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BEST CHEBYSHEV QUADRATURES

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Introduction. The Chebyshev quadrature

$$\int_{-1}^{1} x(s)ds \cong u \sum_{k=1}^{m} x(a_{k-1}), \tag{1}$$

will be considered. Following the method of Sard [1], it will be assumed that $d^{n-1}x/ds^{n-1}$ is absolutely continuous, and that the approximation is precise for degree $\leqq n-1$. Under these assumptions, the remainder, or error term,

$$Rx = \int_{-1}^{1} x(s)dx - u \sum_{k=1}^{m} x(a_{k-1}) \tag{2}$$

may be written in the form (see [1, p. 25])

$$Rx = \int_{-1}^{1} K(t) \frac{d^n x}{ds^n}(t)dt. \tag{3}$$

One possible appraisal of the magnitude of the error term is obtained by applying the Schwarz inequality to (3), obtaining

$$|Rx|^2 \leqq J \int_{-1}^{1} \left[\frac{d^n x}{ds^n}(t) \right]^2 dt,$$

where

$$J = \int_{-1}^{1} [K(t)]^2 dt. \tag{4}$$

Any L_p norm ($p \geqq 1$) of $K(t)$ could be considered. The L_2 norm was chosen because of the resulting simplicity of the calculations.

This paper will obtain "best" Chebyshev quadratures in the sense of Sard, i.e., those which minimize J. We will require that $-1 \leqq a_0 < a_1 < \cdots < a_{m-1} \leqq 1$, and that the a_k be symmetric, i.e., $a_{i-1} = -a_{m-i}$, $i = 1, \cdots, m$. Precision zero will be required in all cases, thus $n \geqq 1$ and $u = 2/m$.

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The cases to be considered numerically are $n = 1$, all m; and $n = 2$, $2 \leqq m \leqq 11$. These choices of n allow minimization of J with no constraints on the a_k, under the above assumptions. The values of J obtained in each case will be compared with the corresponding value for the classical Chebyshev approximations, where the highest possible precision is required. For $m = 8$, 10, and 11, the classical Chebyshev approximations do not exist under our restrictions. For those values of m, Barnhill, Dennis, and Nielson [4] have obtained Chebyshev type quadratures, and the best approximations for $n = 2$ will be compared with them.

The functions $K(t)$ and $J(a_0, \cdots, a_{m-1})$. We define $r^{(i)} = r^i/i!$ and

$$\theta(t, s) = 1 - \varphi(t, s) = 1, \qquad t < s,$$
$$= 0, \qquad t \geqq s.$$

Then, by Sard's Kernel Theorem [**1**, p. 25],

$$\begin{aligned} K(t) &= -\,R[(s-t)^{(n-1)}\theta(t,s)] \\ &= R[(s-t)^{(n-1)}\varphi(t,s)], \end{aligned} \tag{5}$$

for $t \neq a_0, \cdots, a_{m-1}$. Here R acts on the argument only as a function of s.

We consider the two equivalent forms of the kernel $K(t)$. We have

$$\begin{aligned} K(t) &= -R[(s-t)^{(n-1)}\theta(t,s)] \\ &= -\int_{-1}^{1} (s-t)^{(n-1)}\theta(t,s)ds \\ &\quad + u \sum_{k=1}^{m} (a_{k-1}-t)^{(n-1)}\theta(t, a_{k-1}), \end{aligned}$$

which, when simplified, yields

$$\begin{aligned} K(t) &= (-1-t)^{(n)}; \qquad -1 \leqq t < a_0, \\ &= (-1-t)^{(n)} + u \sum_{i=1}^{k} (a_{i-1}-t)^{(n-1)}; \\ &\qquad\qquad a_{k-1} < t < a_k, \quad k = 1, \cdots, m-1, \\ &= (-1-t)^{(n)} + u \sum_{i=1}^{m} (a_{i-1}-t)^{(n-1)}; \\ &\qquad\qquad a_{m-1} < t \leqq 1. \end{aligned} \tag{6}$$

