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REGULARITY THEOREMS AND GERŠGORIN THEOREMS FOR MATRICES OVER RINGS WITH VALUATION

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ABSTRACT. The collection of root-location theorems for matrices of complex numbers is now quite extensive. Since their proofs involve chiefly manipulation of absolute value inequalities, many of these theorems can be extended to noncommutative domains, in particular to quaternion matrices. Secondly, the ring of polynomials has a valuation with properties that differ slightly from those of the ordinary absolute value function. Using this valuation, a different type of regularity theorem is obtainable. With a suitable definition of proper value of a matrix of polynomials, these regularity theorems also lead to root-location theorems. Finally, bounds for determinants can be obtained. These bounds are given in terms of the valuation: for polynomials, they are bounds on the degree.

1. **Introduction.** Let $A = [a_{ij}]_1^n$ be a matrix of complex numbers. The curtate row and column-sums $R_{i,p}$, $C_{i,p}$ are defined by

$$R_{i,p}^p = \sum_{j \neq i} |a_{ij}|^p, \qquad C_{i,p}^p = \sum_{j \neq i} |a_{ji}|^p.$$

Thus $R_i \equiv R_{i,1}$ is the sum of the absolute values of the nondiagonal elements of the ith row; $C_i \equiv C_{i,1}$ is the sum of the absolute values of the nondiagonal elements of the ith column. A is called regular if no nonzero vector x exists such that $Ax = 0$. If A is regular, so is A^T, its transpose; and A is invertible. The constants used below are real and subject to $0 \leqq \alpha, \beta, \gamma \leqq 1$; $1 \leqq p, p', p''$; $\alpha + \beta + \gamma = 1$;

$$p^{-1} + q^{-1} = p'^{-1} + q'^{-1} = p''^{-1} + q''^{-1} = 1;$$
$$0 \leqq k_i, \quad \sum (k_i + 1)^{-1} \leqq 1.$$

$R_{i,\infty} = m_i$ is the maximum of the absolute values of the nondiagonal elements of the ith row of A; $C_{i,\infty} = c_i$.

The following hypotheses are known to guarantee the nonsingularity of A.

(1.1) $$\forall_i \{|a_{ii}| > R_i\} \qquad [8].$$

(1.2) $$\forall_i \{|a_{ii}| > C_i\}.$$

(1.3) $$\forall_i \{|a_{ii}| > R_i^{\alpha} C_i^{1-\alpha}\} \qquad [15].$$

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(1.4) $\forall_i\{|a_{ii}| > k_i{}^{1/q} R^{\alpha}_{i,\alpha p} C^{1-\alpha}_{i,\,(1-\alpha)p}\}$ [16].

(1.5) $\forall_i\{|a_{ii}| > k_i m_i{}^{\alpha} c_i{}^{1-\alpha}\}$ [16].

(1.6) $\forall_i\{|a_{ii}| > k_i{}^{1/e} R^{\alpha}_{i,\alpha p} R^{\beta}_{i,\beta p'} R^{\gamma}_{i,\gamma p''}\}$
$(0 < e = pp'p''/(pp'p'' - pp' - pp'' - p'p''))$ [5].

There are corresponding theorems (see below) that relate to an arbitrary partitioning of A. Let the indices $\{1, \cdots, n\}$ be partitioned into mutually exclusive sets, and let $I(i)$ denote the set to which the index i belongs. (For a slight generalization involving overlapping sets, see [4], [11].) Let

$$b_{ij} = \det A\begin{pmatrix} I(i) \\ I(i)\backslash i, j \end{pmatrix}$$

denote the determinant of the matrix based on rows $I(i)$, and columns $[I(i)$, but with the ith column replaced by the jth$]$. If $B = [b_{ij}]_1^n$, satisfies hypotheses corresponding to (1.1)–(1.6), then both A and B are nonsingular. This gives six more theorems, which we number (1.7)–(1.12).

