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BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS[1]

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1. Consider the nth order differential equation

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)}) \qquad (n \geqq 2) \tag{1.1}$$

where $f(t, x_0, x_1, \cdots, x_{n-1})$ and $f_i(t, x_0, \cdots, x_{n-1})$ are continuous on $I \times R^n$ where I is an open subinterval of R and where

$$f_i(t, x_0, \cdots, x_{n-1}) = \frac{\partial f}{\partial x_i}(t, x_0, \cdots, x_{n-1}), \quad i = 0, 1, \cdots, n-1.$$

After some preliminaries in §1, we devote §2 to establishing necessary and (or) sufficient conditions in order that there exist at most one solution of (1.1) satisfying the boundary conditions

$$y^{(i)}(c) = \alpha_i, \qquad y^{(k)}(d) = \beta, \tag{1.2}$$

or

$$y^{(k)}(c) = \alpha, \qquad y^{(i)}(d) = \beta_i, \tag{1.3}$$

when $c, d \in I$, $c < d$, $i = 0, 1, 2, \cdots, n-2$, $0 \leqq k \leqq n-1$ is fixed, and the α_i and β_i are constants. We shall refer to (1.2) as an $(n; k)$ BVP for equation (1.1) and (1.3) as a $(k; n)$ BVP. (We use n rather than $n-1$ to avoid confusion when $k = n-1$.) Our technique, similar to that used in [1] for $n = 2$, involves studying the behaviour of solutions of the variational equation

$$z^{(n)} = \sum_{i=0}^{n-1} f_i(t, y(t), \cdots, y^{(n-1)}(t)) z^{(i)} \tag{1.4}$$

where $y(t)$ is a solution of (1.1). For $n = 2$ and 3 it has been shown [2], [3] that if solutions to the n-point BVP,

$$y(t_i) = \alpha_i, \qquad t_i < t_{i+1}, \qquad 1 \leqq i \leqq n-1, \tag{1.5}$$

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are unique, when they exist, then all $(0; n)$, $(n; 0)$, and n point BVP's do, in fact, have unique solutions, thus generalizing what is true in the linear case. The assumption is also made that solutions of IVP's for (1.1) extend throughout I. Moreover, for arbitrary $n \geqq 2$ Hartman [4] has shown that if IVP's for (1.1) have unique solutions on I and if the n-point BVP (1.5) has a unique solution for all $t_i \in I$ and all α_i, then for any set of positive integers $\lambda_1, \cdots, \lambda_k$ with $\lambda_1 + \cdots + \lambda_k = n$ and any $t_i \in I$, $t_i < t_{i+1}$, $1 \leqq i \leqq k$, and any constants α_i^j, $0 \leqq j \leqq \lambda_i - 1$, there is a unique solution of (1.1) satisfying

$$y^{(j)}(t_i) = \alpha_i^j, \qquad 1 \leqq i \leqq k, \quad 0 \leqq j \leqq \lambda_i - 1.$$

In general, however, there is no relation between uniqueness of solutions of $(n; k)$ and $(k; n)$ BVP's (see Remark 4.15 below). Therefore, it seems desirable to relate the uniqueness of solutions of these types of BVP's to the behaviour of solutions of (1.4), since, in many cases, this can be more easily determined. When (1.1) is linear these BVP's arise in determining intervals of nonoscillation (see [**20**], [**21**]). In analogy with (1.2) and (1.3), we will say that the interval I is an interval of $(n; k)$ nonoscillation for equation (1.1) in case no nontrivial solution satisfies the BVP (1.2) in which $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-2} = \beta = 0$, $c, d \in I$. We say that I is an interval of strict $(n; k)$ nonoscillation in case no nontrivial solution of (1.1) satisfies the BVP (1.2) in which $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-2} = \beta = 0$ and in which $y^{(k)}$ has an odd order zero at d. Intervals of $(k; n)$ nonoscillation and strict $(k; n)$ nonoscillation are similarly defined.

In addition to the continuity assumptions on f and f_i, we shall assume that the following condition holds:

(H_1) If $y \in C^{(n)}[c, d]$ is a solution of (1.1) with $|y(t)| \leqq M_1$ on $[c, d]$ and $\sum_{i=1}^{n-1} |y^{(i)}(t_0)| \leqq M_2$ for some $c \leqq t_0 \leqq d$ and $M_1, M_2 > 0$, then there exists $N > 0$ depending only on M_1, M_2 and $[c, d]$ such that $\sum_{i=1}^{n-1} |y^{(i)}(t)| \leqq N$ on $[c, d]$.

In §4 of this paper we show that condition (H_1) holds for a certain class of equations. §3 is devoted to the case $n = 2$ where we obtain necessary and sufficient conditions for uniqueness of solution to certain two point BVP's, extending results of Ullrich [**5**]. In §4 we study the case $n = 3$ where we obtain some generalizations of results in [**6**] and [**7**]. In particular, Theorem 4.9 gives sufficient conditions for the existence of positive solutions to a third order nonlinear equation. The final section deals with the case $n \geqq 4$.

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2. LEMMA 2.1. *Let $y \in C^{(n)}[c, d]$ be a solution of* (1.1) *such that* (1.4) *is* $(n; k)$ *nonoscillatory on* $[c, d]$. *Let* y_λ *denote the solution of* (1.1) *with*

$$(2.1) \qquad \begin{aligned} y_\lambda^{(i)}(c) &= y^{(i)}(c), \qquad 0 \leqq i \leqq n-2, \\ y_\lambda^{(n-1)}(c) &= \lambda. \end{aligned}$$

Then there is a $\delta > 0$ depending on y and $[c, d]$ such that $0 < |y^{(n-1)}(c) - \lambda| < \delta$ implies $y_\lambda \in C^{(n)}[c, d]$ and $y_\lambda^{(k)}(t) - y^{(k)}(t) \neq 0$ on $(c, d]$.

PROOF. If not, we may assume without loss of generality, that there is a sequence of real numbers λ_m monotonically decreasing to $y^{(n-1)}(c)$ and a sequence of points $t_m \to t_0 \in [c, d]$ with $y_m^{(k)}(t_m) = y^{(k)}(t_m)$, where $y_m \equiv y_{\lambda_m}$. Define z_m by

$$z_m(t) = (y_m(t) - y(t))/(\lambda_m - y^{(n-1)}(c)), \qquad c \leqq t \leqq d.$$

Then $z_m^{(i)}(c) = 0 = z_m^{(k)}(t_m)$, $0 \leqq i \leqq n-2$ and $z_m^{(n-1)}(c) = 1$. Moreover,

$$z_m^{(n)} = \sum_{i=0}^{n-1} \{f_i(t, y(t), \cdots, y^{(n-1)}(t)) + p_m^{(i)}(t)\} z^{(i)}$$

where $p_m^{(i)} \to 0$ uniformly on $[c, d]$, for $0 \leqq i \leqq n-1$. It follows [8, Theorem 3.2, p. 14] that a subsequence of the sequence z_m converges uniformly on $[c, d]$ to a solution $z(t)$ of (1.4) satisfying $z^{(i)}(c) = z^{(k)}(t_0) = 0$, $0 \leqq i \leqq n-2$, and $z^{(n-1)}(c) = 1$. Therefore, $t_0 = c$ so $k < n-1$ and applying the Mean Value Theorem $n-1-k$ times gives a sequence $s_m \to c$ with $z_m^{(n-1)}(s_m) = 0$, a contradiction. This proves the lemma.

REMARK 2.2. There is an obvious analogue to Lemma 2.1 for the case when (1.4) is $(k; n)$ nonoscillatory for some solution y of (1.1). In this and in most subsequent cases we will not state the analogous results explicitly.

THEOREM 2.3. *For some $[c, d] \subset R$ let the BVP* (1.2) *have two solutions $y_1(t)$, $y_2(t)$ with $y_1(t) > y_2(t)$ on (c, d). Then for some λ, $y_2^{(n-1)}(c) \leqq \lambda \leqq y_1^{(n-1)}(c)$, the equation* (1.4) *with $y_\lambda(t)$ in place of $y(t)$ is $(n; k)$ oscillatory on $[c, d]$ where y_λ satisfies* (2.1).

