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ON THE HURWITZ ZETA-FUNCTION

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1. **Introduction.** Briggs and Chowla [2] and Hardy [3] calculated the coefficients of the Laurent expansion of the Riemann zeta-function $\zeta(s)$ about $s = 1$. Kluyver [4] found a certain infinite series representation for these coefficients. In another paper [1] Briggs found estimates for the coefficients. These estimates were improved by Lammel [6]. Using these estimates, Lammel also gave a simple proof of the fact that $\zeta(s)$ has no zeros on $|s - 1| \leqq 1$.

Using the same technique as in [2] and [6], we derive expressions for the coefficients of the Laurent expansion of the generalized or Hurwitz zeta-function $\zeta(s, a)$, $0 < a \leqq 1$, about $s = 1$. A similar formula for these coefficients has been given by Wilton [11]. We then obtain estimates for these coefficients. Our technique here is somewhat simpler than in [6], and as a special case we obtain improved estimates for the Laurent coefficients of $\zeta(s)$. Next, we use our estimates to show that $\zeta(s, a) - a^{-s}$ has no zeros on $|s - 1| \leqq 1$. We conclude by indicating a new, simple proof of a representation formula for $\zeta(s, a)$ that was first discovered by Hurwitz.

2. **Calculation of the Laurent coefficients.** In the sequel we shall need a slightly different version of the Euler-Maclaurin summation formula from what is usually given. Let $f \in C^n$ on $[\alpha, m]$, where m is an integer. Then,

$$(2.1)\qquad \begin{aligned}\sum_{\alpha<k\leqq m} f(k) = \int_\alpha^m f(x)dx + \sum_{k=1}^{n} (-1)^k \frac{B_k}{k!} f^{(k-1)}(m) \\ + \sum_{k=1}^{n} (-1)^{k+1} P_k(\alpha) f^{(k-1)}(\alpha) + R_n,\end{aligned}$$

where

$$R_n = (-1)^{n+1} \int_\alpha^m P_n(x) f^{(n)}(x)dx.$$

Here, B_k, $1 \leqq k \leqq n$, denotes the kth Bernoulli number, and $P_k(x)$,

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$1 \leqq k \leqq n$, has period 1 and equals $k!B_k(x)$ on $[0, 1]$, where $B_k(x)$ denotes the kth Bernoulli polynomial. The proof of (2.1) follows along the same lines as the proof of a somewhat less general version in [**5**, pp. 520-524].

It is well known that $\zeta(s, a)$ is a meromorphic function on the entire complex plane and that its only pole is a simple pole at $s = 1$ with residue 1. Thus, if we set

$$(s - 1)\zeta(s, a) = 1 + \sum_{n=0}^{\infty} \gamma_n(a)(s - 1)^{n+1},$$

our aim is to prove

THEOREM 1. *For* $0 < a \leqq 1$ *and* $n = 0, 1, 2, \cdots$,

$$\text{(2.2)} \quad \gamma_n(a) = \gamma_n = \frac{(-1)^n}{n!} \lim_{m \to \infty} \left(\sum_{k=0}^{m} \frac{\log^n(k + a)}{k + a} - \frac{\log^{n+1}(m + a)}{n + 1} \right).$$

If $a = 1$, (2.2) *gives the coefficients in the Laurent expansion of* $\zeta(s)$ *about* $s = 1$.

PROOF. Let $f(x) = (x + a)^{-s}$ and $n = 1$ in (2.1). Then,

$$\sum_{n=0}^{m} (n + a)^{-s} = \tfrac{1}{2}(m + a)^{-s} + \tfrac{1}{2} a^{-s} + \int_0^m (x + a)^{-s} dx$$

$$- s \int_0^m (x - [x] - \tfrac{1}{2})(x + a)^{-s-1} dx.$$

If $\sigma = \operatorname{Re} s > 1$, we obtain upon letting m tend to ∞,

$$\text{(2.3)} \qquad \zeta(s, a) = a^{-s} + \frac{a^{1-s}}{s - 1} - s \int_0^{\infty} (x - [x])(x + a)^{-s-1} dx.$$

The proof now parallels that of Lammel [**6**] for the case $a = 1$, and consequently we omit the remainder of the details.

