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VALUATION RINGS WITH ZERO DIVISORS

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Introduction. All rings in this paper are commutative with identity element, and all subrings possess the identity element of the containing ring. An overring of a ring is a subring of the total quotient ring containing the given ring. It is assumed that the reader is acquainted with the concepts of quotient ring, integral extensions and the elementary theorems on valuation rings. Excellent references for this material are the textbooks of Zariski and Samuel [7].

Recent papers by Davis [2] and Manis [6] have extended parts of the theory of valuation domains to include rings with zero divisors. It is our purpose here to study Bezout rings and their relation to a certain type of "valuation ring" which contains zero divisors.

§1 introduces a useful "pre-valuation map". This map reduces to the valuation map of Manis [6] for quasi-valuation rings, and to the usual valuation map of Krull [7] for valuation domains. A few basic properties of this map are derived. These results motivate the author's definitions of the concepts of a ring with few zero divisors and quasi-valuation ring. The properties of quasi-valuation rings are enumerated and their relation to Bezout rings are studied.

§2 employs the results of §1 to study more properties of almost-Bezout rings and the relation between v-closed ideals and regular ideals of a quasi-valuation ring.

§3 compares the author's "pre-valuation" map with another extension of the concept of a valuation ring.

1. Let x be called a regular element of the ring A if x is a nonzero divisor of A. An ideal I of A is called regular if it contains a regular element. The reader is once again reminded that all rings in this paper are commutative with identity element. The expression P is a prime ideal of A shall imply $P \neq A$. A regular maximal ideal is an ideal which is maximal with respect to the property that it is regular. Such ideals are of course also maximal ideals and hence prime. The following results of Manis [6] will be used. A Manis valuation is a map v

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from a ring K onto a linearly ordered abelian group Γ and a symbol $-\infty$ such that for all x, y in K

(1) $v(xy) = v(x)v(y)$,

(2) $v(x + y) \leqq \max \{v(x), v(y)\}$.

THEOREM 1.1 (MANIS). *Let A be a subring of K. Let P be a prime ideal of A. The following conditions are equivalent:*

(1) *if B is a subring of K containing A and M is a prime ideal of B with $M \cap A = P$ then $B = A$,*

(2) *for all x in $K - A$ there exists x' in P such that xx' is in $A - P$,*

(3) *there is Manis valuation (v, Γ) on K such that:*

$$A = R_v = \{x \in K \mid v(x) \leqq v(1)\},$$

$$P = P_v = \{x \in K \mid v(x) < v(1)\}.$$

The pair (A, P) will be called a Manis valuation pair.

DEFINITION 1.2. Let A be a ring with regular prime ideal P. Let K be the total quotient ring of A. A relation v is defined on K in the following way. For all a, b in K

(i) if $aK \cap A \subseteq P$ and $bK \cap A \subseteq P$ then write $v(a) = v(b)$ $(= v(0) = -\infty)$,

(ii) if $aK \cap A \nsubseteq P$ and $bK \cap A \nsubseteq P$ then write $v(a) = v(b)$ if and only if there exist s, t in $A - P$ such that $as = bt$,

(iii) if $aK \cap A \subseteq P$ and $bK \cap A \nsubseteq P$ then write $v(a) \neq v(b)$.

A straightforward calculation shows that v is an equivalence relation on K. The equivalence class of an element a of K is denoted by $v(a)$. The collection of all such equivalence classes is $K/v = \{v(a) \mid a \in K\}$.

In Definition 1.1 we notice that v describes the elements of $A - P$ as being "near units" (i.e. $x \in A - P$ implies $v(x) = v(1)$). In this sense $v(a) = v(b)$ $(\neq v(0))$ if and only if a and b are "near-associates" (i.e. there exists $s, t \in A - P$ such that $as = bt$).

DEFINITION 1.3. Let A be a ring with regular prime ideal P and let v be the map of Definition 1.2. Let a and b be elements of the total quotient ring K of A. Write $v(a) \leqq v(b)$ if and only if $v(a) = v(b)$ or $v(a) \neq v(b)$ and there ixists z in K such that $az \in P$ and $bz \in A - P$.

It is a simple matter for the reader to show that the above ordering is well defined on K/v. In the light of the above discussion the element z of Definition 1.3 is a "near-inverse" of b. Consequently the requirement $v(a) < v(b)$ says that b "nearly-divides" a in A but b is not a "near-associate" of a.