PROOF. If the conclusion is false, then for $y_2^{(n-1)}(c) < \lambda < y_1^{(n-1)}(c)$ define t_λ by

$$t_\lambda = \min\{\inf\{t > c : y_\lambda^{(k)}(t) = y_1^{(k)}(t)\}, \ \inf\{t > c : y_\lambda^{(k)}(t) = y_2^{(k)}(t)\}\}.$$

Condition (H_1) implies t_λ exists for all $y_2^{(n-1)}(c) < \lambda < y_1^{(n-1)}(c)$. Define the set A by

$$A = \{\lambda : y_2^{(n-1)}(c) < \lambda < y_1^{(n-1)}(c), y_\lambda^{(k)}(t_\lambda) = y_1^{(k)}(t_\lambda)\}.$$

By Lemma 2.1, $A \neq \emptyset$ so let $\mu = \sup A$. It follows that $\mu < y_1^{(n-1)}(c)$ and $\mu \in A$. But for $\mu < \lambda < y_1^{(n-1)}(c)$ and $\lambda - \mu$ sufficiently small, Lemma 2.1 implies $y_\lambda^{(k)}(t) > y_\mu^{(k)}(t)$ on $(c, t_\mu]$, which contradicts the definition of μ. This proves the theorem.

We now establish a partial converse to Theorem 2.3.

THEOREM 2.4. *Let $y(t)$ be a solution of* (1.1) *on an interval* (a, b) *and assume for each $a < c < d < b$ that if $y_1(t)$ is any solution of* (1.1) *satisfying*

$$y_1^{(i)}(c) = y^{(i)}(c), \qquad 0 \leqq i \leqq n-2,$$
$$y_1^{(k)}(d) = y^{(k)}(d),$$

then $y_1 \equiv y$ on (a, b). Then (a, b) is an interval of strict $(n; k)$ nonoscillation for equation (1.4).

PROOF. If not, assume $z(t)$ is a solution of (1.4) with $z^{(i)}(c) = 0 = z^{(k)}(d)$, $0 \leqq i \leqq n-2$, and let $z^{(n-1)}(c) = 1$. Let $\lambda_m = 1/m + y^{(n-1)}(c)$ in (2.1) and let y_m denote the corresponding solution. Putting $z_m(t) = m(y_m(t) - y(t))$ we have that for sufficiently large m, $z_m^{(i)}(c) = 0$, $0 \leqq i \leqq n-2$, $z_m^{(n-1)}(c) = 1$ so that $z_m \to z$ uniformly on compact subsets of (c, b). Since $z^{(k)}(t) < 0$ at some points of (c, b), we see that $z_m^{(k)}(t_m) = 0$ for large m, where $c < t_m < b$. Therefore, $y_m \equiv y$ on $[a, b]$ by hypothesis. This is a contradiction and proves the theorem.

The two previous theorems imply the following.

COROLLARY 2.5. *Let D be an open simply-connected subset of $(a, b) \times R$ and assume no solution y of* (1.1) *satisfying*

$$(2.2) \qquad (t, y(t)) \in D, \qquad c \leqq t \leqq d,$$

is such that (1.4) *has a solution $z(t)$ with a zero of multiplicity $n-1$ at c and a zero of $z^{(k)}(t)$ of even order at d. Then the following are equivalent:*

(a) *Equation* (1.4) *is $(n; k)$ nonoscillatory for all solutions $y(t)$ of* (1.1) *satisfying* (2.2).

(b) *All $(n; k)$ BVP's for equation* (1.1) *have at most one solution $y(t)$ satisfying* (2.2).

We shall see in §4 that Corollary 2.5 is not true if one drops the assumption regarding the even order zero of $z^{(k)}(t)$.

3. In this section we shall consider the second order differential equation

$$y'' + f(t, y) = 0 \tag{3.1}$$

where f and f_y are continuous on $[0, +\infty) \times R$ and satisfy $f(t, 0) = 0 = f_y(t, 0)$, $f(t, y) > 0$ and $f_y(t, y) > 0$ for $y > 0$ and $0 \leqq t < +\infty$ and where $f_y(t, y)$ is increasing in y for $y > 0$. An integration from t_0 to t shows that condition (H_1) holds for equation (3.1).

We have the following lemma, due to Moore and Nehari [**9**].

LEMMA 3.1. *Let $u(t)$, $v(t)$, and $w(t)$ be three distinct solutions of* (3.1) *satisfying $0 \leqq u(t) < w(t)$, $0 \leqq v(t) < w(t)$ on $[a, b]$, $0 < a < b < +\infty$. Then $u(t)$ and $v(t)$ cannot intersect in $[a, b]$ more than once.*

In addition to the conditions mentioned above, we shall make the following assumption:

(H_2) For any $0 < a < b < +\infty$ there is a solution $y(t)$ of (3.1) with $y(a) = y(b) = 0$ and $y(t) > 0$ on (a, b).

For example, (H_2) holds for the equation

$$y'' + p(t)y^{2n+1} = 0, \qquad p(t) > 0,\ 0 \leqq t < +\infty,\ n \geqq 1, \tag{3.2}$$

and certain of its generalizations, as considered in [**10**] (but clearly does not hold if (3.1) is linear). The first result of this section is a generalization of a result due to Ullrich [**5**, Theorem 5.1] for which we give a different proof. Our method also shows that certain of his assumptions are unnecessary.

THEOREM 3.2. *Let $y(t)$ be a solution of* (3.1) *satisfying*

$$y(a) = A, \qquad y(b) = B, \qquad A, B \geqq 0,\ A + B > 0, \qquad 0 < a < b < +\infty, \tag{3.3}$$

and $y(t) > 0$ on (a, b). Then $y(t)$ is the unique positive solution of (3.1) *satisfying* (3.3) *if and only if the equation*

$$z'' + f_y(t, y(t))z = 0 \tag{3.4}$$

has a solution with $z(a) = z(b) = 0$, $z(t) > 0$ on (a, b).

PROOF. Assume first that equation (3.4) has a solution $z(t)$ with $z(a) = z(b) = 0$ and $z(t) > 0$ on (a, b). Let $y_1(t)$ be a second solution of (3.1) satsifying $y_1(a) = y(a)$, $y_1(b) = y(b)$, and $y_1(t) > 0$ on (a, b). If $y_1(t) < y(t)$ on an interval (t_0, t_1) with $y_1(t_0) = y(t_0)$, $y_1(t_1) = y(t_1)$, then by Theorem 2.3 for some $y_\lambda(t)$, equation (3.4) with $y_\lambda(t)$ in place of $y(t)$ has a solution with two zeros on $[t_0, t_\lambda]$, where $y_1(t) \leqq y_\lambda(t) \leqq y(t)$

on $[t_0, t_\lambda]$. By the Sturm comparison theorem and our assumption above, we must have $y_\lambda \equiv y$ and $t_0 = a$, $t_\lambda = b$. Hence, equation (3.4) with $y_1(t)$ in place of $y(t)$ is disconjugate on $[a, b]$ so that by Lemma 2.1, for $\lambda - y_1'(a) > 0$ and sufficiently small, $y_\lambda(t) - y_1(t) > 0$ on $(a, b]$ and $y_\lambda(t) < y(t)$ on (a, t_0) with $y_\lambda(t_0) = y(t_0)$, where $a < t_0 < b$. Therefore, arguing as above, we conclude that equation (3.4) has a solution with two zeros on $[a, t_0]$, a contradiction. Therefore, we must have $y_1(t) > y(t)$ on (a, b). Setting $u(t) = y_1(t) - y(t)$ we have, as in [7],

$$u'' + p(t)u = 0, \qquad u(a) = u(b) = 0,$$

where $p(t) = \{f(t, y_1(t)) - f(t, y(t))\}/(y_1(t) - y(t))$. But $p(t) > f_y(t, y(t))$ on (a, b) so the Sturm theorem shows that equation (3.4) is disconjugate on $[a, b]$, a contradiction.

Conversely, assume $y(t)$ is the unique positive solution satisfying (3.3) and, to be specific, let $y(b) = B > 0$. Then applying Theorems 3.2 and 3.3 of [11] (with $\alpha(t) \equiv 0 \leqq y(t) \equiv \beta(t)$) we conclude that equation (3.4) is disconjugate on (a, b). Hence, if the theorem is not true, then equation (3.4) is disconjugate on $[a, b]$ so by Lemma 2.1 there is a $\delta > 0$ such that $0 < \lambda - y'(a) \leqq \delta$ implies $y_\lambda(t) > y(t)$ on $(a, b]$.

