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COUNTABLY RECOGNIZABLE CLASSES OF GROUPS[1]

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I. **Introduction.** A class Σ of groups is a collection of groups containing the unit group E and closed under the taking of isomorphisms. Let Σ be a class of groups:

(i) $s(\Sigma)$ is the class of all groups which are subgroups of Σ groups.

(ii) $q(\Sigma)$ is the class of all groups which are quotients of Σ groups.

(iii) $L(\Sigma)$ is the class of all groups in which every finitely generated subgroup is a Σ group.

If $L(\Sigma) \subset \Sigma$, Σ is said to satisfy the local theorem. If Σ satisfies the local theorem and $s(\Sigma) = \Sigma$, then the class Σ is determined in a certain sense by the finitely generated groups in Σ.

In this paper, we are interested in classes of groups determined by their countable subgroups. In the sequel, the word countable will mean countably infinite or finite.

DEFINITION 1.1. Let Σ be a class of groups. $C(\Sigma)$ is the class of all groups G such that every countable subgroup of G is a Σ group.

DEFINITION 1.2. A class of groups Σ is countably recognizable if $C(\Sigma) \subset \Sigma$.

Observe that if Σ satisfies the local theorem, then Σ is countably recognizable. Further, if $s(\Sigma) = \Sigma$, then Σ is countably recognizable if and only if $C(\Sigma) = \Sigma$.

The notion of a countably recognizable class of groups is due to R. Baer [**1**]. In the paper [**1**], it is shown that many classes of groups which do not satisfy the local theorem are countably recognizable. There are other isolated theorems of this type in the literature: e.g., [**6**, p. 219] shows that the class of ZA groups is countably recognizable: see also [**10**, p. 349] for a theorem of this type.

In this paper, we add several classes to the list of countably recognizable classes. Let Σ be countably recognizable and assume $s(\Sigma) = \Sigma$. Then the following classes are also countably recognizable:

(1) The class of groups G such that every simple factor G is a Σ group (Theorem 4.2).

(2) The class of groups G such that every principal factor of every subgroup of G is a Σ group (Theorem 5.2).

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(3) The class of groups G such that for every subgroup A of G and every maximal subgroup B of A we have $A/\mathrm{Core}_A(B) \in \Sigma$ (Theorem 6.2).

It is also shown (Corollary 7.1) that the classes residually solvable and residually nilpotent are countably recognizable.

Of independent interest may be

THEOREM 4.1. *If G is a simple (characteristically simple) group and Q is a countable subgroup of G, then there exists a countable simple (characteristically simple) group R such that $Q \subset R \subset G$.*

Many of the above results remain true if one replaces the concept of a normal subgroup by that of a characteristic subgroup. The relevant theorems are proved to handle both cases.

II. **Notation.** Let G be a group and $T \subset \mathrm{Aut}\,(G)$:

(1) A subgroup H of G is T invariant if $HT = H$.

(2) G is T simple if the only T invariant subgroups of G are E and G.

(3) If H is a subset of G, $H^T = \langle ht \mid h \in H,\ t \in T\rangle$. H^T is T invariant and if Y is a T invariant subgroup of G such that $H \subset Y$, then $H^T \subset Y$.

(4) Let $H \subset G$. $T\text{-}\mathrm{Core}_G(H) = \bigcap \{Ht \mid t \in T\}$. $T\text{-}\mathrm{Core}_G(H)$ is T invariant and if Y is a T invariant subgroup of G such that $Y \subset H$, then $Y \subset T\text{-}\mathrm{Core}_G(H)$.

Throughout this paper, $S_1(G)$ will denote the group of inner automorphisms of G while $S_2(G)$ will denote the group of automorphisms of G.

III. **Preliminary lemmas.** The following lemmas can easily be proved.

LEMMA 3.1. *Let Σ be a class of groups. If $\{s, q\}\Sigma = \Sigma$, then $\{s, q\}C(\Sigma) = C(\Sigma)$.*

LEMMA 3.2. *Let G be a group and $H \lhd G$. If G/H is countable, then there exists a countable subgroup M of G such that $M/M \cap H \simeq G/H$.*

IV. **Simple factors.**

THEOREM 4.1. *Let $i \in \{1, 2\}$. If G is $S_i(G)$ simple and Q is a countable subgroup of G, then there exists a countable group R such that R is $S_i(R)$ simple and $Q \subset R \subset G$.*

PROOF. Let L be a countable subgroup of G and $e \neq x \in L$.

