilon/2 \quad \text{for all } x.$$

Consequently $|f(t_i + x) - f(t_j + x)| < \epsilon$ for all x. This is equivalent to $|f(t_i - t_j + x) - f(x)| < \epsilon$ for all x. That is, given $t_1, \cdots, t_{n+1}$, there exist $i \neq j$ such that $t_i - t_j \in \{t : |f(t + x) - f(x)| < \epsilon$, all $x \in G\}$.

Conversely, suppose $B_\epsilon = \{x : |f(x + t) - f(t)| < \epsilon$, all $t \in G\}$ is a $\Delta - m$ set for each $\epsilon > 0$. According to Theorem 2, there exist $t_1, \cdots, t_m$ such that $G = \bigcup_{i=1}^m \{t_i + B_\epsilon\}$. If $t \in G$ is given, then $-t \in \{t_i + B_\epsilon\}$ for some i. Then $-t_i - t \in B_\epsilon$ so that $|f(-t_i - t + x) - f(x)| < \epsilon$ for all $x \in G$. This is equivalent to $|f(t + x) - f(-t_i + x)| < \epsilon$ for all $x \in G$. Thus the $-t_i$ serve as the a_i in the definition of almost periodicity.

The $\Delta - m$ version of almost periodicity on a group appears to be a more direct generalization of the Bohr definition on the reals, **R**.

We now turn to the construction of a collection of $\Delta - m$ sets using almost periodic functions on a group that will serve as a basis for a topology. Recall that a collection of sets $\mathcal{B}$ will serve as a neighborhood base for the identity 0 in a group if $\mathcal{B}$ satisfies

(1) if $U \in \mathcal{B}$ then $0 \in U$;

(2) if $U, V \in \mathcal{B}$ then there is a $W \in \mathcal{B}$ such that $W \subset U \cap V$;

(3) if $U \in \mathcal{B}$ then there is a $V \in \mathcal{B}$ such that $-V \subset U$;

(4) if $U \in \mathcal{B}$ then there is a $V \in \mathcal{B}$ such that $V + V \subset U$; and

(5) if $U \in \mathcal{B}$ and $t \in G$ then there is a $V \in \mathcal{B}$ such that $V \subset t + U - t$.

Furthermore, if $U \in \mathcal{B}$ implies that there are elements $a_1, \cdots, a_n$

such that $\bigcup_{i=1}^{n}\{a_i + U\} = G$, then this is a neighborhood base for a totally bounded topology. This is sometimes called precompact. Note that we are not requiring that the topology be Hausdorff. A topological group will be Hausdorff, if, given $0 \neq x \in G$, there is a $U \in \mathcal{B}$ such that $x \notin U$.

If f is a complex valued almost periodic function on G, and $v(x) = \sup\{|f(x + t) - f(t)| : t \in G\}$, then v is called the translation function of f. It has the following properties:

(i) $v(x) \geqq 0, v(0) = 0$;

(ii) $v(-x) = v(x)$;

(iii) $v(x + y) \leqq v(x) + v(y)$;

(iv) v is its own translation function;

(v) if f is continuous in some topology τ, then v is also continuous in that topology;

(vi) $v(x + y - x) = u_x(y)$ where

$$u_x(y) = \sup_t |v(x + y + t) - v(x + t)|,$$

$u_x(y)$ is the translation function of $v_x(y) = v(x + y)$.

The proofs of these properties are given in Besicovitch [3] for $G = \mathbf{R}$, but the proofs carry over to any group without much change.

If (G, τ) is a given topological group then $\mathcal{U}(G, \tau)$ is the collection of complex valued uniformly continuous functions on (G, τ) and $\mathcal{A}(G, \tau) \subset \mathcal{U}(G, \tau)$ is the collection of almost periodic functions on G which are continuous in (G, τ). Let $T(G, \tau)$ be the collection of translation functions v for $f \in \mathcal{A}(G, \tau)$; and for $v \in T(G, \tau)$ we let $N(v, \epsilon) \equiv \{t \in G : v(t) < \epsilon\}$. Then it is easy to verify that

(a) $0 \in N(v, \epsilon)$;

(b) $N(v_1 + v_2, \min(\epsilon_1, \epsilon_2)) \subset N(v_1, \epsilon_1) \cap N(v_2, \epsilon_2)$;

(c) $-N(v, \epsilon) = N(v, \epsilon)$;

(d) $N(v, \epsilon/2) + N(v, \epsilon/2) \subset N(v, \epsilon)$;

(e) $N(v, \epsilon) \in \tau$;

(*f*) $a + N(u_a, \epsilon) - a \subset N(v, \epsilon)$, $u_a =$ translation function of v_a.

