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GENERALIZATIONS OF MIDPOINT RULES

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ABSTRACT. A midpoint rule proposed by Jagermann and improved upon by Stetter is generalized to Hermite-type quadrature rules and to first degree cubature rules. Remainder terms are included in both cases.

1. **Introduction.** This note contains two types of generalizations of a midpoint rule proposed by Jagermann [1] and improved upon by Stetter [3]. The first generalization involves a Hermite type of midpoint rule and is discussed in §2. The second generalization concerns cubature rules for a function of two variables and is in §3. In both cases, error terms are included, from which asymptotic estimates can be derived.

2. **Hermite-type midpoint rules.** The integral to be approximated is $\int_a^b p(x)f(x)dx$, where $p(x) \geqq 0$, $p(x)$ does not vanish identically on any subinterval of $[a, b]$, and $\int_a^b p(x)dx = 1$, Stetter [3] has proved the following:

Let $N \geqq 1$ and

$$S_N(f) \equiv \int_a^b p(x)f(x)dx - \frac{1}{N}\sum_{i=0}^{N-1} f(a_i),$$

where $a_i = N\int_{x_i}^{x_{i+1}} tp(t)dt$, $i = 0, 1, \cdots, N-1$, and the x_i, $a = x_0 < x_1 < \cdots < x_N = b$, are chosen so that $1/N = \int_{x_i}^{x_{i+1}} p(x)dx$. Then $S_N(f) = \frac{1}{2} S_N(x^2)f''(\epsilon)$, $a < \epsilon < b$.

We generalize this theorem as follows:

THEOREM 1. *Let*

$$R_N^{(1)}(f) \equiv \int_a^b p(x)f(x)dx - \left[\frac{1}{N}\sum_{i=0}^{N-1} f(a_i) + \sum_{i=0}^{N-1} E_i(x)f'(a_i)\right],$$

where $p(x)$ *is as above,*

$$E_i(x) = \int_{x_i}^{x_{i+1}} xp(x)dx - a_i/N$$

and the a_i *are chosen so that*

$$\int_{x_i}^{x_{i+1}} p(x)(x - a_i)^2 dx = 0.$$

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Then

$$R_N^{(1)}(f) = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} p(x) f^{(3)}(\epsilon_i(x))(x - a_i)^{(3)} dx,$$

where $x_i < \epsilon_i(x) < x_{i+1}$ *and* $(x - a_i)^{(k)} \equiv (x - a_i)^k/k!$, *k a positive integer.*

Proof. First,

$$f(x) = f(a_i) + f'(a_i)(x - a_i) + f''(a_i)(x - a_i)^{(2)} + f^{(3)}(\epsilon_i(x))(x - a_i)^{(3)}$$

so that

$$\int_{x_i}^{x_{i+1}} p(x) f(x) dx = \frac{f(a_i)}{N} + f'(a_i) \int_{x_i}^{x_{i+1}} p(x)(x - a_i) dx + \int_{x_i}^{x_{i+1}} p(x) f^{(3)}(\epsilon_i(x))(x - a_i)^{(3)} dx.$$

A summation of the last equation on i from 0 to $N - 1$ completes the proof. Q.E.D.

We remark that Stetter's choice of the a_i was made so that

$$\int_{x_i}^{x_{i+1}} p(x)(x - a_i) dx = 0.$$

His definition of the x_i was the same as the above.

The definition of a_i given above is equivalent to

$$a_i = \left[\int_{x_i}^{x_{i+1}} x p(x) dx \pm \left\{ \left[\int_{x_i}^{x_{i+1}} x p(x) dx \right]^2 - \left[\int_{x_i}^{x_{i+1}} p(x) dx \int_{x_i}^{x_{i+1}} x^2 p(x) dx \right] \right\}^{1/2} \right] \Big/ \int_{x_i}^{x_{i+1}} p(x) dx$$

and it follows that the two possible values of a_i are both complex numbers. Either possible value may be used, but the function f must now be analytic at the a_i.