In Lemmas 1.4 and 1.5 it is assumed that A is a ring with regular prime ideal P and the notation of Definition 1.3 is employed.

LEMMA 1.4. *$\{K/v - \{v(0)\}, \leqq\}$ is a partially ordered abelian group* *with the operation* $v(a)v(b) = v(ab)$. Furthermore, $\{K/v, \leqq\}$ *is a partially ordered abelian semigroup, and for all* x *in* K, $v(x) \geqq v(0)$.

PROOF. It is a simple matter for the reader to show that the operation on K/v is well defined. Clearly $v(1)$ is the identity of K/v. Suppose $v(a) \in K/v - \{v(0)\}$. Since $v(a) \neq v(0)$ then $aK \cap A \not\subseteq P$. There exists a' in K such that $aa' \in A - P$. It follows that $v(aa') = v(1)$.

Suppose $v(a) \neq v(0) \neq v(b)$ are elements of K/v. Then there exists a', b' in K such that aa' and bb' are elements of $A - P$. Therefore $aba'b' \in A - P$. It follows that $abK \cap A \not\subseteq P$ and $v(ab) \neq v(0)$. Therefore $K/v - \{v(0)\}$ is closed under the operation. By calculations similar to the above, the reader may show that $\leqq$ is a partial ordering of K/v and if $v(a) \leqq v(b)$ then for any $v(c)$ in K/v it follows that $v(ac) \leqq v(bc)$.

The following terminology is due to Griffin [5]. Let A be a ring with proper prime ideal P and total quotient ring K. The large quotient ring of A with respect to P is the subring $A_{[P]}$ of K consisting of all elements z of K such that zs is in A for some s in $A - P$.

LEMMA 1.5. *If* $A = R_v$, *then* $A = R_v = A_{[P]}$. $P_v = \{x \in K \mid v(x) < v(1)\}$ *is a prime ideal of* R_v *which contains* P. *The map* v *has the following property. For all* a, b *and* c *in* K *if* $v(c) \neq v(0)$ *and* $v(a) \leqq v(c)$ *and* $v(b) \leqq v(c)$ *then* $v(a - b) \leqq v(c)$.

PROOF. The first two assertions follow directly from Definition 1.2. Since $v(c) \neq v(0)$ then by Lemma 1.4 there exists c' in K such that $v(cc') = v(1)$. Since $v(a) \leqq v(c)$ and $v(b) \leqq v(c)$ then $v(ac') \leqq v(1)$ and $v(bc') \leqq v(1)$. Therefore ac' and bc' are elements of R_v and so is $ac' - bc'$. It follows that $v(ac' - bc') \leqq v(1)$ and $v(a - b) \leqq v(c)$.

The following question naturally arises. When is the ordering of Lemma 1.4 linear and when is $A = R_v$ and $P = P_v$? The answer to this question is contained in the next theorem.

Notice that $A = R_v$ implies that $P = P_v$. For if $x \in P_v$ then $v(x) < v(1)$. There exists an element z of K such that $xz \in P$ and $z \in A - P$. Therefore, since P is a prime ideal, $x \in P$. It follows that $P = P_v$.

THEOREM 1.6. *Let* A *be a ring with regular prime ideal* P. *Let* v *be the map of Definition* 1.2. *Then* $A = R_v$ *and* $\{K/v, \leqq\}$ *is linearly ordered if and only if* (A, P) *is a Manis valuation pair. In this case the map* v *is equivalent to the valuation map of Manis.*

PROOF. Assume $A = R_v$ and K/v is linearly ordered. Suppose $x \in K - A$. There exists x' in K such that $v(xx') = v(1)$. It follows that $xx' \in A - P$. Since K/v is linearly ordered then $v(x') \geqq v(1)$ or $v(x') < v(1)$. Clearly $v(x') \geqq v(1)$ leads to a contradiction. It follows that A satisfies the requirements of Theorem 1.1 (2). The reader may show that v is equivalent to the valuation map of Manis.