3. **An upper bound for the Laurent coefficients.** Instead of estimating γ_n, we, in fact, will estimate

$$c_n(a) = c_n = \gamma_n - \frac{(-1)^n \log^n a}{a n!}.$$

For the case $a = 1$ our estimates are better than Lammel's by a factor of $1/n$. We shall now prove

THEOREM 2. *For* $0 < a \leqq 1$ *and* $n \geqq 1$,

$$|c_n| \leqq 4/n\pi^n, \qquad n \text{ even},$$
$$\leqq 2/n\pi^n, \qquad n \text{ odd}.$$

PROOF. Let $\alpha = 1 - a$ and $f(x) = \{\log^n(x+a)\}/(x+a)n!$. Note that $f^{(k)}(1-a) = 0$, $k = 0, \cdots, n-1$, and $\lim_{m\to\infty} f^{(k)}(m) = 0$, $k \geqq 0$. We then obtain from (2.1) upon letting m tend to ∞,

$$(-1)^n c_n = \frac{1}{n!} \lim_{m\to\infty} \left(\sum_{1-a<k\leqq m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} \right) = R_n, \tag{3.1}$$

where

$$R_n = (-1)^{n+1} \int_1^\infty P_n(x-a) f^{(n)}(x-a)\,dx.$$

We must estimate $f^{(n)}(x-a)$.

To that end, put $F_l(x) = \log^l x$. An elementary calculation shows that for $1 \leqq k < l$,

$$F_l^{(k)}(x) = x^{-k} \sum_{j=1}^{k} l \cdots (l+1-j) a_j^{(k)} \log^{l-j} x, \tag{3.2}$$

where the constants $a_j^{(k)}$ satisfy the recursion formulae,

$$\begin{aligned} a_1^{(k)} &= -(k-1)a_1^{(k-1)}, \\ a_j^{(k)} &= -(k-1)a_j^{(k-1)} + a_{j-1}^{(k-1)}, \qquad j = 2, \cdots, k-1, \\ a_k^{(k)} &= 1. \end{aligned} \tag{3.3}$$

We will show by induction on k that

$$|a_j^{(k)}| = \sum_{1\leqq i_1<i_2<\cdots<i_{k-j}\leqq k-1} i_1 i_2 \cdots i_{k-j}, \qquad 1 \leqq j \leqq k, \tag{3.4}$$

where the sum is over all possible choices of the integers $i_1, \cdots, i_{k-j}$ satisfying the given conditions. If $j = k$, the sum is to be interpreted as equaling 1. For $k = 1$, (3.4) is trivial. Now, from (3.3) for $j = 2, \cdots, k-1$,

$$\begin{aligned} |a_j^{(k)}| &= (k-1)|a_j^{(k-1)}| + |a_{j-1}^{(k-1)}| \\ &= (k-1) \sum_{1\leqq i_1<\cdots<i_{k-1-j}\leqq k-2} i_1 \cdots i_{k-1-j} \\ &\quad + \sum_{1\leqq i_1<\cdots<i_{k-j}\leqq k-2} i_1 \cdots i_{k-j} \\ &= \sum_{1\leqq i_1<\cdots<i_{k-j}\leqq k-1} i_1 \cdots i_{k-j}. \end{aligned}$$

For $j = 1$ and $j = k$, (3.4) is trivially established, and thus the proof of (3.4) is complete.

It follows from (3.4) that for $1 \leqq j \leqq k - 1$,

$$|a_j^{(k)}| \leqq \binom{k-1}{k-j} (k-1)(k-2)\cdots j. \tag{3.5}$$

Since $f(x-a) = F'_{n+1}(x)/(n+1)!$, we find from (3.2) and (3.5) that for $x \geqq 1$,

$$|f^{(n)}(x-a)|$$

$$\leqq \frac{1}{(n+1)!} x^{-n-1} \sum_{j=1}^{n+1} \frac{(n+1)!}{(n+1-j)!} \binom{n}{n+1-j} \frac{n!}{(j-1)!} \log^{n+1-j} x \tag{3.6}$$

$$= x^{-n-1} \sum_{j=1}^{n+1} \binom{n}{n+1-j}^2 \log^{n+1-j} x.$$

Now, for $n \geqq 1$,

$$|P_n(x)| \leqq 4/(2\pi)^n, \qquad n \text{ even},$$
$$\leqq 2/(2\pi)^n, \qquad n \text{ odd}. \tag{3.7}$$