The remainder term $R_N^{(1)}$ cannot in general be simplified because the factor $(x - a_i)^{(3)}$ can be of variable sign. However this can be remedied as follows:

Theorem 2. *Let*

$$R_N^{(2)}(f) \equiv \int_a^b p(x)f(x)dx - \sum_{i=1}^{N-1} \left[\frac{f(a_i)}{N} + A_i f'(a_i) + B_i f''(a_i)\right],$$

where the x_i are as before and the a_i are chosen so that

$$\int_{x_i}^{x_{i+1}} p(x)(x - a_i)^3 dx = 0, \qquad A_i = \int_{x_i}^{x_{i+1}} p(x)(x - a_i)dx,$$

and

$$B_i = \int_{x_i}^{x_{i+1}} p(x)(x - a_i)^{(2)} dx.$$

Then $R_N^{(2)}(f) = f^{(4)}(\epsilon)R_N^{(2)}(x^{(4)}),\ a < \epsilon < b.$

PROOF. Now

$$f(x) = f(a_i) + f'(a_i)(x - a_i) + f''(a_i)(x - a_i)^{(2)} + f^{(3)}(a_i)(x - a_i)^{(3)} + f^{(4)}(\epsilon_i(x))(x - a_i)^{(4)}.$$

Multiplication of this equation by $p(x)$, integration from x_i to x_{i+1}, and summation on i from 0 to $N - 1$ yields the desired quadrature sum. The remainder term is

$$R_N^{(2)}(f) = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} p(x)f^{(4)}(\epsilon_i(x))(x - a_i)^{(4)} dx$$

$$= \sum_{i=0}^{N-1} f^{(4)}(\epsilon_i) \int_{x_i}^{x_{i+1}} p(x)(x - a_i)^{(4)} dx$$

$$= f^{(4)}(\epsilon) \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} p(x)(x - a_i)^{(4)} dx$$

by the application of the two mean value theorems. Finally, if $R_N^{(2)}(f) = f^{(4)}(\epsilon)C_N$, C_N a constant, then $C_N = R_N^{(2)}(x^{(4)})$ by inspection. Q.E.D.

The deeper reason that $R_N^{(2)}(f)$ has a simpler form than $R_N^{(1)}(f)$ is that the Peano kernel is of one sign for $R_N^{(2)}(f)$.

Since a_i in Theorem 2 must satisfy a cubic equation, there are three possible choices for a_i. At least one of these is real and it must be in (x_i, x_{i+1}) since, if not, the conditions on p imply that $\int_{x_i}^{x_{i+1}} p(x)(x - a_i)^3 dx = 0$ is impossible. Although Theorem 2 is true for all three choices, the real root is the one that should be used. The fact that there is a real root for a_i makes Theorem 2 an improvement over Theorem 1. Also f need only be in $C^4[a, b]$ for Theorem 2 rather than analytic as in Theorem 1.

We remark that Theorems 1 and 2 can, of course, be generalized to higher-order rules.

3. **Cubature midpoint rules.** We discuss two cubatures to approximate the integral $\int_a^b \int_c^d p(x, y)f(x, y)dydx$, $p(x, y) \geqq 0$ and > 0 except on a set of measure zero. The triangular Taylor's expansion is the following [2]:

$$
\begin{aligned}
f(x, y) &= f(a_i, b_j) + f_{1,0}(a_i, b_j)(x - a_i) + f_{0,1}(a_i, b_j)(y - b_j) \\
&\quad + \tfrac{1}{2}\,[f_{2,0}(\epsilon, \eta)(x - a_i)^2 + 2f_{1,1}(\epsilon, \eta)(x - a_i)(y - b_j) \\
&\qquad + f_{0,2}(\epsilon, \eta)(y - b_j)^2],
\end{aligned} \tag{1}
$$

ϵ between x and a_i and η dually. We must assume that there exist x_i and y_j such that

$$\frac{1}{MN} = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} p(x, y)dydx. \tag{2}$$

(In the case of functions of one variable, $\{x_i\}$ such that $1/M = \int_{x_i}^{x_{i+1}} p(x)dx$ exist because the positivity of $p(x)$ ensures the existence of the appropriate inverse function. Cf. [3].) A special case in which the above always holds is if $p(x, y) = p_1(x)p_2(y)$ where p_1 and p_2 are both $\geqq 0$ and > 0 except on a set of measure zero. However, there are examples in which equation (2) holds and $p(x, y)$ is not of the form $p_1(x)p_2(y)$. Such an example can be constructed as follows: On each subrectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, let $p(x, y)$ be a pyramid that is zero on the boundary of the subrectangle and has positive height h_{ij} such that (2) obtains. Specifically, for the rectangle $[-1, 1] \times [-1, 1]$, $p(x, y)$ is defined in the figure below.

