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PROPERTIES OF PÓLYA PEAKS

HERB SILVERMAN

1. **Introduction.** In this note, $G(t)$ will represent a positive, nondecreasing, unbounded, continuous function defined for all real $t \geqq t_0 \geqq 0$.

The *order* λ and the *lower order* μ of the function $G(t)$ are defined by the relations

$$\limsup_{t\to\infty} \frac{\log G(t)}{\log t} = \lambda, \qquad \liminf_{t\to\infty} \frac{\log G(t)}{\log t} = \mu.$$

An increasing sequence

$$r_1, r_2, \cdots, r_m, \cdots$$

is said to be a *sequence of Pólya peaks of order* α $(0 \leqq \alpha < \infty)$ for the function $G(t)$ if it is possible to find sequences $\{r_m'\}$, $\{r_m''\}$ such that

$$r_m' \to \infty, \qquad r_m/r_m' \to \infty, \qquad r_m''/r_m \to \infty$$

and such that

$$\text{(1.1)} \qquad G(t)/t^\alpha \leqq (1 + o(1))G(r_m)/r_m^\alpha \quad (m \to \infty,\ r_m' \leqq t \leqq r_m'').$$

It is required that the error term $o(1)$ in (1.1) approaches zero uniformly as t tends to infinity in the intervals $[r_m', r_m'']$.

Pólya peaks have been used by Edrei and other authors [**6**], [**7**], [**10**] to establish interesting results on the value distribution theory of meromorphic functions in the plane; in particular, inequality (1.1) is useful in the estimation of integral transforms occurring naturally in this theory.

The following existence theorem relates the order of the function to the orders of its Pólya peaks.

THEOREM A. *Let $G(t)$ have order λ and lower order μ $(\mu < \infty$, $\lambda \leqq \infty)$. Then with every finite α such that*

$$\mu \leqq \alpha \leqq \lambda,$$

it is possible to associate a sequence of Pólya peaks of order α.

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Theorem A was proved by Edrei [3] who introduced "Pólya peaks."

It is sometimes useful to reverse the sense of the inequality (1.1). An increasing sequence

$$r_1^*, r_2^*, \cdots, r_m^*, \cdots$$

is said to be a *sequence of Pólya peaks of the second kind* (of order α) for the function $G(t)$ if it is possible to find sequences $\{r_m'\}$, $\{r_m''\}$ such that

$$r_m' \to \infty, \quad r_m^*/r_m' \to \infty, \quad r_m''/r_m^* \to \infty$$

and such that

$$G(t)/t^\alpha \geqq (1 + o(1))G(r_m^*)/(r_m^*)^\alpha \qquad (m \to \infty, \ r_m' \leqq t \leqq r_m'').$$

In [7], Shea proves the analogue of Theorem A for Pólya peaks of the second kind. (To distinguish the peaks $\{r_m\}$ in (1.1) we shall sometimes refer to them as Pólya peaks of the first kind.) The theorems of Edrei and Shea contain no information pertaining to the existence of Pólya peaks outside the interval $[\mu, \lambda]$.

2. Statement of main results.

THEOREM 1. *Choose any four real numbers α, μ, λ, and β such that $0 \leqq \alpha < \mu \leqq \lambda < \beta$. Then there exists a logarithmically convex function $G(t)$ of order λ, lower order μ, and having Pólya peaks (of both kinds) of orders α and β.*

The additional effort to construct the function $G(t)$ to be logarithmically convex (i.e., $\log G(t)$ is a convex function of $\log t$) seems justifiable due to the following considerations:

If $f(z)$ is a nonconstant, entire function, then from Hadamard's three-circles theorem [8, p. 172] it follows that

$$\log M(r,f) = \max_{|z|=r} \log |f(z)|$$

is a convex function of $\log r$. It can also be shown that Nevanlinna's characteristic function $T(r, f)$ is logarithmically convex, for any meromorphic function $f(z)$ [5]. Edrei and Fuchs [4] have shown that for an unbounded logarithmically convex function $\Omega(r)$ of finite order, there exists an entire function $f(z)$ satisfying

$$T(r,f) \sim \log M(r,f) \sim \Omega(r) \qquad (r \to \infty).$$

(Clunie [1] has removed the restriction of finite order. Compare also Valiron [9, p. 130].)

In view of these results, Theorem 1 may be applicable in the theory of entire and meromorphic functions.