Assume (A, P) is a Manis valuation pair. Let w be the valuation map of Manis. Then $R_w = A$ and $P_w = P$ and K/w is linearly ordered. If $w(x) < w(1)$ then there exists z in K such that $zx \in P$ and $z \notin P$. In fact there exists z' in K such that $w(zz') = w(1)$. Therefore $zz' \in A - P$ and $zz'x \in P$. It follows that $v(x) < v(1)$. Clearly if $v(x) < v(1)$ then $w(x) < w(1)$.

In a similar manner the reader may show that $w(x) = w(1)$ if and only if $v(x) = v(1)$. It follows that $P = P_w = P_v$ and $A = R_w = R_v$. Using the above facts the reader may show that the linear ordering of K/w induces a linear ordering of K/v.

Definition 1.7. A ring is a Bezout ring if every finitely generated ideal is principal. A ring is an almost-Bezout ring if every finitely generated regular ideal is principal.

Definition 1.8. A ring has few zero divisors if it has the following property. If I and J are ideals of the ring such that I is regular and J contains all the regular elements of I then $I \subseteq J$.

Remark. E. D. Davis [2] has used the term few zero divisors to define a ring whose total quotient ring has only finitely many maximal ideals. This definition is more restrictive than the requirements of Definition 1.8. For every ring whose total quotient ring has only finitely many maximal ideals also has few zero divisors (i.e. satisfies Definition 1.8). However, the direct sum of the integers with an infinite collection of fields provides a relatively simple example of a ring which satisfies the requirements of Definition 1.8 but does have infinitely many maximal ideals in its total quotient ring.

Direct sums of integral domains provide us with examples of rings with few zero divisors. A simple argument shows that every almost-Bezout ring has few zero divisors. In fact it can be shown that every overring of an almost-Bezout ring has few zero divisors.

Lemma 1.9. *In a ring with few zero divisors every (finitely generated) regular ideal can be generated by (a finite number of) regular elements.*

Proof. Since the ring has few zero divisors, the set of all regular elements of a regular ideal is a generating set for the ideal.

In Theorems 1.10 and 1.11 the map v refers to the map obtained from Definition 1.2 using the unique regular maximal ideal P of A.

Theorem 1.10. *Let A be an almost-Bezout ring with unique regular maximal ideal P. Then $\{K/v, \leqq\}$ is a linearly ordered abelian group with $-\infty$ adjoined. In this case $A = R_v = \{x \in K \mid v(x) \leqq v(1)\}$ and $P = P_v = \{x \in K \mid v(x) < v(1)\}$.*

PROOF. Let x in K be such that $v(x) \leqq v(1)$. If $x \notin A$ then $x = a/b$ where a and b are elements of A and b is a regular element of P. Since A is almost Bezout it may be assumed that $aA + bA = A$. Therefore $b \in P$ and $a \in A - P$. It follows that $v(1) = v(a) \leqq v(b) < v(1)$. This contradiction proves that $x \in A$ and $A = R_v$. Likewise it follows that $P = P_v$.

Let a and b be elements of A. It remains to show that $v(a) \leqq v(b)$ or $v(b) \leqq v(a)$. From this fact it follows that K/v is linearly ordered. Clearly it may be assumed that $v(a) \neq v(0) \neq v(b)$. By Lemma 1.4 there exists a' in K such that $v(aa') = v(1)$. Consider ba'. Either $ba' \in A$ or $ba' \in K - A$. If $ba' \in A$ then by the fact that $A = A_v$ it follows that $v(ba') \leqq v(1)$ and hence $v(b) < v(a)$. If $ba' \in K - A$ then $ba' = s/t$ where s and t are elements of A and t is a regular element of P. As before, it may be assumed that $sA + tA = A$ and $s \notin P$. Therefore $ba't = s \in A - P$ and $v(ba't) = v(s) = v(1)$. Since $t \in P$ then $v(t) < v(1)$. It follows that $v(ba') = v(1/t) > v(1)$ and $v(b) > v(a)$.

THEOREM 1.11. *Let A be a ring with few zero divisors and unique regular maximal ideal P. If $\{K/v, \leqq\}$ is linearly ordered and $A = R_v = \{x \in K \mid v(x) \leqq v(1)\}$ then A is almost Bezout.*

PROOF. By Lemma 1.9 every finitely generated regular ideal I of A can be generated by a finite set of regular elements of A. The element of this set which has minimal value generates I.