From each of the twelve nonsingularity theorems, and from any other theorem of the same type, a root-location theorem is obtainable. To see this, let λ be a proper value ($\equiv$ root) of A, set $C = A - \lambda I$, and note that C is singular. Thus corresponding to (1.3) for example, the assertion

$$\exists_i\{|a_{ii} - \lambda| \leqq R_i^{\alpha} C_i^{1-\alpha}\} \tag{1.15}$$

is valid. Assertions (1.13)–(1.24) are explained similarly. Note that (1.19)–(1.24) involve minors of the matrix C, not of the matrix A.

2. **Generalization to other domains.** The proofs of the above theorems depend on standard inequalities (triangle, Hölder) satisfied by the absolute value function of a complex number. Such a function exists over quaternions. Moreover, determinant, proper value, and regularity can all be defined for matrices of quaternions [2], [7]. (The vector x ($x \neq 0$) is a proper vector and λ a proper value of A if the relation $Ax = x\lambda$ holds. If λ is a proper value, so is $\rho\lambda\rho^{-1}$.)

Thus assertions (1.1)–(1.24) hold for matrices of quaternions. We number these assertions, applied to quaternions, (2.1)–(2.24). The proofs remain the same: for expository purposes, the proof of (2.1) is given.

(2.1) Let $A = [a_{ij}]$ be a matrix of quaternions; define $R_i = \sum_{j\neq i}|a_{ij}|$. Then $[\forall_i\{|a_{ii}| > R_i\}] \Rightarrow A$ is invertible.

PROOF. The hypothesis guarantees that $a_{11} \neq 0$. Quaternions form a division ring. Thus a multiple of the first column can be subtracted from each of the other columns to form a matrix $B = [b_{ij}]$ with the following property: $b_{1j} = 0$ if $j > 1$. The matrix transformation $A \rightarrow B$ can be represented as a matrix equation

$$AE_2 \cdots E_n = B,$$

where E_j are (elementary) invertible matrices. Since B has the form

$$\begin{bmatrix} a_{11} & 0 \\ * & B_1 \end{bmatrix},$$

either A is invertible as claimed, or else B_1 is not invertible. But B_1 has lower dimension than B; a simple induction shows that if B (and hence B_1) is not invertible, there must exist a nonzero vector y (obtained by appending an initial zero to the nonzero vector connected with B_1) such that $By = 0$. Setting $E_1 \cdots E_n y = x$, it follows that if A is not invertible, a nonzero vector x exists such that $Ax = 0$. With this established, the rest of the proof is standard:

Let $x = (x_1, \cdots, x_n)$; define k so that $\forall_i\{|x_k| \geqq |x_i|\}$. Since $Ax = 0$, the relation $a_{kk}x_k = -\sum_{j \neq k} a_{kj}x_j$ holds. Now divide by x_k (which cannot be 0), take absolute values of both sides, and apply the triangle inequality to obtain the contradiction $|a_{kk}| \leqq R_k$. ||

The root-location theorem 2.13 (corresponding to 1.13) states that every proper value of A is in the union of n hyperspheres.

The above methods can be further generalized. In the first place, any field or division ring that has a valuation can be substituted for the quaternion ring. It should be noted that for noncommutative division rings it is not necessary to distinguish between left- and right-proper values [**2**].

In the second place, the results extend easily to certain commutative rings with valuation: principal ideal rings (in particular euclidean rings) and local rings. (Here, a local ring is defined as a ring with the property that for every pair of elements, at least one is a divisor of the other.) In all cases it is required in addition that the ring have a valuation (so that the hypotheses (1.1) to (1.12) make sense.)

The key lemma is the following:

2.25. LEMMA. *Let A be a matrix with elements from a principal ideal ring. If A is not regular, there exists a nonzero vector x such that $Ax = 0$, and conversely.*

A regular matrix is one with determinant not a zero divisor. The proof of Lemma 2.25 will be indicated in case the ring is euclidean.

For a general principal ideal ring, the arguments are similar; see [**10**], [**19**].