We consider first the case $y(a) = A = 0$. By assumption (H_2), for some $\mu \neq y'(a)$ we have $y_\mu(b) = 0$. Therefore, either $0 < \mu < y'(a)$ or $\mu > \delta + y'(a)$. Since (3.4) is disconjugate on $[a, b]$ and since $f_y(t, y)$ is increasing in y for $y > 0$, it follows by Theorem 2.3, the Sturm comparison theorem, and Lemma 3.1, that $0 < y_\lambda(t) < y(t)$ on $(a, b]$ for all $0 < \lambda < y'(a)$. Hence, $\mu > \delta + y'(a)$. But since the set $\{y_\lambda(b) : \delta + y'(a) < \lambda < \mu\}$ is connected, it follows by continuity with respect to initial conditions that $y_{\lambda_0}(b) = y(b)$ for some $\lambda_0 > \delta + y'(a)$, and this is a contradiction.

Now if $y(a) = A > 0$, let u be the solution of (3.1) satisfying $u(a) = u(b) = 0$, $u(t) > 0$ on (a, b). For any $\lambda > y'(a)$, Lemma 3.1 shows that $y_\lambda(t) \leqq u(t)$ on an interval $(t_\lambda, \tau_\lambda) \subset (a, b)$ with $u(t_\lambda) - y_\lambda(t_\lambda) = u(\tau_\lambda) - y_\lambda(\tau_\lambda) = 0$. Letting $\lambda_n \to +\infty$ and applying the Mean Value Theorem to $u(t) - y_{\lambda_n}(t)$ we obtain a sequence of points $\{\eta_n\}$ with $|y_{\lambda_n}'(\eta_n)| \leqq \max_{a \leqq t \leqq b} |u'(t)|$. Taking subsequences, if necessary, we may assume that $\eta_n \to \eta_0$, $y_{\lambda_n}(\eta_n) \to k_0$, and $y_{\lambda_n}'(\eta_n) \to k_0'$. It follows by continuity with respect to initial conditions that $\{y_{\lambda_n}\}$ converges uniformly on $[a, b]$ to the solution $u_0(t)$ of (3.1) satisfying $u_0(\eta_0) = k_0$, $u_0'(\eta_0) = k_0'$. But then $u_0'(t) \to +\infty$ as $t \to a+$, which contradicts (H_1). This proves the theorem.

By applying Lemma 2.1, Theorem 2.3, and the Sturm comparison theorem, one can easily show that if y is a solution of (3.1) with $y(a) = y(b) = 0$ and $y > 0$ on (a, b), then (3.4) has a solution z with $z(a) = z(t_0) = 0, z > 0$ on (a, t_0), where $a < t_0 < b$.

Our next theorem is the analogue of Theorem 3.2 for the second type of BVP for equation (3.1). We first prove a preliminary lemma.

LEMMA 3.3. *Assume that there exists a unique positive solution of* (3.1) *satisfying*

$$\text{(3.5)}\qquad y(a) = A, \qquad y'(b) = B, \qquad y' > 0 \quad \text{on } (a, b), \qquad A, B \geqq 0, A + B > 0.$$

Then there is a $\delta > 0$ *such that for* $B - \delta < \alpha < B$ *there is a* $\lambda = \lambda(\alpha)$ *with* $y_\lambda(a) = y(a)$, $0 < y_\lambda'(a) = \lambda < y'(a)$, $y_\lambda'(b) = \alpha$ *and* $0 < y_\lambda(t) < y(t)$ *on* (a, b).

PROOF. Assume first that $y(a) = A = 0$. Let $y'(b) = B = \delta > 0$. Now since $y_\lambda \to 0$ and $y_\lambda' \to 0$ uniformly on $[a, b]$ as $\lambda \to 0$ it follows by continuity with respect to initial conditions that for any $0 < \alpha < \delta$ there is $\lambda_0 = \lambda_0(\alpha)$ with

$$\text{(3.6)}\quad y_{\lambda_0}(a) = y(a) = 0, \qquad y_{\lambda_0}'(b) = \alpha, \quad \text{and } 0 < \lambda_0 = y_{\lambda_0}'(a) < y'(a).$$

We claim that $y_{\lambda_0}(t) < y(t)$ on $(a, b]$. For suppose $y_{\lambda_0}(t) < y(t)$ on $(a, c) \subset (a, b)$ with $y_{\lambda_0}(c) = y(c)$ for some $a < c \leqq b$. It follows, by continuity with respect to initial conditions that for some $0 < \lambda_1 \leqq \lambda_0$ we have $y_{\lambda_1}(t) < y(t)$ on (a, b) and $y_{\lambda_1}(b) = y(b)$. Hence, $y_{\lambda_1}'(b) > y'(b)$. But then by a connectedness argument and continuity we must have $y_\mu'(b) = y'(b)$ for some $0 < \mu < \lambda_1$, and this contradicts the uniqueness hypothesis.

We claim next that $y_{\lambda_0}(t) > 0$ on $(a, b]$. For if $y_{\lambda_0}(c) = 0$ for some $a < c \leqq b$, then $y_{\lambda_0}(t) > y(t)$ at some points of (a, c). Otherwise $0 < y_{\lambda_0}(t) < y(t)$ on (a, c) and this contradicts Lemma 3.1. (Note that one may allow u, v, w to agree at one of the endpoints in Lemma 3.1 as is evident from the proof [9].) But we have just shown in the first part of the proof that $y_{\lambda_0}(t) < y(t)$ on $(a, b]$. Hence, we must have $0 < y_{\lambda_0}(t) < y(t)$ on $(a, b]$.

To handle the case $y(a) = A > 0$, $y'(b) = B \geqq 0$, we let $u(t)$ be the solution of (3.1) with $u(a) = u(b) = A$ and $0 < u(t) < y(t)$ on $(a, b]$. (See Theorem 3.1 of [11].) In this case, $u'(b) < 0$ so we may set $\delta = B - u'(b) > 0$. Then an argument similar to the first part of the proof shows that for any $u'(b) < \alpha < B$ there is a $\lambda_0 = \lambda_0(\alpha)$ such that (3.6) holds and, in addition, $u(t) < y_{\lambda_0}(t) < y(t)$ on $(a, b]$. This proves the lemma.

Theorem 3.4. *Let* $y(t)$ *be a solution of* (3.1) *satisfying* (3.5). *Then* $y(t)$ *is the unique positive solution of* (3.1) *satisfying* (3.5) *if and only if* (3.4) *has a solution with*

$$z(a) = z'(b) = 0, \qquad z'(t) > 0 \quad on\ [a, b). \tag{3.7}$$

Proof. Assume first that $y(t)$ is the unique positive solution of (3.1) satisfying (3.5) and let $z(t)$ be the solution of (3.4) with $z(a) = 0$, $z'(a) = 1$. We first show that $z' > 0$ on $[a, b)$. By Lemma 3.3 for all $n \geqq 1$ sufficiently large we may choose $0 < \lambda_n < y'(a)$ such that the solution y_n of (3.1) with $y_n(a) = y(a)$, $y_n'(a) = y'(a) - \lambda_n$ satisfies $y_n'(b) = y'(b) - 1/n$ and $0 < y_n(t) < y(t)$ on $(a, b]$. If we define $z_n(t) \equiv (y(t) - y_n(t))/\lambda_n$, $a \leqq t \leqq b$, it follows that $z_n(a) = 0$, $z_n'(a) = 1$ and $z_n'(b) > 0$. Since $\{y_n\}$ converges to a solution satisfying (3.5), it follows that $y_n \to y$ uniformly on $[a, b]$ so that $z_n \to z$ uniformly on $[a, b]$. Also, $z_n''(t) < 0$ on (a, b) so that z_n' is decreasing on $[a, b]$ and therefore $z_n' > 0$ on $[a, b]$ for all n. Hence, $z' > 0$ on $[a, b)$.