Similarly,

$$K(t) = R[(s-t)^{(n-1)}\varphi(t,s)]$$

$$= \int_{-1}^{1} (s-t)^{(n-1)}\varphi(t,s)ds - u \sum_{k=1}^{m} (a_{k-1}-t)^{(n-1)}\varphi(t,a_{k-1}),$$

which, when simplified, yields

$$K(t) = (1-t)^{(n)} - u \sum_{i=1}^{m} (a_{i-1}-t)^{(n-1)}; \qquad -1 \leqq t < a_0,$$

$$(7) \qquad = (1-t)^{(n)} - u \sum_{i=k+1}^{m} (a_{i-1}-t)^{(n-1)};$$

$$a_{k-1} < t < a_k, \quad k = 1, \cdots, m-1,$$

$$= (1-t)^{(n)}; \qquad a_{m-1} < t \leqq 1.$$

Under the assumption of symmetry of the nodes it is easily verified, using equations (6) and (7), that $K(-t) = (-1)^n K(t)$. Thus

$$J = \int_{-1}^{1} [K(t)]^2 dt = 2\int_{-1}^{0} [K(t)]^2 dt,$$

and we need only to minimize

$$J_1 = \int_{-1}^{0} [K(t)]^2 dt.$$

Let m^* be the number of nodes in $[-1, 0)$, i.e.,

$$m^* = m/2, \qquad m \text{ even}$$

$$= (m-1)/2, \qquad m \text{ odd}.$$

Then make the following definitions:

$$I_0 = \int_{-1}^{a_0} [K(t)]^2 dt,$$

$$I_k = \int_{a_{k-1}}^{a_k} [K(t)]^2 dt, \qquad k = 1, 2, \cdots, m^*-1,$$

$$I_{m^*} = \int_{a_{m^*-1}}^{0} [K(t)]^2 dt.$$

We have

$$J_1 = \sum_{k=0}^{m^*} I_k.$$

TABLE 1

J_{best} and $J_{\text{classical}}$

	$n=1$		$n=2$	
m	J_b	J_c	J_b	J_c
2	1.67×10^{-1}	1.79×10^{-1}	4.08×10^{-3}	4.75×10^{-3}
3	7.41×10^{-2}	7.63×10^{-2}	7.25×10^{-4}	7.80×10^{-4}
4	4.17×10^{-2}	4.76×10^{-2}	2.15×10^{-4}	4.19×10^{-4}
5	2.67×10^{-2}	2.80×10^{-2}	8.48×10^{-5}	1.15×10^{-4}
6	1.85×10^{-2}	2.99×10^{-2}	3.98×10^{-5}	2.28×10^{-4}
7	1.36×10^{-2}	1.58×10^{-2}	2.11×10^{-5}	5.29×10^{-5}
9	8.23×10^{-3}	1.49×10^{-2}	7.50×10^{-6}	6.11×10^{-5}

We must obtain a simplified expression for J_1.

Rewriting the I_k, we have

$$I_0 = \int_{-1}^{a_0} [(-1-t)^{(n)}]^2 dt.$$

$$I_k = \int_{a_{k-1}}^{a_k} \left[(-1-t)^{(n)} + u \sum_{i=1}^{k} (a_{i-1}-t)^{(n-1)} \right]^2 dt$$

$$= \int_{a_{k-1}}^{a_k} [(-1-t)^{(n)}]^2 dt$$

$$+ 2u \sum_{i=1}^{k} \int_{a_{k-1}}^{a_k} (-1-t)^{(n)}(a_{i-1}-t)^{(n-1)} dt$$

$$+ u^2 \sum_{i,j=1}^{k} \int_{a_{k-1}}^{a_k} (a_{i-1}-t)^{(n-1)}(a_{j-1}-t)^{(n-1)} dt,$$

$$k = 1, \cdots, m^* - 1.$$

$$I_{m^*} = \int_{a_{m^*-1}}^{0} \left[(-1-t)^{(n)} + u \sum_{i=1}^{m^*} (a_{i-1}-t)^{(n-1)} \right]^2 dt$$

$$= \int_{a_{m^*-1}}^{0} [(-1-t)^{(n)}]^2 dt$$

$$+ 2u \sum_{i=1}^{m^*} \int_{a_{m^*-1}}^{0} (-1-t)^{(n)}(a_{i-1}-t)^{(n-1)} dt$$

$$+ u^2 \sum_{i,j=1}^{m^*} \int_{a_{m^*-1}}^{0} (a_{i-1}-t)^{(n-1)}(a_{j-1}-t)^{(n-1)} dt.$$