The following preliminary assertion is needed.

2.26. Assertion. Two invertible matrices N, M always exist so that NAM has the form

$$\begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

Proof. For example if the first two elements of the first row of A are nonzero, the euclidean hypothesis shows that at least the absolute value of the larger can be reduced, as suggested by the equation

$$\begin{bmatrix} a & b & * \\ & * & \\ & & \end{bmatrix}\begin{bmatrix} 0 & 1 & \\ 1 & q & * \\ & * & \end{bmatrix}=\begin{bmatrix} b, & a+bq & * \\ & * & \\ & & \end{bmatrix},$$

or, if necessary, by the equation

$$[a\, b]\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} = [a, b+aq].$$

This argument is the main step. In fact suppose no product AM of A by an invertible matrix M could have fewer than k (>1) nonzero elements in the first row. By a permutation AMP, bring two nonzero first row elements to the first two positions; apply the argument just outlined.

If all the elements of the first row of A are 0, the above considerations must be applied to PA, where P is some permutation matrix (this interchanges rows).

Proof of Lemma 2.25. Let N, M be invertible matrices such that NAM has the form

$$\begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

If a is not regular, the assertion is clear. Note that $\det A = a \det B$, so that either B is not regular, or else A is regular, or else a is not regular.

Since B has lower dimension than A, the lemma follows by an inductive argument based on the dimension of the matrix A.

A similar device can be used to prove, for local rings, a lemma similar to Lemma 2.25.

The implements are now at hand to extend theorems (2.1) to (2.12) to principal ideal rings and local rings with valuation. The conclusion of the theorems must be modified, however: invertibility does not follow from the hypotheses, but only regularity. In the more general context, a matrix is called regular if its determinant is neither zero nor a zero divisor.

To obtain root-location theorems for principal ideal and local rings, the root concept must be defined.

2.27. DEFINITION. Let A be a matrix over the ring R. A right-proper value (root) of A is an element λ of R such that, for some nonzero vector x with elements in R, the relation $Ax = x\lambda$ holds.

It is known that if R is a division ring, it is not necessary to distinguish between left- and right-proper values, [2].

Also every matrix over the quaternions does have a proper value. Definition 2.27 does not however assure the existence of a proper value.

2.28. THEOREM. *Let A be a matrix with elements from the commutative ring R. If λ is a proper value of A, then $A - \lambda I$ is not regular. If R is a principal ideal ring or a local ring, the converse is valid.*

PROOF. The fact $(Ax = x\lambda)$ that λ is a proper value can be written in the form $Ax = (\lambda I)x$. ‖ Note that commutativity seems to be an essential hypothesis.

The converse, in the cases mentioned, follows from Lemma 2.25.

3. **Matrices of polynomials.** Let $R = F[x]$ be the ring of polynomials in a single indeterminate over a field—or more generally over any commutative principal ideal ring S without zero divisors. It is not required that the ring S have a valuation. The ring R can be valued in the following sense.

3.1. DEFINITION. The value $|f|$ of the polynomial $f(x)$ is $2^{\deg f}$. The value of the zero polynomial is 0.

3.2. THEOREM. *If f, g are two polynomials, then*

$$\{|f| = 0\} \Longleftrightarrow \{f = 0\}, \tag{3.3}$$

$$|fg| = |f| \cdot |g|, \tag{3.4}$$

$$|f + g| \leqq \max(|f|, |g|). \tag{3.5}$$

Note that (3.3) and (3.4) mirror the corresponding property for multiplication of complex numbers, but (3.5) does not.

Since R is itself a principal ideal ring [**10**], Lemma 2.25 is valid. However the theorems of §1 are only valid if the row-sums $R_{i,p}$ are replaced by something else.

3.6. DEFINITION. For a matrix $A = [a_{ij}]$ of polynomials over a field (or commutative principal ideal domain), the generalized ith row-sum is given by $R_i = \max_{j \neq i}\{|a_{ij}|\}$, where the value $|a_{ij}|$ is given by Definition 3.1. Also, $C_i = \max_{j \neq i}\{|a_{ji}|\}$.