Now suppose $z'(b) > 0$. By Lemma 2.1 for $\lambda - y'(a) > 0$ and sufficiently small, we have $y_\lambda'(t) > y'(t)$ on $[a, b]$. By the uniqueness assumption and continuity it follows that $y_\lambda'(b) > y'(b) \geqq 0$ for all $\lambda > y'(a)$. Now if $u(t)$ denotes the solution of (3.1) with $u(a) = u(b) = 0$, $u(t) > 0$ on (a, b), then by Lemma 3.1 we have that $u(t) > y_\lambda(t)$ at some points of (a, b) for all $\lambda \geqq y'(a)$. Now arguing as in the last half of Theorem 3.2, we obtain a contradiction. Therefore, we conclude $z'(b) = 0$.

Conversely, let (3.4) have a solution with $z(a) = z'(b) = 0$, $z' > 0$ on $[a, b)$. Suppose that $y_1(t)$ is a second solution of (3.1) satisfying (3.5). We show first that $y(t) - y_1(t) \not\equiv 0$ on (a, b). Suppose that $y(t) > y_1(t)$ on (a, τ) and $y(t) < y_1(t)$ on (τ, η), $a < \tau < \eta \leqq b$. Then by Theorem 2.3 and the Sturm comparison theorem, $z(t_0) = 0$ for some $a < t_0 \leqq \tau$, a contradiction. If $y(t) < y_1(t)$ on (a, τ) and $y(t) > y_1(t)$ on (τ, η), $a < \tau < \eta \leqq b$, then Theorem 2.3 again shows that there is a solution $z_0(t)$ of (3.4) with $z_0(\tau) = z_0'(t_1) = 0$, $z_0' > 0$ on $[\tau, t_1)$, where $\tau < t_1 \leqq b$. This is a contradiction by the Sturm theorem (see [12]). Hence, $y(t) - y_1(t) \not\equiv 0$ on (a, b). If $y_1(t) > y(t)$ on (a, b), then setting $u = y_1 - y$ we see that $u(a) = u'(b) = 0$, $u(t) > 0$ on $[a, b)$, and $u'' + p(t)u = 0$, where

$$p(t) = (f(t, y_1) - f(t, y))/(y_1(t) - y(t)) > f_y(t, y(t)). \tag{3.8}$$

Also, $u'(t) > 0$ on $[a, b)$ for if $u'(\tau) = 0$ for some $a < \tau < b$, then by the Mean Value Theorem we have $u''(\eta) = 0$ for some $\tau < \eta < b$. Therefore, $p(\eta) = 0$ which is a contradiction since $f(t, y)$ is strictly increasing in y for $y > 0$. But then if $u'(t) > 0$ on $[a, b)$ it follows that $z'(t) > 0$ on $[a, b]$, a contradiction.

Finally, if $y_1(t) < y(t)$ on (a, b), then (3.4) with $y_1(t)$ in place of $y(t)$ has a solution $z_1(t)$ with $z_1(a) = 0$ and $z_1'(t) > 0$ on $[a, b]$. Hence, for $\lambda > y_1'(a)$ and $\lambda - y_1'(a)$ sufficiently small, $y_\lambda'(t) - y_1'(t) > 0$ on $[a, b]$ and $y_1(t) < y_\lambda(t) < y(t)$ on (a, b), where y_λ is the solution of (3.1) with $y_\lambda(a) = y(a)$, $y_\lambda'(a) = \lambda$. But then $y_\lambda'(b) - y'(b) > 0 > y_\lambda'(a) - y'(a)$ so that for some $a < t_0 < b$ we have $y(a) - y_\lambda(a) = 0 = y'(t_0) - y_\lambda'(t_0)$ and $y'(t) - y_\lambda'(t) > 0$ on $[a, t_0)$. But now by Theorem 2.3 and the Sturm comparison theorem, we see that $z(a) = z'(t_1) = 0$, where $a < t_1 \leqq t_0$. This contradiction proves the theorem.

Remark 3.5. Whereas it was obvious that one could not allow $A = B = 0$ in Theorem 3.2, it is not obvious that Theorem 3.4 is false if $A = B = 0$. In fact, the second half of the proof of Theorem 3.4 yields:

Corollary 3.6. *Let $y(t)$ be a solution of* (3.1) *with*

$$y(a) = A, \qquad y'(b) = B, \qquad y(t) > 0 \quad \text{on } (a, b). \tag{3.9}$$

Assume equation (3.4) *has a solution with $z(a) = z'(b) = 0$, $z'(t) > 0$ on $[a, b)$. Then $y(t)$ is the unique positive solution satisfying* (3.9).

Whether or not the converse of Corollary 3.6 is valid is undecided. For a certain class of equations, Coffman [13] has given sufficient conditions which guarantee the uniqueness of the positive solutions satisfying $y(a) = y(b) = 0$ and $y(a) = y'(b) = 0$. Obviously, if $B < 0$ in (3.9) we must have $A > 0$.

4. In this section we shall consider the third order nonlinear equation

$$y''' = f(t, y, y', y''). \tag{4.1}$$

We begin with a definition which, when (4.1) is linear, is due to Hanan [6].

Definition 4.1. Equation (4.1) is said to be of class I in case every (0; 3) BVP for (4.1) has at most one solution. Similarly (4.1) is said to be of class II in case every (3; 0) BVP for (4.1) has at most one solution.

Our first result is a slight generalization of a sufficient condition, due to Hanan [6], in order that the equation

$$y''' + py'' + qy' + ry = 0, \qquad p, q, r \in C(0, +\infty), \tag{4.2}$$

belong to class I or II. (Hanan considered (4.2) when $p \equiv 0$.)

THEOREM 4.2. *Let $p \in C^2$, $q \in C'$, and $r \in C$ on $(0, +\infty)$. Then equation (4.2) is of class I (II) in case $p \leqq 0$ and $2r - q' + p'' > 0$ ($p \geqq 0$ and $2r - q' + p'' < 0$) except at isolated points.*

PROOF. Let $y(t)$ be a solution of (4.2) with $y(a) = y(b) = y'(b) = 0$, $a < b$. Multiply (4.2) by $y(t)$ and integrate by parts to get

$$\begin{aligned}(4.3)\qquad &[pyy' - \tfrac{1}{2}(y')^2]_a^b - \int_a^b p(y')^2 dt \\ &\qquad + \int_a^b (q - p')yy'dt + \int_a^b ry^2 dt \equiv 0.\end{aligned}$$

Since $\int_a^b (q - p')yy'dt = -\frac{1}{2}\int_a^b (q' - p'')y^2 dt$, (4.3) gives

$$(4.4)\qquad \tfrac{1}{2}(y'(a))^2 - \int_a^b p(y')^2 dt + \tfrac{1}{2}\int_a^b (2r - q' + p'')y^2 dt \equiv 0$$

which is a contradiction. Similarly, one shows (4.2) is of class II if the inequalities are reversed.

THEOREM 4.3. *Let $p \in C'$, q, $r \in C$ on $(0, +\infty)$. Then (4.2) is of class I (II) in case $r \geqq 0$ ($r \leqq 0$) and the second order differential equation*

$$(4.5)\qquad y'' + (q - p'/2)y = 0$$

is disconjugate on $(0, +\infty)$.

PROOF. Suppose $r \geqq 0$ and let $y(t)$ be a solution of (4.2) satisfying

$$y'(a) = y(b) = y'(b) = 0, \qquad y > 0,\ y' < 0 \ \text{ on } (a, b).$$

Then multiplying (4.2) by $y'(t)$ and integrating gives

$$\begin{aligned}(4.6)\qquad &\left[y''y' + \frac{p}{2}(y')^2\right]_a^b - \int_a^b (y'')^2 \\ &\qquad + \int_a^b \left(q - \frac{p'}{2}\right)(y')^2 + \int_a^b ryy' \equiv 0.\end{aligned}$$

Now if u is a solution of (4.5) with $u(a) = 0$, $u'(a) > 0$ then $u > 0$ on $(a, +\infty)$, so that expanding $\int_a^b (y'' - y'u'/u)^2 dt > 0$ shows that $\int_a^b (y'')^2 dt > \int_a^b (q - p'/2)(y')^2 dt$. (See [**14**, p. 431].) Substituting this into (4.6) we get

$$(4.7)\qquad \int_a^b ryy'dt > 0,$$

which is a contradiction. Likewise one shows (4.2) is of class II if $r \leqq 0$.