The estimate $4/(2\pi)^n$, $n \geqq 1$, is given in [5, p. 525]. Ostrowski [8] has observed that for n odd, the 4 can be replaced by 2. That this cannot be done for n even can be seen from a theorem of Lehmer [7, p. 534]. Hence, from (3.6) and (3.7) for n even,

$$|R_n| \leqq \frac{4}{(2\pi)^n} \sum_{j=0}^{n} \binom{n}{j}^2 \int_1^\infty \frac{\log^j x}{x^{n+1}}\, dx$$

$$= \frac{4}{(2\pi)^n} \sum_{j=0}^{n} \binom{n}{j}^2 \frac{j!}{n^{j+1}}$$

$$\leqq \frac{4}{(2\pi)^n n} \sum_{j=0}^{n} \binom{n}{j} = \frac{4}{n\pi^n}.$$

For n odd, the 4 may be replaced by 2. Thus, the proof is complete by (3.1).

4. **Zeros in a neighborhood of** $s = 1$. We shall prove

THEOREM 3. *$\zeta(s,a) - a^{-s}$ has no zeros on $|s-1| \leqq 1$, where now we can take $0 \leqq a \leqq 1$.*

Note that if $a = 0$, $\zeta(s,a) - a^{-s} = \zeta(s)$.

PROOF. From the definition of c_n, we have for $|s-1| \leqq 1$,

$$
(4.1)\qquad
\begin{aligned}
|(s-1)(\zeta(s,a)-a^{-s})| &= \left|1+\sum_{n=0}^{\infty} c_n(s-1)^{n+1}\right| \\
&\geqq 1-\sum_{n=0}^{\infty} |c_n|.
\end{aligned}
$$

Thus, we shall be done if we can show that the right-hand side of (4.1) is positive. We shall need to obtain precise estimates for c_0, c_1 and c_2.

To estimate c_0 we put $f(x) = 1/(x+a)$, $\alpha = 1-a$ and $n = 3$ in (2.1). Upon letting m tend to ∞, we obtain

$$
\begin{aligned}
c_0 &= \lim_{m->\infty}\left(\sum_{k=1}^{m} \frac{1}{k+a} - \log(m+a)\right) \\
&= P_1(1-a) + P_2(1-a) + 2P_3(1-a) + R_3.
\end{aligned}
$$

Now, on $[0,1]$, $|P_1(x)| = |x-\frac{1}{2}| \leqq \frac{1}{2}$ and $|P_2(x)| = |\frac{1}{2}x^2 - \frac{1}{2}x + 1/12| \leqq 1/12$. Thus, using (3.7) we obtain

$$
(4.2)\qquad |c_0| \leqq \tfrac{1}{2} + 1/12 + 4/(2\pi)^3 + 4/(2\pi)^3 \leqq .617.
$$

To estimate c_1 we put $f(x) = \{\log(x+a)\}/(x+a)$, $\alpha = 1-a$ and $n = 3$ in (2.1). Upon letting m tend to ∞, we obtain

$$
c_1 = P_2(1-a) + 3P_3(1-a) - \int_1^{\infty} P_3(x-a)x^{-4}(11 - 6\log x)dx.
$$

Thus, from (3.7) and the above estimate of P_2,

$$
(4.3)\qquad |c_1| \leqq 1/12 + 6/(2\pi)^3 + 2/(2\pi)^3 \cdot 13/3 \leqq .144.
$$

To estimate c_2 we put $f(x) = \{\log^2(x+a)\}/2(x+a)$, $\alpha = 1-a$ and $n = 3$ in (2.1). Upon letting m tend to ∞, we obtain

$$
c_2 = P_3(1-a) + \int_1^{\infty} P_3(x-a)x^{-4}(-6 + 11\log x - 3\log^2 x)dx.
$$