From (a)–(f) it follows that $\mathcal{B} = \{N(v, \epsilon) : \epsilon > 0,\ v \in T(G, \tau)\}$ satisfies (1)–(5) and thus is a base for a topology which makes G a topological group. We now have

THEOREM 6. *Let (G, τ) be a topological group. Then there exists a topology $\bar{\tau}$ for G such that*

(1) *$(G, \bar{\tau})$ is a totally bounded topological group*;

(2) $\bar{\tau} \subset \tau$;

(3) $\mathcal{A}(G, \tau) = \mathcal{U}(G, \bar{\tau})$;

(4) *$(G, \bar{\tau})$ is Hausdorff if and only if $\mathcal{A}(G, \tau)$ separates points*;

(5) *if* (G, τ) *is totally bounded, then* $\tau = \bar{\tau}$;
(6) *if* $\tau_1 \subset \tau_2$ *then* $\bar{\tau}_1 \subset \bar{\tau}_2$.

PROOF. We take for $\bar{\tau}$, the topology generated by the collection $\mathcal{B}$ described above. Then $(G, \bar{\tau})$ is a topological group. Each $N(v, \epsilon)$ is a $\Delta - m$ set by Theorem 5 so by Theorem 2, $(G, \bar{\tau})$ is a totally bounded topological group. By (e) above $\bar{\tau} \subset \tau$.

To show (3) we let $f \in \mathcal{A}(G, \tau)$ and $\epsilon > 0$. If v is the translation function of f, then $v \in T(G, \tau)$ so that if $x - y \in N(v, \epsilon)$, then

$$|f(x) - f(y)| \leqq \sup_t |f(x + t) - f(y + t)|$$
$$= \sup_t |f(x - y + t) - f(t)| = v(x - y) < \epsilon.$$

Thus $f \in \mathcal{U}(G, \bar{\tau})$. To show the converse, it is sufficient to show that $\mathcal{A}(G, \bar{\tau}) \subset \mathcal{A}(G, \tau)$ and that $\mathcal{U}(G, \bar{\tau}) \subset \mathcal{A}(G, \bar{\tau})$ holds for totally bounded topologies. The above inclusion for $\mathcal{A}$ follows from the fact that the definition of almost periodic does not depend on the topology. Thus the inclusion is just the continuity inclusion. To see that $\mathcal{U}(G, \bar{\tau}) \subset \mathcal{A}(G, \bar{\tau})$ for $(G, \bar{\tau})$ totally bounded, let $f \in \mathcal{U}(G, \bar{\tau})$. Then there exists a symmetric neighborhood V of 0 such that $x - y \in V$ implies that $|f(x) - f(y)| < \epsilon$. Let $G = \bigcup_{i=1}^n \{a_i + V\}$. If $x \in G$, there exists a_i such that $x - a_i \in V$. Thus for any $t \in G$, $(x + t) - (t + a_i) \in V$ and hence $|f(x + t) - f(t + a_i)| < \epsilon$ for all $t \in G$. Thus f is von Neumann almost periodic. This finishes the proof of (3). To see that (4) is right, suppose $(G, \bar{\tau})$ is Hausdorff. Then for $a \neq b$, there is an $N(v, \epsilon)$, $v \in T(G, \tau)$ such that $a - b \notin N(v, \epsilon)$. That is $v(a - b) \geqq \epsilon > 0$. Then $v(x - b)$ satisfies $v \in \mathcal{A}(G, \tau)$, $v(b) = 0$, and $v(a - b) \geqq \epsilon > 0$. Conversely, if $\mathcal{A}(G, \tau)$ separates points, let $a \neq 0$. Then there is an $f \in \mathcal{A}(G, \tau)$ such that $|f(a) - f(0)| = \epsilon > 0$. Then for v the translation function of f, we have $v(a) = \sup_t |f(t + a) - f(t)| \geqq |f(a) - f(0)| = \epsilon$. Hence $a \notin N(v, \epsilon/2)$. But a topological group is Hausdorff if every nonzero point is excluded by some neighborhood of 0.

To see that $(G, \tau) = (G, \bar{\tau})$ if τ is totally bounded, note first that it suffices to prove that $\tau \subset \bar{\tau}$. Let U be a neighborhood of 0 in (G, τ). Since (G, τ) is completely regular there exists a uniformly continuous [4, p. 14] function $f: (G, \tau) \to [0, 1]$ such that $f(0) = 0$, $f(x) = 1$ on $G - U$. Since (G, τ) is totally bounded, $f \in \mathcal{A}(G, \tau)$ by the proof of (3). Let v be the translation function of f. Then for $x \notin U$, $v(x) = \sup_t |f(x + t) - f(t)| \geqq |f(x) - f(0)| = 1$. Hence $N(v, \frac{1}{2}) \subset U$. Hence U is a neighborhood of 0 in $\bar{\tau}$.