Thus (2) is equivalent to the following:

$$1/MN = [(x_{i+1} - x_i)(y_{j+1} - y_j)h_{ij}]/3,$$

so that

$$h_{ij} = 3/[MN(x_{i+1} - x_i)(y_{j+1} - y_j)].$$

Now we multiply equation (1) by $p(x, y)$ and integrate from x_i to x_{i+1} and y_j to y_{j+1}. Since

$$\frac{1}{MN} = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} p(x, y)dydx$$

the cubature sum

$$\frac{1}{MN} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f(a_i, b_j)$$

is exact for constant functions. Let the a_i and b_j be so chosen that

$$\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} p(x, y)(x - a_i)dxdy = 0$$

and

$$\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} p(x, y)(y - b_j)dxdy = 0,$$

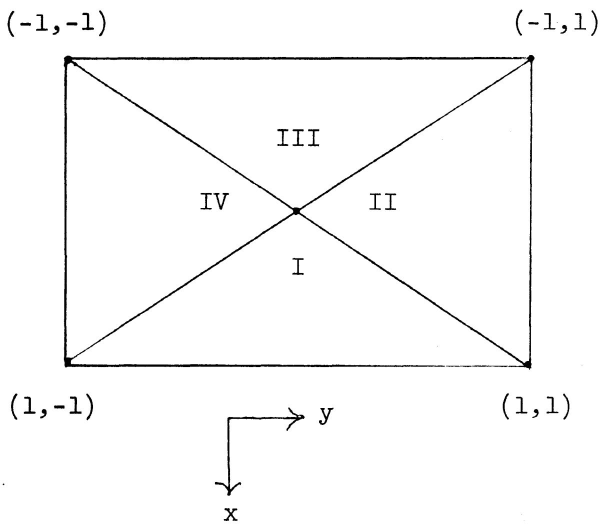
i.e., if

$$I_{ij}(g) \equiv \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} p(x, y)g(x, y)dydx,$$

then

$$(3) \qquad a_i = \frac{\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} p(x, y)x\, dxdy}{\iint p(x, y)dxdy} \equiv \frac{I_{ij}(x)}{I_{ij}(1)} \quad \text{and} \quad b_j = \frac{I_{ij}(y)}{I_{ij}(1)}.$$

$$p(x,y) = \begin{cases} h(1-x) & \text{on I} \\ h(1-y) & \text{on II} \\ h(1+x) & \text{on III} \\ h(1+y) & \text{on IV} \end{cases}$$



THEOREM 3. *Let*

$$R_{MN}(f) = \int_a^b \int_c^d p(x, y)f(x, y)dydx - \frac{1}{MN}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(a_i, b_j),$$

where $p(x, y)$ is as above, the a_i and b_j are as in equation (3) *and the x_i and y_j as in* (2). *Then $R_{MN}(f)$ has the following representations*:

$$\begin{aligned} R_{MN}(f) &= f_{2,0}(\tilde{\epsilon}, \tilde{\eta})R_{MN}(x^{(2)}) + f_{0,2}(\gamma, \delta)R_{MN}(y^{(2)}) \\ &+ \sum_{i,j}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} p(x, y)f_{1,1}(\epsilon_i(x), \eta_j(y))(x - a_i)(y - b_j)dxdy, \end{aligned} \tag{4}$$

$a < \tilde{\epsilon}, \gamma < b$ *and* $\tilde{\eta}$, δ *dually.*

$$\begin{aligned} R_{MN}(f) = &\sum_{i,j} f_{1,1}(a_i, b_j) \\ &\cdot \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} p(x, y)(x - a_i)(y - b_j)dydx \\ &+ f_{2,0}(\alpha_1, \beta_1)R_{MN}(x^{(2)}) + f_{0,2}(\alpha_2, \beta_2)R_{MN}(y^{(2)}) \\ &- f_{2,2}(\alpha_3, \beta_3)R_{MN}(x^{(2)}y^{(2)}), \end{aligned} \tag{5}$$

$a < \alpha_i < b$, $i = 1, 2, 3$ *and the* β_i *dually.*

PROOF. 1. Equation (4) follows from (1), since

$$\begin{aligned} R_{MN}(f) = \frac{1}{2}\sum_{i,j}\Big[&\int\int p(x, y)f_{2,0}(\epsilon, \eta)(x - a_i)^2dxdy \\ &+ 2\int\int p(x, y)f_{1,1}(\epsilon, \eta)(x - a_i)(y - b_j)dxdy \\ &+ \int\int p(x, y)f_{0,2}(\epsilon, \eta)(y - b_j)^2dxdy\Big], \end{aligned}$$

where the integrals are over $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and $\epsilon = \epsilon_i(x)$, $\eta = \eta_j(y)$. The two mean value theorems can be used on the first and third integrals, but not on the second one. E.g., the first sum becomes $f_{2,0}(\tilde{\epsilon}, \tilde{\eta}) \sum_{i,j} I_{ij}[(x - a_i)^{(2)}]$ and this equals $f_{2,0}(\tilde{\epsilon}, \tilde{\eta})R_{MN}(x^{(2)})$, as can be seen by applying R_{MN} to $x^{(2)}$.