We shall now apply these results to the equation

$$y''' + py'' + qy' + f(t, y) = 0 \tag{4.8}$$

where $f(t, y)$ and $f_y(t, y)$ are continuous on $[0, +\infty) \times R$. We first show that (4.8) satisfies condition (H_1). In fact, we have

LEMMA 4.4. *Condition* (H_1) *holds for the equation*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + f(t, y) = 0 \tag{4.9}$$

where $p_i \in C(0, +\infty)$ *and* $f(t, y) \in C(0, \infty) \times R$.

PROOF. Let M_1, M_2, and $[c, d]$ be given as in condition (H_1) and let $K = \max\{|f(t, y)| : c \leqq t \leqq d, |y| \leqq M_1\}$. Then for $t_0 \leqq t \leqq d$ we have

$$|y^{(n-1)}(t)| \leqq k_0 + k_1 \sum_{i=1}^{n-1} \int_{t_0}^{t} |y^{(i)}(s)| ds \tag{4.10}$$

where k_0 and k_1 are positive constants depending on M_2, K, $[c, d]$, and $\max\{|p_i(t)| : c \leqq t \leqq d, 1 \leqq i \leqq n-1\}$. Since $|y^{(i)}(t)| \leqq M_2 + \int_{t_0}^{t} |y^{(i+1)}(s)| ds$, $1 \leqq i \leqq n-2$, we have

$$\sum_{i=1}^{n-1} |y^{(i)}(t)| \leqq k_0' + k_1' \sum_{i=1}^{n-1} \int_{t_0}^{t} |y^{(i)}(s)| ds,$$

where k_0' and k_1' are positive constants. Hence, Gronwall's inequality yields the result. A similar argument holds on $[c, t_0]$.

THEOREM 4.5. *Equation* (4.8) *is of class* I (II) *in case* $p \leqq 0$, $p'' - q' \geqq 0$, *and* $f_y(t, y) > 0$, $y \neq 0$. ($p \geqq 0$, $p'' - q' \leqq 0$, *and* $f_y(t, y) < 0$, $y \neq 0$.)

PROOF. To be specific, assume $f_y(t, y) > 0$, $y \neq 0$, $p \leqq 0$, and $p'' - q' \geqq 0$. If the conclusion is not true, then let y_1 and y_2 be two solutions of the same (0; 3) BVP for (4.8). By Theorem 2.3 there is an interval $[t_\mu, d]$ and a solution y_μ of (4.8) with y_μ between y_1 and y_2 such that

$$z''' + pz'' + qz' + f_y(t, y_\mu(t))z = 0 \tag{4.11}$$

is (0; 3) oscillatory on $[t_\mu, d]$. Therefore, by Theorem 4.2 we must have $f_y(t, y_\mu(t)) \equiv 0$ and hence $y_\mu(t) \equiv 0$ on $[t_\mu, d]$. Thus, we may assume $y_1(c) = y_1(d) = y_1'(d) = 0$ and $y_1 > 0$ on (c, d) and that

$y_2 \equiv 0$. By Lemma 2.1 (since the equation $z''' = 0$ is (0; 3) non-oscillatory), it follows that for $\lambda > 0$ and sufficiently small, we have $0 < y_\lambda(t) < y_1(t)$ on (t_λ, d) and $y_\lambda(t_\lambda) = y_1(t_\lambda)$ where $c < t_\lambda < d$ and where $y_\lambda(t)$ is the solution of (4.8) satisfying $y_\lambda(d) = y_\lambda'(d) = 0$, $y_\lambda''(d) = \lambda$. But now Theorem 2.3 implies that (4.11) is (0; 3) oscillatory on $[t_\mu, d]$ for some $\lambda \leqq \mu \leqq y_1''(d)$ and this contradicts Theorem 4.2 since in this case $y_\mu(t) \not\equiv 0$.

THEOREM 4.6. *Equation* (4.8) *is of class* I (II) *in case* $f_y(t, y) \geqq 0$ $(f_y(t, y) \leqq 0)$ *for all* t, y *and* (4.5) *is disconjugate on* $(0, +\infty)$. *In fact, all* (1; 3) ((3;1)) *BVP's for* (4.8) *have at most one solution.*

PROOF. Apply Theorem 4.3 and Theorem 2.3.

EXAMPLE 4.7. This example shows that the assumption in Corollary 2.5 regarding adjacent double zeros of solutions of (1.4) (when $n = 3$) cannot be dropped in general. Consider the differential equation

$$y''' + 4y' + r(t)y^\gamma = 0, \tag{4.12}$$

where $\gamma > 1$ is the quotient of odd integers and $r(t) > 0$ on $(0, +\infty)$. Theorem 4.5 implies (4.12) is of class I so that all (0; 3) BVP's for (4.12) have at most one solution. However, the variational equation with respect to the zero solution is

$$y''' + 4y' = 0 \tag{4.13}$$

whose solutions are linear combinations of $\sin^2 t$, $\cos^2 t$, and $\sin t \cos t$. That is, (4.13) has solutions with adjacent double zeros on any interval of length greater than π.

The hypotheses of Theorem 4.5 or 4.6 imply the existence of non-oscillatory solutions for the important case when $f(t, 0) = 0$. The proof is somewhat more detailed than that for the linear case (see [**15**], for example). We first prove a lemma.

LEMMA 4.8. *Consider equation* (4.8) *where* $f(t, 0) = 0$, *solutions of IVP's for* (4.8) *extend to all of* $(0, +\infty)$ *and either*

(a) $p \leqq 0$, $p'' - q' \geqq 0$, *and* $f_y(t, y) > 0$, $y \neq 0$, *or*

(b) $f_y(t, y) \geqq 0$ *for all* t, y *and* (4.5) *is disconjugate on* $(0, +\infty)$.

Then for any $0 < a < b < +\infty$ *and any* $k > 0$ *there is a solution* $y(t)$ *of* (4.8) *satisfying the BVP*

$$(y(a))^2 + (y'(a))^2 + (y''(a))^2 = k, \qquad y(b) = y'(b) = 0. \tag{4.14}$$

PROOF. Either (a) or (b) implies that $y_\lambda(t) > 0$ on (a, b) where y_λ is the solution of (4.8) with $y_\lambda(b) = y_\lambda'(b) = 0$, $y_\lambda''(b) = \lambda > 0$. By continuity with respect to initial conditions it is clear that (4.14) can be satisfied for k sufficiently small. Let $k(\lambda) = (y_\lambda(a))^2 + (y_\lambda'(a))^2$

$+ (y_\lambda''(a))^2$. If (a) holds, an integration of (4.8) gives for $\lambda > 0$

$$\lambda + \int_a^b f(t, y_\lambda(t))dt + \int_a^b p(t)y_\lambda''(t)dt + \int_a^b q(t)y_\lambda'(t)dt = y_\lambda''(a). \tag{4.15}$$

Since $\int_a^b p(t)y_\lambda''(t)dt = -y_\lambda'(a)p(a) - p'(a)y_\lambda(a) + \int_a^b p''(t)y_\lambda(t)dt$ and $\int_a^b q(t)y_\lambda'(t)dt = -q(a)y_\lambda(a) - \int_a^b y_\lambda(t)q'(t)dt$, (4.15) implies

$$\lambda \leqq |y_\lambda''(a)| + |y_\lambda'(a)p(a)| + |q(a)y_\lambda(a)| + \int_a^b (q'(t) - p''(t))y_\lambda(t)dt$$

so

$$\lambda \leqq |y''(a)| + |y_\lambda'(a)p(a)| + |q(a)y_\lambda(a)|. \tag{4.16}$$

Since the right-hand side of (4.16) cannot remain bounded for all λ, we see that $k(\lambda) \to +\infty$ as $\lambda \to +\infty$ which proves (4.14).

If (b) holds, then by Theorem (4.6) we have $y_\lambda'(t) < 0$ on (a, b) so that by condition (H_1), there is an $N > 0$, depending only on $[a, b]$ and $k(\lambda)$, such that $|y_\lambda'(t)| + |y_\lambda''(t)| \leqq N$ on $[a, b]$. This shows that $k(\lambda) \to +\infty$ as $\lambda \to +\infty$.

THEOREM 4.9. *Let the hypotheses of Theorem* 4.5 *or* 4.6 *hold, assume* $f(t, 0) = 0$, *and for each* $0 < t_0 < +\infty$ *assume that solutions of IVP's for* (4.8) *extend to all of* $(0, +\infty)$. *Then for any* $0 < \tau < +\infty$ *there is a solution of* (4.8) *with* $y(t) > 0$ *on* $(\tau, +\infty)$.