The last statement of the theorem is obvious.

COROLLARY 1. *If* (G, τ) *is a totally bounded topological group, then* $\mathcal{A}(G, \tau) = \mathcal{U}(G, \tau)$. *This follows immediately from* (3) *and* (5).

4. **The reals.** In the case when G is the reals one can say a few more things. We will be interested in the two cases when τ is the discrete topology D or τ is the usual topology u.

In the case of the discrete topology $(\mathbf{R}, D)$, we have $\mathcal{U}(\mathbf{R}, \bar{D}) = \mathcal{A}(\mathbf{R}, D) \equiv$ {all von Neumann almost periodic functions}. This set contains nonmeasurable functions so that some sets in $\bar{D}$ are nonmeasurable $\Delta - m$ sets.

In the case $(\mathbf{R}, u)$ we have the following properties of $(\mathbf{R}, \bar{u})$. $(\mathbf{R}, \bar{u})$ is

(1) A Hausdorff totally bounded topological group.

(2) a Lindelöf completely regular topological group, hence normal,

(3) not compact, not locally compact, nor even pseudocompact.

The discussion of the previous section and Theorem 6 immediately give property (1). It is completely regular by a general theorem on topological groups, Lindelöf since $\bar{u} \subset u$ and hence normal. Since a Lindelöf topological group is pseudocompact if and only if it is compact and a totally bounded locally compact group is compact, the proof of (3) is completed by showing that $(\mathbf{R}, \bar{u})$ is not compact. We exhibit an almost periodic function, i.e., a uniformly continuous function on $(\mathbf{R}, \bar{u})$, whose range is not compact. Let $f(x) = \sin x + \sin \sqrt{2}x$, then $-2 < f(x) < 2$. But $\sup f = 2$. In fact, if $\epsilon > 0$ is given, there exists a $\delta > 0$ such that $|x - (4k + 1)\pi/2| < \delta$ implies that $|\sin x - 1| < \epsilon/2$. By a well-known approximation theorem there exists an integer n such that, for some integral k and m, $|2n/\pi - (4k + 1)| < \delta\pi/2$ and $|2n\sqrt{2}/\pi - (4m + 1)| < \delta\pi/2$. Then $|n - (4k + 1)\pi/2| < \delta$ and $|n\sqrt{2} - (4k + 1)\pi/2| < \delta$ so that $|\sin n - 1| < \epsilon/2$ and $|\sin \sqrt{2}n - 1| < \epsilon/2$. Thus $|f(n) - 2| < \epsilon$.

Beside $\bar{D}$ and $\bar{u}$, what other totally bounded topologies for $\mathbf{R}$ are there? To look at this question, suppose $(\mathbf{R}, \tau)$ is a totally bounded group. Then $\mathcal{U}(\mathbf{R}, \tau)$ is an algebra of functions that are almost periodic. Now one goes back to the construction of $\bar{\tau}$ to see that one does not need $\mathcal{A}(\mathbf{R}, \tau)$ to do this. What one needs is an algebra of functions in $\mathcal{A}(\mathbf{R}, D)$. In fact, if one starts with $\mathcal{U}(\mathbf{R}, \tau)$, considered as in $\mathcal{A}(\mathbf{R}, D)$, and constructs the topology $\bar{\tau}$ by using the translation functions from this algebra, one gets a totally bounded topology $(\mathbf{R}, \bar{\tau})$ such that $\mathcal{U}(\mathbf{R}, \tau) \subset \mathcal{U}(\mathbf{R}, \bar{\tau})$. But $\bar{\tau} \subset \tau$ implies the reverse. So $\mathcal{U}(\mathbf{R}, \tau) = \mathcal{U}(\mathbf{R}, \bar{\tau})$. This implies $\tau = \bar{\tau}$. We have thus shown that

THEOREM 7. *If* $(\mathbf{R}, \tau)$ *is a totally bounded group, then there exists an algebra* A *of almost periodic functions such that* τ *is generated by* $\{N(v, \epsilon) : v$ *is a translation function of a function in* $A\}$.