DEFINITION 1.12. Let A be a ring with prime regular ideal P. Let v be the map of Definition 1.2 corresponding to P. Then A is a pre-valuation ring if $A = R_v = \{x \in K \mid v(x) \leqq v(1)\}$. A pre-valuation ring for which K/v is linearly ordered is called a Manis valuation ring. A pre-valuation ring which is almost Bezout is called a quasi-valuation ring.

The term quasi-valuation ring as used by Davis [2] requires that the ring have only finitely many ideals maximal with respect to the property of being nonregular. This requirement has been avoided in Definition 1.12. However, a ring which is a quasi-valuation ring in the sense of Definition 1.12 is a valuation ring in the sense of Manis's paper. In this case the "pre-valuation" map of Definition 1.2 coincides with a map which Manis obtained by a different construction. If K/v is not linearly ordered these maps do not necessarily coincide.

The basic properties of quasi-valuation rings are the following. The regular ideals are linearly ordered by inclusion. Every overring is a quasi-valuation ring. Every quasi-valuation ring is integrally closed in its total quotient ring. These properties are proved in a manner which parallels the "standard" proofs given by Zariski and Samuel [7].

2. The expression P is a prime ideal of the ring A shall imply $P \neq A$. If P is a prime ideal of the ring A, let $S(P)$ be the set of all regular elements of A which are not in P. The following theorem indicates the relationship between an almost-Bezout ring and its overrings which are quasi-valuation rings.

LEMMA 2.1. *Let A be an almost-Bezout ring. Every overring T of A has the form $T = A_U$ where U is a multiplicatively closed set of regular elements of A.*

PROOF. This lemma may be proved by an adaptation of some results of Gilmer and Ohm [4] to rings with zero divisors.

COROLLARY 2.2 *Let A be an almost-Bezout ring. If T and W are overrings of A such that T contains every regular element of W then $W \subseteq T$.*

THEOREM 2.3. *Let A be an almost-Bezout ring. The overring T of A is a quasi-valuation ring if and only if T is of the form $T = A_{S(P)}$ where P is a regular prime ideal of A.*

PROOF. Since P is a regular prime ideal of A a routine calculation shows that $A_{S(P)}$ is a quasi-valuation ring.

If T is an overring of A and T is a quasi-valuation ring then T must have a unique regular maximal ideal M. Hence $M \cap A = P$ is a regular prime ideal of A. Clearly $R_{S(P)} \subseteq T_{S(P)} = T$.

Let x be a regular element of T. If $x \in R_{S(P)}$ then by Corollary 2.2 it follows that $T \subseteq R_{S(P)}$. Since A is almost Bezout $x = a/b$ where a and b are relatively prime elements of A and b is regular. If $b \in M$ then $bx = a \in M$. This statement is a contradiction, since $A = aA + bA \subseteq M$ implies that $M = T$.

The author is gratefully indebted to the referee of this paper for Lemma 2.4 and Corollary 2.5. The proofs of these results follow closely the suggested proofs given by the referee.

LEMMA 2.4. *If A is an almost-Bezout ring, and if P is a regular prime ideal of A, then $A_{S(P)} = A_{[P]}$.*

PROOF. Clearly $A_{S(P)} \subseteq A_{[P]}$. Let $x \in A_{[P]}$. It follows that there exists y in $A - P$ such that xy is in A. Also $x = a/b$ where a and b are elements of A and b is regular. If b is in $A - P$ then x is in $A_{S(P)}$. If b is in P then xb is in A. Since A is almost Bezout $(y, b) = (d)$ where d is regular. Since y is not in P then d is in $S(P)$. Also xd is in A. Hence $x = xd/d$ is in $A_{S(P)}$.

COROLLARY 2.5. *Let A be an almost-Bezout ring. The overring T of A is a quasi-valuation ring if and only if T is a Manis valuation ring.*

PROOF. Using Theorem 1.1 (2) the reader may show that every quasi-valuation ring is also a Manis valuation ring. Conversely, suppose T is a Manis valuation ring. An almost-Bezout ring is a Prüfer ring. If M is the regular maximal ideal of T and if $P = M \cap A$, then by Lemma 2.4 and some remarks preceding Proposition 14 of Griffin's paper [5] it follows that $T = A_{[P]} = A_{S(P)}$. Furthermore, P is regular, so T is a quasi-valuation ring by Theorem 2.3.