PROOF. It is clear that we need only consider the case when either (a) or (b) of Lemma 4.8 holds since otherwise (4.8) is of class II and the result is obvious. Therefore, let $t_n \to +\infty$ and let y_n be the solution of (4.8) satisfying

$$\begin{gathered} y_n(t_n) = y_n'(t_n) = 0 < y_n''(t_n), \\ (y_n(\tau))^2 + (y_n'(\tau))^2 + (y_n''(\tau))^2 = 1. \end{gathered} \tag{4.17}$$

Since $y_n(t) > 0$ on (τ, t_n), it follows, by (4.17) and a standard argument, that a subsequence of y_n converges to a solution y_0 of (4.8) with $y_0(t) > 0$ on $(\tau, +\infty)$.

Since the oscillatory or nonoscillatory character of (4.2) is determined in many cases by the signs of the coefficients alone ([**6**], [**16**], [**20**], [**21**]), it is therefore possible to obtain criteria in order that (4.1) be of class I or II on the basis of the signs of the partial derivatives of (4.1). As an example, we prove a result due to L. Jackson (unpublished), which is a generalization of a result due to Kim [**16**].

THEOREM 4.10. *If* $q \leqq 0$ *and* $r \geqq 0$ *on* $(0, +\infty)$ *then no nontrivial solution of* (4.2) *satisfies* $y(a) = y(b) = y'(b) = 0$, $0 < a < b < +\infty$.

PROOF. Let $y(t)$ be a solution of (4.2) with $y(b) = y'(b) = 0$ and $y''(b) > 0$. If $y'(t) = 0$ at some points $t < b$, let $t = \tau$ be the nearest such point to b. Then $y'(\tau) = y'(b) = 0$, $y(t) \geqq 0$ on $[\tau, b]$, and $y'(t) < 0$ on (τ, b). Hence,

$$y''' + py'' + qy' = -ry \leqq 0 \qquad \text{on } [\tau, b].$$

Since $q \leqq 0$, the second order equation

$$z'' + pz' + qz = 0$$

is disconjugate on $[\tau, b]$ [8, p. 362]. But with $\alpha(t) \equiv -y'(t)$ and $\beta(t) \equiv M \equiv \max_{\tau \leq t \leq b} |y'(t)|$, Theorem 3.1 of [**11**] yields a solution $v(t)$ of the BVP,

$$v'' + pv' + qv = 0, \qquad v(\tau) = v(b) = 0,$$

with $\alpha(t) \leqq v(t) \leqq \beta(t)$ on $[\tau, b]$. This contradiction proves the theorem.

COROLLARY 4.11. *Equation* (4.1) *is of class* I *in case* $f_0(t, x_0, x_1, x_2) \geqq 0$ *and* $f_1(t, x_0, x_1, x_2) \leqq 0$ *for all* t, x_0, x_1, x_2.

As an example, using a comparison theorem for the third order linear equation, we have

THEOREM 4.12. *Assume equation* (4.2) *is of class* I *and that* $f_y(t, y) \geqq r(t)$ *for all* t, y. *Then equation* (4.8) *is of class* I.

PROOF. Apply [**6**, Theorem 3.10] and Theorem 2.3.

We conclude this section with a generalization of a result due to Heidel [7] which also serves to illustrate the fact that uniqueness of solutions of (0; 3), (3; 0) and the three point BVP for (4.1) are independent. We also obtain a qualitative description of the types of zeros that (4.8) may possess under certain conditions. Parts of the proof are adapted from those in [7] and [**17**]. As in [7], we shall have occasion to use the following lemma, essentially due to Kiguradze [**18**].

LEMMA 4.13. *Let* $f(t)$ *be a continuous nonnegative function defined for* $t \geqq t_0$ *with* $f^{(n)}(t) \leqq 0$, $n \geqq 2$, *and* $f^{(i)}(t) \geqq 0$, $t \geqq t_0$, $i = 0, 1, \cdots, n-1$. *Then there is a constant* $k > 0$ *such that for all large* t

$$f(t)/f^{(i)}(t) \geqq kt^i, \qquad i = 1, 2, \cdots, n-1. \tag{4.18}$$

THEOREM 4.14. *Consider equation* (4.8) *in which we assume*

$$p(t) \leqq 0, \quad q(t) \geqq 0, \quad q(t) - p'(t) \geqq 0, \quad q'(t) - p''(t) \leqq 0, \tag{4.19}$$

and $yf(t, y) > 0$, $y \neq 0$. Let there exist a nonnegative function $\beta(t)$ and a number $M > 0$ such that

$$f(t, y)/y \geqq \beta(t)|y|^{\gamma}, \qquad |y| \geqq M, \tag{4.20}$$

where $\gamma > 0$. Furthermore, assume for any $k > 0$ there exists $T_k > 0$ such that

$$|kp(t)| \leqq q(t) + t^{1+\gamma}\beta(t), \qquad t \geqq T_k. \tag{4.21}$$

Finally, assume

$$\int^{\infty} |p(t)|dt < +\infty, \qquad \int^{\infty} (t^{1-\gamma}q(t) + t^{2+\gamma}\beta(t))dt = +\infty. \tag{4.22}$$

Then any (continuable) solution of (4.8) which has a zero is oscillatory.

PROOF. Let $y(t)$ be a continuable solution of (4.8) with $y(t_0) = 0$ and $y(t) \neq 0$ for $t > t_0$. We may suppose, without loss of generality, that $y(t) > 0$ for $t > t_0$. Multiplying (4.8) by $y(t)$ and integrating yields:

$$\begin{aligned} y''(t)y(t) - \tfrac{1}{2}(y'(t))^2 + \tfrac{1}{2}(y'(t_0))^2 + \int_{t_0}^{t} p(s)y(s)y''(s)ds \\ + \int_{t_0}^{t} q(s)y(s)y'(s)ds + \int_{t_0}^{t} y(s)f(s, y(s))ds \equiv 0. \end{aligned} \tag{4.23}$$

Since

$$\begin{aligned} \int_{t_0}^{t} p(s)y(s)y''(s)ds = \; & y(t)y'(t)p(t) - \int_{t_0}^{t} p(s)(y'(s))^2ds \\ & - \int_{t_0}^{t} p'(s)y(s)y'(s)ds \end{aligned}$$

and

$$\begin{aligned} & \int_{t_0}^{t} (q(s) - p'(s))y(s)y'(s)ds \\ & \quad = \tfrac{1}{2}(q(t) - p'(s))(y(t))^2 - \tfrac{1}{2}\int_{t_0}^{t} (q'(s) - p''(s))(y(s))^2ds, \end{aligned}$$

we see that (4.23) gives

$$y''(t)y(t) - \tfrac{1}{2}(y'(t))^2 + y(t)y'(t)p(t) + H(t) \equiv 0, \tag{4.24}$$

where $H(t) > 0$. Therefore, if $y'(t_1) = 0$ for some $t_1 > t_0$, then $y''(t_1) < 0$ and hence $y'(t)$ can vanish at most once on $(t_0, +\infty)$. If $y'(t_1) = 0$ and $y'(t) < 0$ for $t > t_1$ and if $y''(t) \leqq 0$ for all large t, we obtain an immediate contradiction to the assumption that $y(t)$ is nonoscillatory. If $y''(t) \geqq 0$ for all large t, then (4.24) shows that $y'(t) \to -\alpha < 0$, a contradiction. If $y''(t)$ oscillates, then since $\limsup_{t\to\infty} y'(t) = 0$, we may choose $t_n \to +\infty$ such that $\lim_{n\to\infty} y'(t_n) = 0$ and $y''(t_n) = 0$. But then (4.24) implies $(y'(t_n))^2 \geqq k > 0$ for all n, where k is some positive constant. This is a contradiction. Therefore, $y(t) > 0$ and $y'(t) > 0$ for all $t > t_0$.