This result is particularly interesting in view of a result of Halmos [5]. He shows that there exists a topology c such that $(\mathbf{R}, c)$ is a compact Hausdorff topological group. According to Theorem 7, this topology could be constructed in the above way. In particular, if the algebra $A \subset \mathcal{A}(\mathbf{R}, u)$, then $c \subset u$. However, this is impossible, as can be easily seen by a simple Baire category argument (pointed out to the second author by Larry Baggett). For suppose that $c \subset u$ and that $(\mathbf{R}, c)$ is a compact Hausdorff topological group. Then each closed interval $[a, b]$ in $\mathbf{R}$ is a compact set in $(\mathbf{R}, c)$, and since $(\mathbf{R}, c) = \bigcup_{n=-\infty}^{+\infty} [n, n+1]$ and $(\mathbf{R}, c)$ is a Baire space, there exists an integer m such that $[m, m+1]$ contains a nonempty open subset $U \in c$. Choose $t_0 \in U$ and define $V = U - t_0$. Then V is an open neighborhood of zero and $V \subset [-t_0 + m, -t_0 + m + 1]$. Since $(\mathbf{R}, c)$ is compact it is also totally bounded and so there exist real numbers $t_1, \cdots, t_n$ such that

$$\mathbf{R} = \bigcup_{j=1}^{n} (V + t_j) \subset \bigcup_{j=1}^{n} [t_j - t_0 + m, t_j - t_0 + m + 1]$$

which is clearly a contradiction.

If $\tau \subset u$, then $(\mathbf{R}, \bar{\tau})$ has another interesting property. The Arzela-Ascoli Theorem is true. If $F \subset \mathcal{U}(\mathbf{R}, \bar{\tau})$ is a family which is uniformly bounded and uniformly-equicontinuous in $(\mathbf{R}, \bar{\tau})$, then F is a family in $\mathcal{A}(\mathbf{R}, u)$ that is homogeneous in Bochner's [6] terminology. This means that for each $\epsilon > 0$ and $T(F)$ the translation functions of the family F, $\bigcap_{v \in T(F)} \{x \,|v(x) < \epsilon\}$ is a $\Delta - m$ set containing an interval about 0. This implies that $\bar{F}$ is compact in the uniform norm, see [6] or [8].

It is not necessary that $(\mathbf{R}, \tau)$ be totally bounded for the Arzela-Ascoli Theorem to hold in $\mathcal{A}(\mathbf{R}, \tau)$. In particular, if Γ is a semigroup of nonnegative real numbers with no finite limit point, and $0 \in \Gamma$ then the set of almost periodic functions with exponents in Γ and continuous in $(\mathbf{R}, u)$ is an algebra A containing the constants and separating points. The Arzela-Ascoli Theorem holds in A, see Fink [7] or [8].

5. **Concluding remarks.** For some purposes $(G, \bar{\tau})$ may be a substitute for the Bohr compactification of a group. If one is concerned with arguments about uniformly continuous functions, then $(G, \bar{\tau})$ has essentially all the structure that the Bohr compactification has, and one does not need to embed G into a larger group.

The generation of totally bounded group topologies by functions can also be done with almost automorphic functions. In fact Veech [**10**, Lemma 2.12] has proved the hard part, that the $\Delta - m$ sets arising from ϵ-translation sets of almost automorphic functions satisfy condition (4) of the axioms for a neighborhood base.

References

1. A. Markov, *Stabilität im Liapounoffschen Sinne und Fastperiodizität,* Math. Z. **36** (1933), 708–738.

2. H. J. Ryser, *Combinatorial mathematics,* Carus Monograph Series, no. 14, Wiley, New York, 1963. MR **27** #51.

3. A. S. Besicovitch, *Almost periodic functions,* Dover, New York, 1955. MR **16**, 817.

4. André Weil, *Sur les espaces à structure uniforme et sur la topologie générale,* Hermann, Paris, 1937.

5. P. R. Halmos, *Comment on the real line,* Bull. Amer. Math. Soc. **50** (1944), 877–878. MR **6**, 145.

6. S. Bochner, *Beiträge zur Theorie der fastperiodischen Funktionen.* I. *Funktionen einer Variablen,* Math. Ann. **96** (1926), 119–147.

7. A. M. Fink, *Compact families of almost periodic functions,* Bull. Amer. Math. Soc. **75** (1969), 770–771. MR **40** #640.

8. ———, *Compact families of almost periodic functions and an application of the Schauder fixed point theorem,* SIAM J. Appl. Math. **17** (1969), 1258–1262. MR **41** #2299.

9. A. M. Turing, *Equivalence of left and right almost periodicity,* J. London Math. Soc. **10** (1935), 284–285.

10. W. A. Veech, *Almost automorphic functions on groups,* Amer. J. Math. **87** (1965), 719–751. MR **32** #4469.

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