Then,

$$J_1 = \sum_{k=0}^{m^*} I_k = \int_{-1}^{0} [(-1-t)^{(n)}]^2 dt$$

$$+ 2u \left\{ \sum_{k-1}^{m^*-1} \sum_{i=1}^{k} \int_{a_{k-1}}^{a_k} (-1-t)^{(n)}(a_{i-1}-t)^{(n-1)} dt \right.$$

$$\left. + \sum_{i=1}^{m^*} \int_{a_{m^*-1}}^{0} (-1-t)^{(n)}(a_{i-1}-t)^{(n-1)} dt \right\}$$

$$+ u^2 \left\{ \sum_{k=1}^{m^*-1} \sum_{i,j=1}^{k} \int_{a_{k-1}}^{a_k} (a_{i-1}-t)^{(n-1)}(a_{j-1}-t)^{(n-1)} dt \right.$$

$$\left. + \sum_{i,j=1}^{m^*} \int_{a_{m^*-1}}^{0} (a_{i-1}-t)^{(n-1)}(a_{j-1}-t)^{(n-1)} dt \right\}$$

$$= \int_{-1}^{0} [(-1-t)^{(n)}]^2 dt$$

$$+ 2u \sum_{k=1}^{m^*} \int_{a_{k-1}}^{0} (-1-t)^{(n)}(a_{k-1}-t)^{(n-1)} dt$$

$$+ u^2 \sum_{k=1}^{m^*} \int_{a_{k-1}}^{0} \left\{ [(a_{k-1}-t)^{(n-1)}]^2 \right.$$

$$\left. + 2(a_{k-1}-t)^{(n-1)} \sum_{i=1}^{k-1} (a_{i-1}-t)^{(n-1)} \right\} dt.$$

The sum in the last term is to be interpreted as zero when $k-1=0$, as are any sums in the following work where the upper limit is smaller than the lower limit. Integrating the above expressions, by parts several times in the last two terms, one obtains

$$J_1 = \frac{1}{(n!)^2(2n+1)} + 2u(-1)^{n+1} \sum_{k=1}^{m^*} \sum_{\ell=0}^{n} (1)^{(n-\ell)} a_{k-1}^{(n+\ell)}$$

$$(8) \qquad + u^2 \sum_{k=1}^{m^*} \left[-\frac{a_{k-1}^{2n-1}}{[(n-1)!]^2(2n-1)} \right.$$

$$\left. + 2 \sum_{i=1}^{k-1} \sum_{\ell=1}^{n} (-1)^{\ell} a_{k-1}^{(n+\ell-1)} a_{i-1}^{(n-\ell)} \right]$$

TABLE 2

Nodes: $n = 2$, and classical case

m	$n=2$		Classical [3, p. 920]	
2	±.55051	02572	±.57735	02692
3	±.69722	43623	±.70710	67812
	.0		.0	
4	±.77315	45618	±.79465	44723
	±.24721	76225	±.18759	24741
5	±.81850	45407	±.83249	74870
	±.39799	68704	±.37454	14096
	.0		.0	
6	±.84875	53923	±.86624	68181
	±.49831	19555	±.42251	86538
	±.16685	25367	±.26663	54015
7	±.87036	16256	±.88386	17008
	±.56998	33013	±.52965	67753
	±.28585	74954	±.32391	18105
	.0		.0	
9	±.89917	01631	±.91158	93077
	±.66554	24529	±.60101	86554
	±.44455	69674	±.52876	17831
	±.22221	09597	±.16790	61842
	.0		.0	

The case $n = 1$. Making the appropriate simplifications to (8), we find

$$J_1 = \frac{1}{3} + \frac{4}{m} \sum_{k=1}^{m^*} [a_{k-1}^{(2)} + a_{k-1}] + \frac{4}{m^2} \sum_{k=1}^{m^*} (-2k+1)a_{k-1}. \tag{9}$$