We claim next that $y''(t) > 0$ for all large t. If $y''(t) < 0$ for $t \geqq T > t_0$, then (4.8) shows $y'''(t) = -p(t)y''(t) - q(t)y'(t) - f(t, y(t)) \leqq 0$, $t \geqq T$, so that $y''(t)$ is decreasing. But then

$$y'(t) = y'(T) + \int_T^t y''(s)ds \leqq y'(T) + y''(T)(t - T)$$

so that $y'(t) \to -\infty$, a contradiction. This also shows that $y''(t) \leqq 0$, $t \geqq T$, cannot hold either. Now if $y''(t)$ has arbitrarily large zeros, choose $t_1 < t_2$ so that $y''(t_1) = y''(t_2) = 0$ and $y''(t) \neq 0$ on (t_1, t_2). Then

$$y'''(t_i) = -q(t_i)y'(t_i) - f(t_i, y(t_i)) < 0, \qquad i = 1, 2, \tag{4.25}$$

which is a contradiction. This shows $y''(t) > 0$ for all large t.

We now show that $y'''(t) \leqq 0$ for all large t.

Multiplying (4.8) by $1/y''(t)$ and integrating, we get for any $t > T$,

$$y''(t) \leqq y''(T) \exp\left(- \int_T^\infty p(s)ds \right) < +\infty. \tag{4.26}$$

Hence, since $y > 0$, $y' > 0$, and $y'' > 0$ for all large t, we may choose $T > 0$ and $k > 0$ such that $y(t) \geqq kt \geqq M$ for $t \geqq T$. Then applying (4.20) and (4.26) to (4.8) we have

$$-y'''(t) \geqq k_0 p(t) + k_1 q(t) + k_2 t^{1+\gamma}\beta(t), \qquad t \geqq T, \tag{4.27}$$

where $k_0 = y''(T) \exp\left(-\int_T^\infty p(s)ds\right)$, $k_1 = y'(T) > 0$ and $k_2 = k^{1+\gamma}$. By (4.21) the right-hand side of (4.27) is eventually nonnegative and this shows that $y'''(t) \leqq 0$ for large t.

Now multiplying (4.8) by $t/(y'(t))^{1+\gamma}$, integrating, and applying (4.20) yields

$$\begin{aligned} &\int_T^t \frac{sy'''(s)ds}{(y'(s))^{\gamma+1}} + \int_T^t \frac{sp(s)y''(s)ds}{(y'(s))^{\gamma+1}} \\ &\qquad + \int_T^t \frac{sq(s)ds}{(y'(s))^{\gamma}} + \int_T^t s\beta(s) \left(\frac{y(s)}{y'(s)} \right)^{\gamma+1} ds \leqq 0. \end{aligned} \tag{4.28}$$

Expanding the first integral on the left, we have

$$\int_T^t \frac{sy'''(s)ds}{(y'(s))^{\gamma+1}} = \frac{ty''(t)}{(y'(t))^{\gamma+1}} - c_1 + c_2 \int_T^t \frac{sy''(s)ds}{(y'(s))^{\gamma+2}} - \int_T^t \frac{y''(s)ds}{(y'(s))^{\gamma+1}} \tag{4.29}$$

where c_1 and c_2 are positive constants. But since $y'(t)$ is increasing for $t \geqq T$,

$$\int_T^t \frac{y''(s)ds}{(y'(s))^{\gamma+1}} = \gamma^{-1}[(y'(T))^{-\gamma} - (y'(t))^{-\gamma}] < +\infty.$$

This shows that the first integral in (4.28) converges. We next show that the second integral in (4.28) converges. To see this, choose $k > 0$ by Lemma 4.13 so that $(y'(t)/y''(t)) \geqq kt$ for large t, say $t \geqq T$. Since $y'(t) \geqq y'(T)$, we have for all $t \geqq T$,

$$\int_T^t \frac{sp(s)y''(s)ds}{(y'(s))^{1+\gamma}} \geqq (1/k(y'(T))^{\gamma}) \int_T^\infty p(s)ds > -\infty. \tag{4.30}$$

Now since $y''(t)$ is decreasing on $[T, +\infty)$ we see that $y'(t) = y'(T) + \int_T^t y''(s)ds \leqq y'(T) + (t - T)y''(T)$ so that $y'(t) \leqq \alpha t$ for some $\alpha > 0$ and all large t. Therefore, applying Lemma 4.13 to the fourth integral in (4.28) and using the above estimate for $y'(t)$ in the third integral we have that

$$\int_T^\infty s^{1-\gamma}q(s)ds + \int_T^\infty s^{2+\gamma}\beta(s)ds < +\infty,$$

which is a contradiction.

REMARK 4.15. We see, therefore, that if the hypotheses of Theorem 4.14 hold and if $f_y(t, y)$ is continuous and positive for $y \neq 0$, then (4.8) is of class I by Theorem 4.5. Hence, any nontrivial solution of (4.8) which has a zero is oscillatory and has at most one (necessarily the first) double zero on $(0, +\infty)$.

5. In this final section we shall show how some of the results which have been obtained for the linear differential equation of order n, $n \geqq 4$, yield uniqueness results for equation (1.1). In [19] Leighton and Nehari have made an extensive study of the selfadjoint fourth order linear differential equation of the form

$$(r(t)y'')'' + p(t)y = 0, \qquad r(t) > 0, \quad p(t) > 0, \tag{5.1}$$

or

$$(5.2) \qquad (r(t)y'')'' - p(t)y = 0, \qquad r(t) > 0, \quad p(t) > 0,$$

where $r \in C^{(2)}(0, +\infty)$ and $p \in C(0, +\infty)$. As is shown, the oscillatory behavior of (5.1) and (5.2) is quite dissimilar — all solutions of (5.1) are either oscillatory or nonoscillatory whereas (5.2) will always have some nonoscillatory solutions. To be specific, we have the easily established

LEMMA 5.1 [19, LEMMAS 2.1, 2.2].

(a) *If $y(t)$ is a solution of* (5.2) *and the values of y, y', y'', and $(ry'')'$ are nonnegative (but not all zero) for $t = a$, then the functions $y(t)$, $y'(t)$, $y''(t)$, and $(r(t)y''(t))'$ are positive for $t > a$.*

(b) *If $y(a) \geqq 0$, $y''(a) \geqq 0$, $y'(a) \leqq 0$ and $(ry'')'(a) \leqq 0$, then $y(t)$ and $y''(t)$ are positive and $y'(t)$ and $(r(t)y''(t))'$ are negative for $0 < t < a$.*

Consider now the equation

$$(5.3) \qquad (r(t)y'')'' + f(t, y) = 0, \qquad r(t) > 0.$$

We obtain, from Lemma 5.1 and Theorem 2.3

THEOREM 5.2. *All $(4; j)$ and $(j; 4)$ BVP's, $0 \leqq j \leqq 3$, for equation* (5.3) *have at most one solution provided $f_y(t, y) < 0$, $y \neq 0$.*

For the classical fourth order equation

$$(5.4) \qquad y^{\text{iv}} + py''' + qy'' + ry' + sy = 0,$$

where $p \in C'$, $q, r, s \in C$ on $(0, +\infty)$, we have

THEOREM 5.3. *All $(4; 2)$ BVP's for* (5.4) *have unique solutions if $r(t) \leqq 0$, $r(t) + ts(t) \leqq 0$, $t > 0$, and the second order equation*

$$(5.5) \qquad x'' + (q - p'/2)x = 0$$

is disconjugate on $(0, +\infty)$.

PROOF. If not, we may assume $y(t)$ is a solution of (5.4) with $y(a) = y'(a) = y''(a) = 0 = y''(b)$, $y > 0$, $y' > 0$, and $y'' > 0$ on (a, b). Multiplying (5.4) by y'' and integrating by parts we obtain, after rearranging,

$$(5.6) \qquad \int_a^b (y''')^2 dt = \int_a^b (q - p'/2)(y'')^2 dt + \int_a^b (ry' + sy)y'' dt.$$

Since (5.5) is nonoscillatory, it follows, as in Theorem 4.3, that

$$(5.7) \qquad \int_a^b (ry' + sy)y'' dt > 0.$$

However, we have $y(t) \leqq y'(t)(t-a)$ so that $y(t)/y'(t) \leqq t$ on (a, b). Hence, the integrand in (5.7) is nonpositive on (a, b), a contradiction.

Under slightly different assumptions we have

THEOREM 5.4. *All* (2; 4) *BVP's for* (5.4) *have unique solutions provided* (5.5) *is disconjugate and* $s(t) \leqq 0$, $r(t) \geqq 0$.