Let $A = [a_{ij}]_1^n$ be a matrix of polynomials. Any one of the following hypotheses is sufficient to guarantee the regularity of A ($0 \leqq \alpha \leqq 1$).

(3.7) $\forall_i\{|a_{ii}| > R_i\}$,

(3.8) $\forall_{i>k}\{|a_{ii}| \cdot |a_{kk}| > R_i R_k\}$,

(3.9) $\forall_i\{|a_{ii}| > R_i^\alpha C_i^{1-\alpha}\}$,

(3.10) $\forall_{i>h}\{|a_{ii}| \cdot |a_{hh}| > R_i^\alpha R_h^\alpha C_i^{1-\alpha} C_h^{1-\alpha}\}$.

Since the arguments differ somewhat from the proofs of the corresponding theorems for complex numbers, some detail is given.

Proof that (3.7) $\Rightarrow A$ is regular.

Assume A not regular, apply Lemma 2.25, and obtain $Ax = 0$, where $x = [x_1, x_2, \cdots, x_n]$ is a nonzero (column-) vector of polynomials. Choose k so that $\forall_j\{|x_k| \geqq |x_j|\}$ (see 3.1). From $Ax = 0$, the relation

(3.11) $$a_{kk}x_k = -\sum_{j \neq k} a_{kj}x_j$$

follows. The proof is finished by noting the consequent relations

(3.12) $$|a_{kk}x_k| \leqq \max_j |a_{kj}x_j| \leqq \max_j |a_{kj}x_k| = |x_k| \max_j |a_{kj}| = |x_k| R_k,$$

which contradict $|a_{kk}| > R_k$. ||

Proof that (3.8) $\Rightarrow A$ is regular. Similar to the preceding proof, with x_k, x_ℓ chosen so that

(3.13) $$\forall_{j\neq k}\{|x_k| \geqq |x_\ell| \geqq |x_j|\}.$$

Note that $x_\ell \neq 0$, since otherwise x_k would be the only nonzero component of x. This would force $a_{kk} = 0$ because $Ax = 0$. But if a_{kk} were 0, (3.8) could not hold.

The condition $Ax = 0$ does result in

(3.14) $$a_{kk}x_k = -\sum_{j\neq k} a_{kj}x_j,$$

(3.15) $$a_{\ell\ell}\, x_\ell = -\sum_{j\neq \ell} a_{\ell j}x_j.$$

If (3.8) is now applied, along with the argument (3.12), the intermediate results $|a_{kk}| \cdot |x_k| \leqq |x_\ell| \cdot R_k$, $|a_{\ell\ell}| \cdot |x_\ell| \leqq |x_k| \cdot R_\ell$ are obtained. Since $|x_k x_\ell| \neq 0$, this shows (by multiplication and cancellation) that $|a_{kk}a_{\ell\ell}| \leqq R_k R_\ell$, which contradicts the hypothesis. Thus A is regular. ||

Proof that (3.9) $\Longleftrightarrow A$ is regular. If A is singular, a column-vector $[x_1, \cdots, x_n]$ of polynomials exists so that $Ax = 0$. Thus if k is chosen so that $\forall_k\{|x_k| \geqq |x_i|\}$, then it must be true that

$$|a_{ii}x_i| \leqq \max_j |a_{ij}|\,|x_j| \leqq R_i^{\alpha} \max_j \{|a_{ij}|^{1-\alpha}|x_j|\}. \tag{3.16}$$

Thus if the hypothesis is satisfied, and if $|x_i| \neq 0$, then

$$C_i^{1-\alpha}|x_i| < \max_j \{|a_{ij}|^{1-\alpha}|x_j|\}. \tag{3.17}$$

Applying $\max_i$ to both sides of (3.17) gives the relation

$$\max_i \{C_i^{1-\alpha}|x_i|\} < \max_j \max_i \{|a_{ij}|^{1-\alpha}|x_j|\} = \max_j \{C_j^{1-\alpha}|x_j|\}. \tag{3.18}$$