The a_{k-1} are independent in this case, so setting the partial derivatives to zero we obtain the equations

$$\begin{gathered} \frac{\partial J_1}{\partial a_{k-1}} = \frac{4}{m}(a_{k-1}+1) + \frac{4}{m^2}(-2k+1) = 0, \\ a_{k-1} = \frac{2(k-1) - m + 1}{m}, \qquad k = 1, 2, \cdots, m^*. \end{gathered} \tag{10}$$

This is the repeated midpoint rule, a result previously noted by Krylov [2, pp. 138-140]. The value of J_1 can be obtained by substituting (10) into (9), and after laborious simplification, it is found that

TABLE 3

J_{best} and $J_{\text{Barnhill, Dennis, Nielson}}$

m	J_b	$J_{B,D,N}$
8	1.22×10^{-5}	1.06×10^{-4}
10	4.87×10^{-6}	2.88×10^{-4}
11	3.30×10^{-6}	1.00×10^{-4}

TABLE 4

Nodes: $m = 8, 10, 11$; $n = 2$, and Barnhill, Dennis, Nielson

m	$n=2$		Barnhill, Dennis, Nielson [4]	
8	±.88656	64347	±.90044	55323
	±.62373	52450	±.55898	89280
	±.37512	67322	±.45850	78272
	±.12498	59060	±.0	
10	±.90925	31469	±.92106	80558
	±.69898	82064	±.63181	13569
	±.50010	12823	±.58191	74110
	±.29998	97486	±.38366	17341
	±.10000	11389	±.0	
11	±.91750	28608	±.92676	50132
	±.72635	29150	±.70492	41194
	±.54554	66192	±.51792	29707
	±.36362	70548	±.45740	29197
	±.18181	91126	±.0	
	.0		.0	

$J_1 = 1/3m^2$. Table 1 compares the values of J corresponding to (10) and to the classical Chebyshev approximations.

The case $n = 2$. Equation (8) reduces to

$$J_1 = \frac{1}{20} - \frac{4}{m} \sum_{k=1}^{m^*} \left[\frac{a_{k-1}^2}{4} + \frac{a_{k-1}^3}{6} + \frac{a_{k-1}^4}{24} \right]$$
$$+ \frac{4}{m^2} \sum_{k=1}^{m^*} \left[\frac{k-2}{3} a_{k-1}^3 - a_{k-1}^2 \sum_{i=1}^{k-1} a_{i-1} \right]. \tag{11}$$

There are no constraints on the a_{k-1} since symmetry assures precision $n - 1 = 1$. Equating the partial derivatives to zero, we have the system of equations

$$\frac{\partial J_1}{\partial a_{k-1}} = -\frac{4}{m}\left[\frac{a_{k-1}}{2}+\frac{a_{k-1}^2}{2}+\frac{a_{k-1}^3}{6}\right]$$

$$+\frac{4}{m^2}\left[(k-2)a_{k-1}^2 - 2a_{k-1}\sum_{i=1}^{k-1} a_{i-1} - \sum_{i=k+1}^{m^*} a_{i-1}^2\right] = 0$$

$$\text{for } k = 1, \cdots, m^*.$$

Simplifying, we obtain the system

$$3a_{k-1} + 3a_{k-1}^2 + a_{k-1}^3$$

(12)
$$-\frac{6}{m}\left[(k-2)a_{k-1}^2 - 2a_{k-1}\sum_{i=1}^{k-1} a_{i-1} - \sum_{i=k+1}^{m^*} a_{i-1}^2\right] = 0,$$

$$k = 1, \cdots, m^*.$$

Using Newton's Method for systems, (12) was solved for the a_{k-1} for $2 \leqq m \leqq 9$. $J = 2J_1$ was evaluated for those values, as well as for the classical approximations. The results are shown in Tables 1 and 2.