The concept of a v-closed ideal is due to Manis.

DEFINITION 2.6 (MANIS). Let A be a quasi-valuation ring. A v-closed ideal I of A is an ideal such that $x \in I$ and $v(y) \leqq v(x)$ implies $y \in I$.

THEOREM 2.7. *I is a v-closed ideal of the quasi-valuation ring A if and only if $I = v^{-1}(v(0))$ or I is regular.*

PROOF. Clearly $v^{-1}(v(0))$ is a v-closed ideal of A. Let I be a regular ideal of A with regular element u. Let y be an element of A and let x be an element of I such that $v(y) \leqq v(x)$. Since A is almost Bezout there is a regular element d of A such that

$$dA = xA + uA \subseteq I.$$

It follows that $v(y) \leqq v(x) \leqq v(d)$ and $y/d \in A$. Therefore $y \in dA \subseteq I$ and I is v-closed

Let I be a v-closed ideal of A which is not equal to $v^{-1}(v(0))$. Then there exists x in I such that $v(x) \neq v(0)$. Consequently, there exist elements s and t in A with t regular such that $x \cdot s/t \notin P$ (where P is the unique regular maximal ideal of A). It follows that $v(t) \leqq v(xs) \leqq v(x)$. Since I is v-closed then t is a regular element I.

For the remainder of this section the following notation will be used. A is a quasi-valuation ring. Δ is a proper segment of K/v.

$$L_\Delta = \{\alpha \in K/v \mid \alpha < v(1) \text{ and } \alpha \notin \Delta\},$$

$$\Theta = \{v^{-1}(L_\Delta) \mid \Delta \text{ is a proper segment of } K/v\}.$$

Clearly the elements of Θ are ideals of A which contain $v^{-1}(v(0))$. If I is an ideal of A which contains $v^{-1}(v(0))$ then let

$$\Gamma_I = K/v - \{v(I) \cup (v(I))^{-1}\}$$

where

$$(v(I))^{-1} = \{\alpha^{-1} \mid v(0) \neq \alpha \in v(I)\}.$$

THEOREM 2.8. *I is a v-closed ideal of A if and only if $v^{-1}(v(I)) = I$.*

PROOF. If I is v-closed the reader may show that $v^{-1}(v(I)) = I$. Assume $v^{-1}(v(I)) = I$ and let P be the unique regular maximal ideal of A. If $I \not\subseteq P$ then $v(I) = v(A)$ and $I = v^{-1}(v(I)) = v^{-1}(v(A)) = A$. Therefore $I = A$ is v-closed If $I \subseteq P$ then $\Delta = \Gamma_I$ is a segment of K/v and it follows that $I \subseteq v^{-1}(L_\Delta)$. However $v(I) = v(v^{-1}(L_\Delta))$. Therefore

$$I = v^{-1}(v(I)) = v^{-1}(v(v^{-1}(L_\Delta))) = v^{-1}(L_\Delta).$$

Since Δ is a segment of K/v it follows by a result of Manis [**6**] that $I = v^{-1}(L_\Delta)$ is v-closed.

THEOREM 2.9. *Θ is the set of all proper ideals of A which are v-closed.*

PROOF. It has been shown by Manis [**6**] that all ideals of Θ are v-closed. If I is a v-closed proper ideal and $I \neq v^{-1}(v(0))$ then by Theorem 2.7 I is regular. The reader may show that every regular ideal of A contains $v^{-1}(v(0))$. If $I \not\subseteq P$ it follows that $I = A$. But this statement contradicts the assumption that I is a proper ideal. $\Delta = \Gamma_I$ is a proper segment of K/v and $v(I) = L_\Delta$. Therefore

$$I = v^{-1}(v(I)) = v^{-1}(L_\Delta) \in \Theta.$$

COROLLARY 2.10 (SUMMARY). *For an ideal I of a quasi-valuation ring A the following are equivalent.*

(i) *I is regular or* $I = v^{-1}(v(0))$,

(ii) *I is v-closed,*

(iii) $v^{-1}(v(I)) = I$.

Furthermore, there is a one-to-one correspondence between proper segments of K/v and proper regular ideals of A. Such a segment is an isolated subgroup if and only if the corresponding ideal is a regular prime ideal.