The inequality sign can never degenerate to equality. This contradiction shows that A is regular. ||

Proof that (3.10)$\Longrightarrow A$ is regular. This proof is similar to the preceding one. It is first necessary to remark that in this case, at least two components x_k, x_h of x are nonzero, since otherwise $Ax = 0$ implies $a_{kk} = 0$ for some k, contradicting (3.10). Finally $Ax = 0$ contradicts (3.10) anyway; multiply two forms of (3.17) (with i replaced by s on the one hand and by t on the other hand), take $\max_{s>t}$ in this putative product, and reverse the order of the max operations. ||

REMARK. The assertion just proved includes the others as special cases.

These four theorems seem to be the natural generalizations of (1.1) to (1.6) to matrices of polynomials. There are also generalizations of (1.7) to (1.12). In fact the quantities b_{ij} of §1 still make sense (with the same formal definition), and it is still the case that det B is equal to det A multiplied by a polynomial function of the elements a_{ij}. The details of this assertion, which is proved in [5], are:

3.19. THEOREM. *Let $A = [a_{ij}]_1^n$ be a matrix of indeterminates. Let the indices* 1, $\cdots$, *n be partitioned into disjoint sets* $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_t$. *Let b_{ij} be defined by*

$$b_{ij} = \det A \begin{pmatrix} \mathcal{S} \\ \mathcal{S}\backslash i,j \end{pmatrix},$$

where $\mathcal{S}$ is the set to which the index i belongs, and $\mathcal{S}\backslash i, j$ denotes the column-set $\mathcal{S}$, but with the ith column replaced by the jth column. The determinant of $B = [b_{ij}]$ is equal to

$$\det B = \det A \prod_1^t \left[\det A \begin{pmatrix} \mathcal{S}_u \\ \mathcal{S}_u \end{pmatrix}\right]^{|\mathcal{S}_u|-1}, \tag{3.20}$$

where $|\mathcal{S}_u|$ denotes the cardinality of $\mathcal{S}_u$.

Since the postfactor in (3.20) is never 0 if no diagonal element of the matrix B is 0, the following four theorems are valid. They are companions, applied to B, of theorems (3.7) to (3.10) applied to A.

3.21. THEOREM. *Let $A = [a_{ij}]$ be a matrix of polynomials. Define b_{ij} as the determinant*

$$\det \begin{pmatrix} I(i) \\ I(i)\backslash i, j \end{pmatrix}$$

of an almost principal minor as explained in §1. Let $R_i(B)$, $C_i(B)$ be defined by

$$R_i(B) = \max_{j \neq i} |b_{ij}|, \qquad C_i(B) = \max_{j \neq i} |b_{ji}|, \tag{3.22}$$

with $|f| = 2^{\deg f}$, $|0| = 0$. Any one of the following conditions is sufficient that A be regular [α *fixed*, $0 \leqq \alpha \leqq 1$]:

$$\forall_i \{ |b_{ii}| > R_i(B) \}, \tag{3.23}$$

$$\forall_{i>j} \{ |b_{ii} b_{jj}| > R_i(B) R_j(B) \}, \tag{3.24}$$

$$\forall_i \{ |b_{ii}| > R_i(B)^\alpha C_i(B)^{1-\alpha} \}, \tag{3.25}$$

$$\forall_i \{ |b_{ii} b_{jj}| > R_i(B)^\alpha R_j(B)^\alpha C_i(B)^{1-\alpha} C_j(B)^{1-\alpha} \}. \tag{3.26}$$

4. **Bounds for determinants.** For a matrix of complex numbers that satisfies one of the hypotheses (1.1) to (1.12), bounds can be given for the (absolute value of) its determinant. See [**1**], [3], [**18**]. The proofs of these results carry over to the case of quaternions: without change for hypotheses (1.1) to (1.6); with some changes in the case of (1.7) to (1.12). The definition of determinant of a matrix of quaternions is nonstandard: see [**2**]. Although this question is not without interest, it is not discussed further here.