I will also prove the following "intermediate value" property for Pólya peaks:

THEOREM 2. *Assume $G(t)$ has a sequence $\{r_m\}$ of Pólya peaks (of the first kind) of order γ and a sequence $\{s_m\}$ of Pólya peaks (of the first kind) of order β $(\beta < \gamma)$. Let α be given in the interval $[\beta, \gamma]$. Then $G(t)$ has a sequence of Pólya peaks (of the first kind) of order α.*

The theorem still holds if peaks of the first kind are replaced by peaks of the second kind.

It should be pointed out that Theorem 2 is implicit in a paper of Drasin [**2**, p. 337].

We have stated some sufficient conditions for the existence of Pólya peaks. Are there any necessary conditions? I shall show that given $G(t)$, there exist upper and lower bounds for the possible orders of Pólya peaks associated with the quantities

$$\limsup_{t\to\infty} \frac{G(\sigma t)}{G(t)} = \sigma^{\Omega(\sigma)}, \qquad \liminf_{t\to\infty} \frac{G(\sigma t)}{G(t)} = \sigma^{\kappa(\sigma)} \qquad (1 < \sigma < \infty). \tag{2.1}$$

Define

$$\Omega = \inf_{\sigma>1} \Omega(\sigma), \qquad \kappa = \sup_{\sigma>1} \kappa(\sigma).$$

THEOREM 3. *If $G(t)$ has a sequence of Pólya peaks (of either kind) of order α, then*

$$\kappa \leqq \alpha \leqq \Omega.$$

In particular, Theorem 3 together with Theorem A yields that

$$\kappa \leqq \mu \leqq \lambda \leqq \Omega.$$

3. **Proof of Theorem 1.** Let $\{x_m\}_{m=1}^{\infty}$ be chosen so that

$$x_1 > 3, \qquad x_{m+1} = x_m + N \log_2 x_m \qquad (m = 1, 2, \cdots), \tag{3.1}$$

where $N > 0$ is a given constant, and $\log_2 x = \log(\log x)$. Simple computations yield

$$x_m \geqq x_1 + (m-1)N \log_2 x_1,$$

$$x_m < Km \log m \qquad (m \geqq m_0).$$

Choose

$$N = (\beta + \lambda)/\mu, \tag{3.2}$$

and a sequence $\{\nu_m\}$ such that

$$\limsup_{m\to\infty} \nu_m = \lambda, \qquad \liminf_{m\to\infty} \nu_m = \mu, \tag{3.3}$$

and

$$\nu_{m+1} - \nu_m \leqq 1/x_m \qquad (m = 1, 2, 3, \cdots). \tag{3.4}$$

This is clearly always possible since $\sum_{m=1}^{\infty} 1/x_m$ diverges. Conditions (3.2) and (3.4) will enable us to prove the logarithmic convexity part of the theorem.

We now define a continuous function $\psi(r)$ on $[x, \infty]$ by writing

$$[x_m, x_{m+1}] = I_m \cup J_m \cup K_m \qquad (m = 1, 2, \cdots),$$

where

$$I_m = [x_m, x_m + \log_2 x_m], \quad J_m = [x_m + \log_2 x_m, x_m + 2\log_2 x_m],$$
$$K_m = [x_m + 2\log_2 x_m, x_{m+1}].$$

We then define $\psi(x_1) = e^{x_1\nu_1}$ and, for each $m \geqq 1$,

$$\begin{aligned}\psi(r) &= \psi(x_m)e^{\alpha(r-x_m)}\log\{e^{r-x_m} + (e-1)\} \quad (r \in I_m)\\ &= \psi(x_m + \log_2 x_m)e^{\beta(r-x_m - \log_2 x_m)} \qquad (r \in J_m)\\ &= \psi(x_m + 2\log_2 x_m)\\ &\quad + \frac{e^{x_{m+1}\nu_{m+1}} - \psi(x_m + 2\log_2 x_m)}{x_{m+1} - (x_m + 2\log_2 x_m)}(r - (x_m + 2\log_2 x_m))\\ &\qquad\qquad (r \in K_m).\end{aligned}$$

Clearly,

$$\psi(x_m) = e^{x_m\nu_m} \qquad (m = 1, 2, \cdots). \tag{3.5}$$

We now make the following substitutions:

$$r = \log t, \qquad x_m = \log y_m, \qquad G(t) = \psi(\log t).$$

We will show that $G(t)$ satisfies all the conditions of the theorem.