REMARK. In view of the last corollary the reader may define the rank of a quasi-valuation ring A to be the ordinal type of the set of all isolated subgroups of K/v. It follows that if A has the ascending chain condition on regular ideals, then the rank of A is one. Furthermore, $K/v - \{v(0)\}$ is isomorphic to the additive ordered group of integers if and only if A has the ascending chain condition on regular ideals. The proof of these facts parallels the corresponding proof in Zariski and Samuel [7].

3. Let A be a ring with total quotient ring K. Consider the following relation defined on K. For all x, y in K write $w(x) = w(y)$ if and only if x divides y and y divides x in A. The reader may show that this relation is an equivalence relation on K which is "finer" than that of Definition 1.2 (i.e. $w(x) = w(y)$ implies $v(x) = v(y)$ relative to any regular prime ideal of A). An ordering may be defined on K/w in the following way. For all a, b in K, $w(a) \leqq w(b)$ if and only if b divides a in A. It follows that $\{K/w, \leqq\}$ is a partially ordered semi-

group. Furthermore, if $w(a) \leqq w(c)$ and $w(b) \leqq w(c)$ then $w(a-b) \leqq w(c)$. It is clear that $A = \{x \in K \mid w(x) \leqq w(1)\}$.

In the context where A is an integral domain and K is a field, the map w and the partially ordered group K/w have been studied by L. Fuchs [3]. The purpose of this section is to indicate the relationship between the "pre-valuation" maps of Definition 1.2 and the map w.

DEFINITION 3.1. An embedded ring is a ring with few zero divisors and the property that the set of all zero divisors of the ring is contained in every regular maximal ideal of the ring.

LEMMA 3.2. *Let A be an embedded ring with unique regular maximal ideal. Let $\mathfrak{U} = \{w(x) \in K/w \mid v(x) = v(1)\}$ where v is the map of Definition 1.2. Then K/v is order isomorphic to*

$$(K/w)/\mathfrak{U} = \{w(a)\mathfrak{U} \mid w(a) \in K/w\}$$

and the following diagram commutes

$$\begin{tikzcd}
A_v \arrow[dr] & & K/v \\
& K \arrow[ur] \arrow[dr] & \\
A_w \arrow[u] \arrow[ur] & & K/w \arrow[uu]
\end{tikzcd}$$

where the maps on the left are inclusion maps.

The proof of this lemma is left to the reader. It is straightforward but tedious.

Let A be an embedded almost-Bezout ring. Let Λ be the set of all regular maximal ideals of A. For each P in Λ let $S(P)$ be the set of regular elements of A which are not in P. Then $A_{S(P)}$ is a quasi-valuation ring with unique regular maximal ideal $PA_{S(P)}$. Let the map corresponding to Definition 1.2 be $v_p \colon K \to K/v_p$. The reader may apply Lemma 3.2 to prove the following theorem.

THEOREM 3.3. *If A is an embedded almost-Bezout ring, then there is an order preserving homomorphism Ψ of K/w into a subdirect sum of the linearly ordered semigroups K/v_p. The set U of invertible elements of K/w is a partially ordered group. The homomorphism Ψ restricted to U is an order preserving isomorphism of U into a subdirect sum of the linearly ordered groups $K/v_p - \{v_p(0)\}$.*

REFERENCES

1. N. Bourbaki, *Algèbre commutative.* Fasc. XXXI, Hermann, Paris, 1965.

2. E. D. Davis, *Overrings of commutative rings.* II, Trans. Amer. Math. Soc. **110** (1964), 196-212. MR **28** #111.

3. L. Fuchs, *The generalization of the valuation theory,* Duke Math. J. **18** (1951), 19-26. MR **12**, #669.

4. R. W. Gilmer, Jr. and J. Ohm, *Integral domains with quotient overrings,* Math. Ann. **153** (1964), 97-103. MR **28** #3051.

5. M. Griffin, *Prüfer rings with zero divisors,* J. Reine Angew. Math. **239/240** (1969), 55-67. MR **41** #188.

6. M. E. Manis, *Valuations on a commutative ring*, Proc. Amer. Math. Soc. **20** (1969), 193-198. MR **38** #2134.

7. O. Zariski and P. Samuel, *Commutative algebra.* Vols. 1, 2, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1958, 1960. MR **19**, 833; MR **22** #11006.

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