PROOF. If not, let $y(t)$ be a solution of (5.4) with $y''(a) = y(b) = y'(b) = y''(b) = 0$, $y'' > 0$, $y' < 0$, $y > 0$ on (a, b). Then proceeding as in Theorem 5.3 we obtain (5.7). Since $r(t) \geqq 0$ and $s(t) \leqq 0$, we have an immediate contradiction.

The two previous theorems imply uniqueness results for nonlinear fourth order equations. As an example, consider

$$y^{\text{iv}} + py''' + qy'' + ry' + f(t, y) = 0. \tag{5.8}$$

If (5.5) is nonoscillatory and if $r \leqq 0$, $r + tf_y \leqq 0$, then Theorem 5.3 implies that all (4; 2) BVP's for (5.8) have at most one solution. In particular, the equation

$$y^{\text{iv}} + ry' + p(t) \sin y = 0 \tag{5.9}$$

satisfies these hypotheses if $r(t) + tp(t) \leqq 0$, $p(t) > 0$.

Finally, we wish to cite some results of Levin [20], [21] for the nth order linear differential equation

$$Ly = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \tag{5.10}$$

where $p_i \in C[a, b]$. The interval $[a, b]$ is said to be an interval of $(k, n-k)$ oscillation for (5.10) in case $b \geqq \eta_1(a)$ and there exists a solution of (5.10) which is positive on $(a, \eta_1(a))$ and has a zero of order $\geqq k$ at a and a zero of order $\geqq n-k$ at $\eta_1(a)$. (Here $\eta_1(a)$ is the first conjugate point of a and is defined as the infimum of those $c > a$ for which some solution of (5.10) has at least n zeros on $[a, c]$.) The interval $[a, b]$ is said to be an interval of odd (even) oscillation type in case $n - k$ is odd (even). We have the following theorem (Levin [20]).

THEOREM 5.5. *Consider the operator defined by*

$$L_1 y = Ly + r(t)y, \tag{5.11}$$

where $r(t) \geqq 0$ *and L is the operator defined in* (5.10). *Then*:

(a) *every interval of odd oscillation for* (5.10) *is an interval of odd oscillation for* (5.11),

(b) *every interval of even oscillation for* (5.11) *is an interval of even oscillation for* (5.10).

In particular, using $Ly = y^{(n)} = 0$, one obtains the result [20]:

COROLLARY 5.6. *If* $c(t) \geqq 0$, *the intervals of oscillation for the equation* $y^{(n)} + c(t)y = 0$ *are all of odd type, and if* $c(t) \leqq 0$ *they are all of even type.*

By $T_{i,k}$ [21] we denote the class of all operators L for which $Ly = 0$ has no nontrivial solution with a zero of order $\geqq i$ at t_1 and a zero of order $\geqq k$ at t_2, $a \leqq t_1 < t_2 \leqq b$. We are particularly interested in the classes $T_{1,n-1}$ and $T_{n-1,1}$, for which one can be more specific than in Theorem 5.5. Levin [21] has shown that if $L \in T_{n-1,1}$ and if $q(t) \geqq 0$, then $Ly - qy \in T_{n-1,1}$. We also have

THEOREM 5.7 [21]. *Let the numbers* $i_1, i_2 - i_1, \cdots, i_{n-1} - i_{n-2}$ *be either one or zero. If* $q_0(t), q_1(t), \cdots, q_{n-2}(t) \geqq 0$, *then* $L_0 \in T_{n-1,1}$ *where* L_0 *is defined by*

$$L_0y \equiv y^{(n)} - \sum_{k=1}^{n-1} (q_k y^{(i_k)})^{k-i_k} - q_0 y.$$

For example, $y^{(n)} + c_{n-1}y^{(n-1)} - c_{n-2}y^{(n-2)} - \cdots - c_0 y \in T_{n-1,1}$, *provided* $c_i(t) \geqq 0, 0 \leqq i \leqq n-2$.

Using these preceding results and Theorem 2.3, we have

THEOREM 5.8. *Let* $L \in T_{n-1,1}$, *let* f, f_y *be continuous, and let* $f_y(t, y) \leqq 0$, *for all* t, y. *Then all* $(n; 0)$ *BVP's for the equation* $Ly + f(t, y) = 0$ *have at most one solution.*

THEOREM 5.9. *Consider equation* (1.1):

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)}), \tag{1.1}$$

where $f_i(t, x_0, x_1, \cdots, x_{n-1}) \geqq 0$, $0 \leqq i \leqq n-2$. *Then all* $(n; 0)$ *BVP's for* (1.1) *have at most one solution.*

PROOF. Apply Theorem 2.3 and Theorem 5.7.

REFERENCES

1. Lynn Erbe, *A uniqueness theorem for second order differential equations,* Math. Z. **109** (1969), 92–96. MR **39** #7183.

2. A. Lasota and Z. Opial, *L'existence et l'unicité des solutions du problème d'interpolation pour l'équation différentielle d'ordre n,* Ann. Polon. Math. **15** (1964), 253–271. MR **30** #4012.

3. L. Jackson and K. Schrader, *Existence and uniqueness of solutions of boundary value problems for third order differential equations,* J. Differential Equations **9** (1971), 46-54.

4. P. Hartman, *Unrestricted n-parameter families,* Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142. MR **21** #4211.

5. D. F. Ullrich, *Boundary value problems for a class of nonlinear second-order differential equations,* J. Math. Anal. Appl. **28** (1969), 188–210. MR **39** #7203.

6. M. Hanan, *Oscillation criteria for third-order linear differential equations,* Pacific J. Math. **11** (1961), 919-944. MR **26** #2695.

7. J. W. Heidel, *Qualitative behaviour of solutions of a third order nonlinear differential equation,* Pacific J. Math. **27** (1968), 507–526. MR **39** #1738.

8. P. Hartman, *Ordinary differential equations,* Wiley, New York, 1964. MR **30** #1270.

9. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations,* Trans. Amer. Math. Soc. **93** (1959), 30–52. MR **22** #2755.

10. Z. Nehari, *On a class of second-order differential equations,* Trans. Amer. Math. Soc. **95** (1960), 101–123. MR **22** #2756.

11. L. K. Jackson and K. W. Schrader, *Comparison theorems for nonlinear differential equations,* J. Differential Equations **3** (1967), 248–255. MR **34** #6206.

12. W. Leighton, *Some elementary Sturm theory,* J. Differential Equations **4** (1968), 187–193. MR **37** #506.

13. C. V. Coffman, *On the positive solutions of boundary-value problems for a class of nonlinear differential equations,* J. Differential Equations **3** (1967), 92–111. MR **34** #4593.

14. Z. Nehari, *Oscillation criteria for second-order linear differential equations,* Trans. Amer. Math. Soc. **85** (1957), 428–445. MR **19**, 415.

15. M. Švec, *On various properties of the solutions of third and fourth-order linear differential equations,* Proc. Conf. Differential Equations and Their Applications (Prague, 1962), Publ. House Czechoslovak Acad. Sci., Prague; Academic Press, New York, 1963, pp. 187–198. MR **30** #5018.

16. W. J. Kim, *Oscillatory properties of linear third order differential equations,* Proc. Amer. Math. Soc. **26** (1970), 286-293.

17. P. Waltman, *Oscillation criteria for third order nonlinear differential equations,* Pacific J. Math. **18** (1966), 385–389. MR **34** #422.

18. I. T. Kiguradze, *Oscillation properties of solutions of certain ordinary differential equations,* Dokl. Akad. Nauk SSSR **144** (1962), 33–36 = Soviet Math. Dokl. **3** (1962), 649–652. MR **25** #278.

19. W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order,* Trans. Amer. Math. Soc. **89** (1958), 325–377. MR **21** #1429.

20. A. Ju. Levin, *Some problems bearing on the oscillation of solutions of linear differential equations,* Dokl. Akad. Nauk SSSR **148** (1963), 512–515 = Soviet Math. Dokl. **4** (1963), 121–124. MR **26** #3972.

21. ——, *Distribution of the zeros of solutions of a linear differential equation,* Dokl. Akad. Nauk SSSR **156** (1964), 1281–1284 = Soviet Math. Dokl. **5** (1964), 818–821. MR **29** #1378.

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