For $y_m \leqq t \leqq y_{m+1}$,

$$\frac{\log G(y_m)}{\log y_{m+1}} \leqq \frac{\log G(t)}{\log t} \leqq \frac{\log G(y_{m+1})}{\log y_m}.$$

In view of (3.5), this yields

$$\nu_m \frac{x_m}{x_{m+1}} \leqq \frac{\log G(t)}{\log t} \leqq \nu_{m+1}\frac{x_{m+1}}{x_m}.$$

Hence from (3.1) and (3.3) we obtain

$$\limsup_{t\to\infty} \frac{\log G(t)}{\log t} = \lambda, \qquad \liminf_{t\to\infty} \frac{\log G(t)}{\log t} = \mu.$$

$\{y_m(\log_2 y_m)^{1/2}\}$ is a sequence of Pólya peaks (of both kinds) of order α.

$\{y_m(\log_2 y_m)^{3/2}\}$ is a sequence of Pólya peaks (of both kinds) of order β.

It remains to prove that $G(t)$ is logarithmically convex. This is equivalent to proving that $\psi(r)$ is convex. We must show that given $r \geqq r_0$,

$$2\psi(r) \leqq \psi(r+a) + \psi(r-a) \tag{3.6}$$

for every $a > 0$.

If $[r-a, r+a]$ is contained in one of the intervals I_m, J_m, or K_m the verification is straightforward.

If $[r-a, r+a]$ is in two of these intervals, the computations are similar to those used in verifying the values

$$r = x_m + \log_2 x_m, \qquad r = x_m + 2\log_2 x_m, \qquad r = x_{m+1}. \tag{3.7}$$

We verify (3.6) for these three critical values.

For $r = x_m + \log_2 x_m$, we must show

$$2 \leqq e^{a\beta} + e^{-a\alpha}\, \frac{\log\{e^{\log_2 x_m - a} + (e-1)\}}{\log\{e^{\log_2 x_m} + (e-1)\}}. \tag{3.8}$$

If $a \geqq 1/\beta$, the inequality is trivial. If $a \leqq 1/\beta$, choose m large enough so that

$$\frac{\log\{e^{\log_2 x_m - a} + (e-1)\}}{\log\{e^{\log_2 x_m} + (e-1)\}} \geqq e^{-a(\beta-\alpha)}.$$

Inequality (3.8) now follows immediately.

For $r = x_m + 2\log_2 x_m$, we must show

$$\psi(x_m + 2\log_2 x_m) \leqq \frac{e^{x_{m+1}\nu_{m+1}} - \psi(x_m + 2\log_2 x_m)}{x_{m+1} - (x_m + 2\log_2 x_m)}\, a + \psi(x_m + \log_2 x_m) e^{\beta(2\log_2 x_m - a)}.$$

This reduces to

$$1 \leqq A_m a + e^{-a\beta}, \tag{3.9}$$

where

$$A_m = \frac{e^{x_{m+1}\nu_{m+1}} - \psi(x_m + 2\log_2 x_m)}{\psi(x_m + 2\log_2 x_m)(N-2)\log_2 x_m}.$$

Using (3.1), (3.2), (3.3), and (3.4) we obtain

$$A_m \geqq \frac{e^{x_m(\nu_{m+1}-\nu_m)+(N\mu-(\beta+\alpha))\log_2 x_m}}{K(\log_2 x_m)^2} \to \infty .$$

When $A_m \geqq \beta$, inequality (3.9) is true.

Finally, for $r = x_{m+1}$ it is sufficient to show that for m large,

$$(3.10) \qquad 1 \leqq e^{\alpha a} \log \{e^a + (e-1)\} - a/3.$$

Inequality (3.10) follows on observing that the right-hand side is an increasing function of a $(a \geqq 0)$.

4. **Proof of Theorem 2.** By assumption,

$$(4.1) \qquad G(t)/t^\gamma \leqq (1+o(1))\, G(r_m)/r_m{}^\gamma \qquad (r_m{}' \leqq t \leqq r_m{}''),$$

where

$$(4.2) \qquad r_m{}' \to \infty, \qquad r_m/r_m{}' \to \infty, \qquad r_m{}''/r_m \to \infty$$

and

$$(4.3) \qquad G(t)/t^\beta \leqq (1+o(1))\, G(s_m)/s_m{}^\beta \qquad (s_m{}' \leqq t \leqq s_m{}''),$$

where

$$(4.4) \qquad s_m{}' \to \infty, \qquad s_m/s_m{}' \to \infty, \qquad s_m{}''/s_m \to \infty .$$