From (3.7),

$$
(4.4)\qquad |c_2| \leqq 2/(2\pi)^3 + 2/(2\pi)^3 \cdot 31/9 \leqq .036.
$$

Now, by Theorem 2,

$$
(4.5)\qquad \sum_{n=3}^{\infty} |c_n| \leqq 4\sum_{n=3}^{\infty} 1/n\pi^n \leqq 4/3\pi^2(\pi-1) \leqq .064.
$$

From (4.2)-(4.5),

$$1 - \sum_{n=0}^{\infty} |c_n| \geqq 1 - .617 - .144 - .036 - .064 = 1 - .861 > 0,$$

and the proof of Theorem 3 is complete.

5. **A formula of Hurwitz.** The following representation was proved by Hurwitz.

THEOREM 4. *For* $0 < a \leqq 1$ *and* $\sigma < 0$,

$$\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ \sin\left(\tfrac{1}{2}\pi s\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi n a)}{n^{1-s}} + \cos\left(\tfrac{1}{2}\pi s\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi n a)}{n^{1-s}} \right\}. \tag{5.1}$$

The proofs of (5.1) given in [**9**, p. 37] and [**10**, p. 268-269] depend upon the evaluation of a loop integral. The following simple proof does not appear to have been previously noticed.

PROOF. Again, from (2.1) for $\sigma > 1$,

$$\zeta(s, a) - a^{-s} = \frac{1}{s-1} + \left(1 - a - [1-a] - \tfrac{1}{2}\right) + s \int_{1-a}^{\infty} \frac{[x] - x + \frac{1}{2}}{(x+a)^{s+1}} dx. \tag{5.2}$$

By analytic continuation (5.2) is valid for $\sigma > -1$. Now, if $\sigma < 0$,

$$s \int_{-a}^{1-a} \frac{[x] - x + \frac{1}{2}}{(x+a)^{s+1}} dx = \tfrac{1}{2} + a^{-s} - a - \frac{1}{1-s}.$$

Thus, for $-1 < \sigma < 0$,

$$\zeta(s, a) = s \int_{-a}^{\infty} \frac{[x] - x + \frac{1}{2}}{(x+a)^{s+1}} dx. \tag{5.3}$$

The proof now proceeds as in [**9**, p. 15]. We merely use (5.3) instead of (2.1.6) in [**9**].

REFERENCES

1. William E. Briggs, *Some constants associated with the Riemann zeta-function*, Michigan Math. J. 3 (1955/56), 117-121. MR **17**, 955.

2. W. E. Briggs and S. Chowla, *The power series coefficients of* $\zeta(s)$, Amer. Math. Monthly **62** (1955), 323-325. MR **16**, 999.

3. G. H. Hardy, *Note on Dr. Vacca's series for* γ, Quart. J. Pure Appl. Math. **43** (1912), 215–216.

4. J. C. Kluyver, *On certain series of Mr. Hardy*, Quart. J. Pure Appl. Math. **50** (1927), 185–192.

5. Konrad Knopp, *Theory and application of infinite series*, Blackie and Sons, London, 1951.

6. Ernst Lammel, *Ein Beweis, dass die Riemannsche Zetafunktion* $\zeta(s)$ *in* $|s-1| \leqq 1$ *keine Nullstelle besitzt*, Univ. Nac. Tucumán Rev. Ser. A. **16** (1966), 209–217. MR **36** #5090.

7. D. H. Lehmer, *On the maxima and minima of Bernoulli polynomials*, Amer. Math. Monthly **47** (1940), 533–538. MR **2**, 43.

8. Alexander Ostrowski, *Note on Poisson's treatment of the Euler-Maclaurin formula*, Comment. Math. Helv. **44** (1969), 202–206. MR **41** #2092.

9. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1951. MR **13**, 741.

10. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge Univ. Press. Cambridge, 1962. MR **31** #2375.

11. J. R. Wilton, *A note on the coefficients in the expansion of* $\zeta(s, x)$ *in powers of* $s - 1$, Quart. J. Pure Appl. Math. **50** (1927), 329–332.

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