For a matrix of polynomials satisfying (3.7) to (3.10) or (3.25) to (3.28), a bound for the value of the determinant can also be given. It is amusing that, in all cases, this bound is exact.

4.1. THEOREM. *Let $A = [a_{ij}]$ be a matrix of polynomials that satisfies the hypotheses of any one of the theorems* (3.7), (3.10), (3.14), (3.18). *Then $F = \det A$ is a polynomial, and the relation*

$$|F| = \prod |a_{ii}| \tag{4.2}$$

holds.

FIRST PROOF. Comparison of the principal term $\prod a_{ii}$ with the other $n! - 1$ terms in the expansion of det A shows that the other terms all have degree lower than the principal one. ||

For the hypothesis of (3.14), for example,

$$|a_{11}a_{22}a_{33}| > |a_{12}|^{\alpha}|a_{31}|^{1-\alpha}\cdot|a_{31}|^{\alpha}|a_{23}|^{1-\alpha}\cdot|a_{23}|^{\alpha}|a_{12}|^{1-\alpha}$$
$$= |a_{12}a_{23}a_{31}|.$$

For the hypothesis of (3.18), write

$$|a_{11}a_{22}a_{33}| = |a_{11}a_{22}|^{1/2}|a_{22}a_{33}|^{1/2}|a_{33}a_{11}|^{1/2}. \qquad \|$$

Remark. The argument shows why a matrix satisfying the hypotheses of (3.7) to (3.10) is said to have "dominant diagonal." The principal term in the expansion, the product of the diagonal elements, "dominates" the other terms in the expansion, since (3.5) is operative. Although the term "dominant diagonal" had been used for many years, I could not accept this term until after I made this discovery [**6**].

Second Proof (valid for (3.7) and (3.8) only). This proof is based on a polynomial identity expressing the determinant of a matrix in terms of the determinant of a matrix, the elements of which are certain 2×2 minors of the original matrix. The following identity for the 3×3 matrix $A = [a_{ij}]_1^3$ is an example:

$$(4.3) \qquad a_{11} \det A = \det \begin{bmatrix} \det A\begin{pmatrix} 12 \\ 12 \end{pmatrix}, & \det A\begin{pmatrix} 12 \\ 13 \end{pmatrix} \\ \det A\begin{pmatrix} 13 \\ 12 \end{pmatrix}, & \det A\begin{pmatrix} 13 \\ 13 \end{pmatrix} \end{bmatrix}.$$

(4.3) is a special case of the identity

$$(4.4) \qquad a_{11}^{n-2} \det A = \det B,$$

where the (i, j) element b_{ij} of the $n - 1 \times n - 1$ matrix B is given by the formula

$$b_{ij} = \det A\begin{pmatrix} 1 & i+1 \\ 1 & j+1 \end{pmatrix}.$$

Formula (4.4) appears in disguised form in several places; its proof is not difficult. Form the product $A \operatorname{diag}(1, a_{11}, a_{11} \cdots a_{11})$; in this product add $-a_{ij}$ times the first column to the jth column, $j = 2(1)n$. The resulting matrix has the form

$$\begin{bmatrix} a_{11} & 0 \\ * & B \end{bmatrix};$$

its determinant is clearly $a_{11} \det B$. From this fact, (4.4) follows.

The following lemma amounts to a statement that, if A has dominant diagonal, then B has dominant diagonal.