Considering, if necessary, a subsequence of the original sequence of peaks in (4.1) and (4.3), we may assume that

$$(4.5) \qquad s_1 < r_1 < s_2 < r_2 < \cdots .$$

Define a sequence $\{t_m\}$ by

$$(4.6) \qquad \frac{G(t_m)}{t_m{}^\alpha} = \sup_{I_m} \frac{G(t)}{t^\alpha},$$

where

$$(4.7) \qquad I_m = [r_m{}', s_{m+1}''].$$

If we can construct a sequence $\{v_m\}$ with v_m in the interval J_m,

$$(4.8) \qquad J_m = [r_m, s_{m+1}]$$

having the property that

$$(4.9) \qquad G(v_m)/v_m{}^\alpha \geqq (1+o(1))\, G(t_m)/t_m{}^\alpha,$$

then by (4.2), (4.4), (4.5), (4.6), (4.7), and (4.8), $\{v_m\}$ will be a sequence of Pólya peaks of the first kind of order α.

Define v_m as follows:

$$
\begin{aligned}
(4.10)\qquad v_m &= r_m && \text{if } r_m' \leqq t_m \leqq r_m,\\
&= t_m && \text{if } r_m < t_m \leqq s_{m+1},\\
&= s_{m+1} && \text{if } s_{m+1} < t_m \leqq s''_{m+1}.
\end{aligned}
$$

Clearly, v_m is in the interval (4.8). We will show that v_m satisfies (4.9).

If $v_m = t_m$, the result is obvious.

If $v_m = r_m$, by (4.1) and (4.10)

$$\frac{G(v_m)}{v_m{}^{\alpha}} \geqq (1 + o(1)) \frac{G(t_m)}{t_m{}^{\gamma}} v_m{}^{\gamma-\alpha} = (1 + o(1)) \frac{G(t_m)}{t_m{}^{\alpha}} \left(\frac{v_m}{t_m}\right)^{\gamma-\alpha}$$

which, in view of (4.10) and the definitions of γ and α, yields (4.9).

If $v_m = s_{m+1}$, by (4.3) and (4.10)

$$\frac{G(v_m)}{v_m{}^{\alpha}} \geqq (1 + o(1)) \frac{G(t_m)}{t_m{}^{\beta}} v_m{}^{\beta-\alpha} = (1 + o(1)) \frac{G(t_m)}{t_m{}^{\alpha}} \left(\frac{v_m}{t_m}\right)^{\beta-\alpha},$$

and using (4.10) and the definitions of β and α we again obtain (4.9).

The proof for Pólya peaks of the second kind is analogous and will be omitted.

5. **Proof of Theorem 3.** Assume there exists a $\sigma > 1$ and a sequence of Pólya peaks $\{r_m\}$ of the first kind of order $\Omega(\sigma) + \epsilon$ $(\epsilon > 0)$. Then for $m \geqq M_0$,

$$\frac{G(r_m/\sigma)}{(r_m/\sigma)^{\Omega(\sigma)+\epsilon}} \leqq (1 + o(1)) \frac{G(r_m)}{r_m{}^{\Omega(\sigma)+\epsilon}} . \tag{5.1}$$

Letting $\tilde{r}_m = r_m/\sigma$ in (5.1), we obtain

$$G(\sigma\tilde{r}_m)/G(\tilde{r}_m) \geqq (1 + o(1))\sigma^{\Omega(\sigma)+\epsilon}$$

which leads to

$$\liminf_{m\to\infty} \frac{G(\sigma\tilde{r}_m)}{G(\tilde{r}_m)} \geqq \sigma^{\Omega(\sigma)+\epsilon}$$

and contradicts (2.1).

If there existed a $\sigma > 1$ and a sequence of Pólya peaks $\{s_m\}$ of the second kind of order $\Omega(\sigma) + \epsilon$, we would have for $m \geqq M_1$,

$$\frac{G(\sigma s_m)}{(\sigma s_m)^{\Omega(\sigma)+\epsilon}} \geqq (1 + o(1)) \frac{G(s_m)}{s_m{}^{\Omega(\sigma)+\epsilon}}$$

which leads again to a contradiction of (2.1).

This shows that $\alpha \leqq \Omega(\sigma)$ for every σ. Hence $\alpha \leqq \Omega$.

The proof that $\alpha \geqq \kappa$ is similar and will be omitted.

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Clark University, Worcester, Massachusetts 01610