2. We could consider the $f_{1,1}$ terms in (4) as part of the cubature sum, analogous to §2. If not, then it is desirable to get $f_{1,1}$ outside the integral and this is the motivation for (5).

Use a rectangular Taylor's expansion [2] as follows:

$$f(x, y) = f(a_i, b_j) + f_{1,0}(a_i, b_j)(x - a_i) + f_{0,1}(a_i, b_j)(y - b_j) + f_{1,1}(a_i, b_j)(x - a_i)(y - b_j) + R(f),$$

where

$$R(f) = (x - a_i)^{(2)} f_{2,0}(\epsilon, y) + (y - b_j)^{(2)} f_{0,2}(x, \eta) - (x - a_i)^{(2)}(y - b_j)^{(2)} f_{2,2}(\epsilon, \eta).$$

Multiply by $p(x, y)$, integrate over $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and sum on i and j to obtain:

$$\iint pR(f) = \iint p(x, y)(x - a_i)^{(2)} f_{2,0}(\epsilon(x), y)dxdy + \iint p(x, y)(y - b_j)^{(2)} f_{0,2}(x, \eta(y))dxdy - \iint p(x, y)(x - a_i)^{(2)}(y - b_j)^{(2)} f_{2,2}(\epsilon(x), \eta(y))dxdy.$$

The application of the two mean value theorems yields the conclusion. Q.E.D.

We remark that the idea used for one variable of using the a_i to move out further in the Taylor's expansion (e.g., to include f' terms in the quadrature sum) is not effective for two variables because of the binomial effect inherent in two-dimensional Taylor's expansions.

4. **Example.** Let $p(x,y) \equiv 1$, $[a, b] = [c, d] = [0, 1]$. We consider Theorems 1–3 for this case. In Theorem 1, $x_i = i/N$, $i = 0, \cdots, N$, and

$$a_i = \frac{x_{i+1} + x_i}{2} \pm \frac{x_{i+1} - x_i}{2.3^{1/2}}(-1)^{1/2}, \qquad i = 0, \cdots, N - 1.$$

I.e.,

$$a_i = \frac{2i + 1}{2N} \pm \frac{(-1)^{1/2}}{2 \cdot 3^{1/2} N}.$$

In Theorem 2, the equation for a_i is the following:

$$a_i^3 - 3a_i^2(x_{i+1} + x_i)/2 + a_i(x_{i+1}^2 + x_{i+1}x_i + x_i^2) - (x_{i+1}^3 + x_{i+1}^2 x_i + x_{i+1}x_i^2 + x_i^3) = 0.$$

If $N = 1$, then $a_0 = 1/2$, for example.

In Theorem 3, equation (3) can be simplified. In fact, if $p(x, y) = p_1(x)p_2(y)$, then a_i is the same as in [3], i.e.,

$$a_i = \int_{x_i}^{x_{i+1}} p_1(x)x\,dx \Big/ \int_{x_i}^{x_{i+1}} p_1(x)\,dx.$$

Then the cubature rule in Theorem 3 is a cross-product rule.

Recalling the notation of §2, we note that, for the above case, Stetter showed that

$$S_N(x^2) = 1/12N^2.$$

The remainder terms in Theorems 1–3 permit us to determine analogous results for the integration rules concerned. Noting the above equation for a_i in Theorem 1, we see that

$$R_N^{(2)}(f) = O(N^{-3}\|f^{(3)}\|),$$

where norm on $f^{(3)}$ is the sup norm on $[0, 1]$. Similarly, in Theorem 2,

$$R_N^{(2)}(x^{(4)}) = O(N^{-4}),$$

so that

$$R_N^{(2)}(f) = O(N^{-4}\|f^{(4)}\|).$$

In Theorem 3, by the quadrature results, equation (4) yields

$$R_{MN}(f) = O\left(\frac{1}{M^2}\|f_{2,0}\|\right) + O\left(\frac{1}{N^2}\|f_{0,2}\|\right) + O\left(\frac{1}{MN}\|f_{1,1}\|\right),$$

where the norm is the sup norm on $[0, 1] \times [0, 1]$. Equation (5) yields

$$R_{MN}(f) = O\left(\frac{1}{MN}\|f_{1,1}\|\right) + O\left(\frac{1}{M^2}\|f_{2,0}\|\right) + O\left(\frac{1}{N^2}\|f_{0,2}\|\right) + O\left(\frac{1}{M^2N^2}\|f_{2,2}\|\right).$$

These asymptotic estimates illustrate the idea motivating Theorems 1 and 2, as well as showing a connection between the cubature and quadrature results.

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