4.5. LEMMA. *Let A be a matrix of polynomials that satisfies the hypotheses* (3.7), (3.8). *Then* $B = [b_{ij}]$, *with*

$$b_{ij} = \det A\begin{pmatrix} 1 & i+1 \\ 1 & j+1 \end{pmatrix},$$

satisfies the corresponding hypothesis. (*As noted below, the special subscript* 1 *has at times to be replaced by another.*)

PROOF. Consider (3.7): $\forall_i\{|a_{ii}| > R_i\}$. It has to be proved that $|b_{jj}| > \max_k |b_{jk}|$ for $i, j > 1$. This amounts to $\deg(a_{11}a_{jj} - a_{1j}a_{j1}) > \deg(a_{11}a_{jk} - a_{1k}a_{j1})$. ||

Consider (3.8). The hypothesis is $\forall_{i>j}\{|a_{ii}a_{jj}| > R_iR_j\}$. The conclusion of Lemma 4.5 is certainly valid if the stronger hypothesis $\forall_i\{|a_{ii}| > R_i\}$ is valid. Now this stronger hypothesis can fail for at most one value of i. If the value of i for which failure occurs is $i = 1$, the above proof goes through; the hypothesis $|a_{11}a_{jj}| > |a_{1k}a_{j1}|$ is needed. (If failure occurs for some other value of i, the lemma is not true, but we modify the lemma in such a case.)

The second proof of Theorem 4.1 is now easily completed by induction. Exchange two indices of A if necessary so that Lemma 4.5 applies. The principal diagonal of B dominates. But B has lower dimension than A. Thus, $|\det B| = \prod |b_{jj}|$. But $|b_{ii}| = |a_{11}a_{jj}|$. Relation (4.4) now completes the proof.

5. **Complements to the above results.**

(5.1) In §1, the matrix B is defined from A, after the indices $\{1, \cdots, n\}$ have been partitioned into mutually exclusive sets. A slight generalization is available; it is not required that the sets be mutually exclusive, but only for each index i, there be a collection of indices $I(i)$ that contains i [**4**], [**11**].

(5.2) A generalization of (1.1) is known [**9**], [**12**]. The hypothesis is altered by redefining R_i as follows:

(5.3)
$$R_1 = \sum_{j>1} |a_{1j}|,$$
$$R_i = \sum_{j<i} |a_{ij}| \cdot R_j/|a_{jj}| + \sum_{j>i} |a_{ij}|, \qquad j = 2(1)n.$$

Since the hypothesis includes $|a_{jj}| > R_j$ $(j < i)$ the new hypothesis (5.3) is weaker than (1.1). However, nothing like (1.3) to (1.6) can be found.

But there is no analog for matrices of polynomials either. The analogous definitions for R_i would be

(5.5)
$$R_1 = \max_{j>1} |a_{1j}|,$$

(5.6) $$R_j = \max \left\{ \max_{t<j} \left[a_{jt}R_t/|a_{tt}|, \max_{t>j} |a_{jt}| \right] \right\}.$$

The matrix

$$\begin{bmatrix} x^2 & x \\ x^3 & x^2 \end{bmatrix}$$

does satisfy the conditions $|a_{ii}| > R_i$ ($2 > 1$; $2 > 3/2$), but is certainly not regular.

This counterexample seems to be related to the nonexistence of theorems (on matrices of complex numbers) of analogs to (1.3) to (1.6).

Conditions using (5.3) do guarantee regularity of a matrix of quaternions. The proof can be patterned after the proof in [1].

(5.7) More general conditions sufficient for regularity of a matrix can be obtained by using an idea attributed to Müller [13], [17]. If M is any (regular) matrix and AM is known to be regular, then A must be regular also. As a first example, suppose M is a permutation matrix.

5.8. THEOREM. *Let $A = [a_{ij}]$ be a matrix of polynomials. Let $i \longmapsto \phi(i)$ be a permutation of the indices $i = 1(1)n$.*

$$\left[\forall_i \left\{ |a_{i\phi(i)}| > \max_{j \neq \phi(i)} |a_{ij}| \right\} \right] \Rightarrow \begin{array}{l} \text{(a) } A \textit{ is regular;} \\ \text{(b) } |\det A| = \prod |a_{i\phi(i)}|. \end{array}$$

Next let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$, be a regular diagonal matrix of polynomials. If AD is regular, then A is regular.