During the computations on the system of equations (12), as expected, increasing m necessitates more accurate initial guesses at the solution. For $m = 10$ and 11, convergence for all initial guesses attempted was to repeated nodes. An alternative, and in fact simpler procedure than the above, is as follows. We want

$$\frac{\partial}{\partial a_{k-1}}\int_{-1}^{0} [K(t)]^2 dt = 2u\int_{a_{k-1}}^{0} K(t)dt = 0,$$

where $K(t)$ is given by equation (6), with $n = 2$. Since $\int_{a_{k-1}}^{0} K(t)dt = 0$ must hold for $k = 1, \cdots, m^*$, we have $\int_{a_{k-1}}^{a_k} K(t)dt = 0$ for $k = 1, \cdots, m^* - 1$. Simplifying, we find

$$\int_{a_{k-1}}^{a_k} K(t)dt = \int_{a_{k-1}}^{a_k}\left[\frac{(-1-t)^2}{2!} + u\sum_{i=1}^{k}(a_{i-1}-t)\right]dt$$

$$= (a_k - a_{k-1})\left[\frac{(1+a_k)^2 + (1+a_k)(1+a_{k-1}) + (1+a_{k-1})^2}{6} + u\sum_{i=1}^{k} a_{i-1} - \frac{ku}{2}(a_k + a_{k-1})\right]$$

$$= 0.$$

Because we require that $a_{k-1} < a_k$, the quantity in brackets must be zero, and this then simplifies to

$$a_k^2 + a_k(3 - 3ku + a_{k-1})$$

$$(13) \qquad + \left(3 + 3a_{k-1} + a_{k-1}^2 + 6u \sum_{i=1}^{k} a_{i-1} - 3kua_{k-1}\right) = 0,$$

$$k = 1, 2, \cdots, m^* - 1.$$

We note that if the nodes $a_0, \cdots, a_{k-1}$ are known, a_k can be computed from (13). Equation (13) is a quadratic in a_k, hence has two solutions. For $n = 2$, $K(t)$ can be made continuous by properly defining it at the a_k, and is piecewise parabolic, the parabolas all opening upward. We recall that the solutions of (13) correspond to $\int_{a_{k-1}}^{a_k} K(t)dt = 0$. Proceeding inductively on k, and assuming that the two solutions are real and distinct, the smaller solution corresponds to a point where $K(t) < 0$ and the larger to a point where $K(t) > 0$. Noting the shape of $K(t)$ and that $dK/dt|_{a_k^-} > dK/dt|_{a_k^+}$, one is convinced that the larger of the two solutions leads to the smaller value of J_1.

The numerical procedure is as follows. For an initial guess at a_0, one can compute in turn, $a_1, a_2, \cdots, a_{m^*-1}$ from equation (13). Then let

$$F(a_0) = \int_{a_{m^*-1}}^{0} K(t)dt$$

$$= \frac{a_{m^*-1}}{6}\left[3 + 3a_{m^*-1} + a_{m^*-1}^2 + 6u \sum_{i=1}^{m^*} a_{i-1} - 3um^*a_{m^*-1}\right].$$

From above, we must have $\int_{a_{m^*-1}}^{0} K(t)dt = 0$, thus we want to solve the equation $F(a_0) = 0$. The procedure is quite sensitive, and some choices for a_0 result in complex nodes. However, searching for an appropriate initial guess is considerably simplified over that for equation (13), since only one variable is involved. This procedure was successfully applied for $m = 10$, 11, as well as for previously solved values, as a check on the two methods. The results for $m = 8$, 10, and 11 are shown in Tables 3 and 4, along with the approximations of Barnhill, Dennis, and Nielson.

Conclusion. Some observations may be made from the tables. First note that the classical approximation for $m = 6$ seems to be relatively poorer than the others, and in fact for the cases considered, the corresponding value for J exceeds that for $m = 5$. Without exception, the nodes move toward their classical position from their position for $n = 1$, when n is increased to 2.

The existence of approximations for values of m for which the classical approximations do not exist is noted for $n = 1$ and 2. It should be mentioned that the criterion used by Barnhill, Dennis, and Nielson to obtain their approximations is different from that used here. The approximations obtained here would not compare favorably against theirs, using their criterion.

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