5.9. THEOREM. *Let $A = [a_{ij}]$ be a matrix of polynomials. Suppose n nonzero polynomials $d_1, \cdots, d_n$ exist so that the conditions*

$$|a_{ii}| \cdot |d_i| > \max_{j \neq i} |a_{ij}d_j|, \qquad i = 1(1)n$$

hold. Then (a) *A is regular,* (b) $|\det A| = \prod |a_{ii}|$.

If M is taken as a triangular matrix with units on the diagonal, the following theorem is obtained.

5.10. THEOREM. *Let $A = [a_{ij}]$ be a matrix of polynomials. Suppose polynomials m_{ij} exist so that for $i = 1(1)n$ the relations*

(5.11) $$|a_{ii}| > \max_{j \neq i} \left| \sum_{t<j} a_{it}m_{tj} + a_{ij} \right|$$

hold. Then (a) *A is regular,* (b) $|\det A| = \prod |a_{ii}|$.

If M is not required to have units on the diagonal, (5.11) is replaced by

$$(5.12) \qquad |a_{ii}| \cdot |m_{ii}| > \max_{j \neq i} \left| \sum_{t<j} a_{it} m_{tj} \right|.$$

The conclusions remain the same.

The idea behind (5.11), (5.12) is that cancellation might occur among the terms of highest degree in the summations.

6. **Proper values of matrices of polynomials.** For a matrix $A = [a_{ij}]$ the straightforward definition of proper vector and proper value is the following.

6.1. DEFINITION. A vector x of polynomials is a proper vector, and the polynomial λ is a corresponding proper value, of a matrix $A = [a_{ij}]$ of polynomials if the relation $Ax = x\lambda$ holds, ($x \neq 0$).

This definition indeed makes sense over any ring. If ρ is an invertible element in the ring, and λ is a proper value, then so is $\rho\lambda\rho^{-1}: Ay = y(\rho\lambda\rho^{-1})$ (if $y = x\rho$). It is not always the case that a proper vector of length 1 exists; length may not even always be defined. Lemma 2.25 states that over a principal ideal ring, a singular matrix has at least one proper value, namely 0. A local ring has the same property. However not every matrix need have a proper value; it is well known that if a field is not algebraically closed, matrices over that field exist with no proper values in the field. Even more, a matrix of polynomials may have nonpolynomial proper values in the domain's algebraic closure. Nevertheless the following "root-location theorems" may be of interest.

6.2. THEOREM. *Let $A = [a_{ij}]$ be a matrix of polynomials over a commutative principal ideal ring. Every (polynomial) proper value λ of A satisfies each of the following conditions*:

$$(6.3) \qquad \begin{aligned} &\exists_i \{ |a_{ii} - \lambda| \leqq R_i \}, \\ &\exists_{i>j} \{ |a_{ii} - \lambda| \cdot |a_{jj} - \lambda| \leqq R_i R_j \}, \\ &\forall_{\alpha, 0 \leqq \alpha \leqq 1} \exists_i \{ |a_{ii} - \lambda| \leqq R_i^{\alpha} C_i^{1-\alpha} \}, \\ &\forall_{\alpha, 0 \leqq \alpha \leqq 1} \exists_{i>j} \{ |a_{ii} - \lambda| \cdot |a_{jj} - \lambda| \leqq R_i^{\alpha} R_j^{\alpha} C_i^{1-\alpha} C_j^{1-\alpha} \}. \end{aligned}$$

Here

$$|p| = 2^{\deg p}, \quad R_i = \max_{j \neq i} |a_{ij}|, \quad C_i = \max_{j \neq i} |a_{ji}|; \qquad |0| = 0.$$

The same theorem is valid over any ring (with a valuation that satisfies (3.3), (3.4), (3.5)).

Note that (6.3), for example, really gives a useful search bound for $\deg \lambda$. In fact, either the terms of highest degree in λ coincide with those of one of the a_{ii}, or else $|\lambda| < R_i$ for some i.

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