Case 1: $(i = 1)$. Since G is simple, $e \neq y \in G$ implies $y^G = G$.

Thus, there exists a countable subgroup $H(x, L)$ of G such that $x \in H(x, L)$ and $x^{H(x,L)} \supset L$. Let $H(L) = \langle H(x, L) \mid x \in L \rangle$. Then $e \neq y \in L$ implies $y^{H(L)} \supset L$. Further, $H(L)$ is countable and $L \subset H(L)$.

Let $Q_0 = E$ and $Q_1 = Q$. Inductively, define Q_n by $Q_n = H(Q_{n-1})$ for every positive integer $n > 1$. Then

(a) each Q_n is countable,

(b) if $e \neq y \in Q_{n-1}$, then $y^{Q_n} \supset Q_{n-1}$ for $n > 1$,

(c) $Q_0 \subset Q_1 \subset \cdots \subset Q_n \subset \cdots$.

Let $R = \bigcup_{n=1}^{\infty} Q_n$. R is countable. Let $e \neq y \in R$. There is a first integer k such that $y \in Q_k$. Then $y^R = y^{\bigcup_{n=k+1}^{\infty} Q_n} \supset \bigcup_{n=k}^{\infty} Q_n = R$. Thus, R is simple and Case 1 is proved.

Case 2: ($i = 2$). Since G is characteristically simple $e \neq y \in G$ implies $y^{\mathrm{Aut}(G)} = G$. Thus, there exists a countable subgroup $H(x, L)$ of Aut (G) such that $x^{H(x,L)} \supset L$. Let $H(L) = \langle H(x, L) \mid x \in L \rangle$. Then $e \neq y \in L$ implies that $y^{H(L)} \supset L$. Further, $H(L)$ is countable.

Let $Q_1 = Q$ and $H_1 = H(Q_1)$. Suppose we have constructed groups $Q_1 \subset Q_2 \subset \cdots \subset Q_n$ and $H_1 \subset H_2 \subset \cdots \subset H_n$ such that for all i, $1 \leqq i \leqq n$,

(i) $Q_i \subset G$, $H_i \subset \mathrm{Aut}\,(G)$,

(ii) Q_i and H_i are countable,

(iii) $e \neq y \in Q_i$ implies $y^{H_i} \supset Q_i$.

Let $Q_{n+1} = Q_n^{H_n}$ and $H_{n+1} = \langle H(Q_{n+1}), H_n \rangle$. Then both Q_{n+1} and H_{n+1} are countable, $Q_n \subset Q_{n+1}$ and $H_n \subset H_{n+1}$, and $e \neq y \in Q_{n+1}$ implies $y^{H_{n+1}} \supset y^{H(Q_{n+1})} \supset Q_{n+1}$.

Let $R = \bigcup_{n=1}^{\infty} Q_n$ and $H = \bigcup_{n=1}^{\infty} H_n$. Both R and H are countable. It is easily shown that $R^H = R$. We may then view H as a subgroup of Aut (R).

Let $e \neq y \in R$. There is a first k such that $y \in Q_k$. $y^H = y^{\bigcup_{n=k}^{\infty} H_n} \supset \bigcup_{n=k}^{\infty} Q_n = R$. Thus, $y^{\mathrm{Aut}(R)} = R$ and R is characteristically simple.

Definition 4.1. Let G be a group. A factor of G is a group A/B where $B \lhd A \subset G$.

Definition 4.2. Let Σ be a class of groups and $i \in \{1, 2\}$. S_i-Σ_1 is the class of all groups G such that every S_i simple factor of G is a Σ group.

We note that for any class Σ, $\{s, q\}\,(S_i\text{-}\Sigma_1) = S_i\text{-}\Sigma_1$.

Theorem 4.2. *Let Σ be a class of groups and $i \in \{1, 2\}$. If $\{s, C\}\Sigma = \Sigma$, then $C(S_i\text{-}\Sigma_1) = S_i\text{-}\Sigma_1$.*

Proof. Since $s(S_i\text{-}\Sigma_1) = S_i\text{-}\Sigma_1$, we have $S_i\text{-}\Sigma_1 \subset C(S_i\text{-}\Sigma_1)$.

Suppose $G \notin S_i\text{-}\Sigma_1$. Then there exists an S_i simple factor A/B of G such that $A/B \notin \Sigma$. Since $C(\Sigma) = \Sigma$, there is a countable subgroup

Q/B of A/B such that $Q/B \notin \Sigma$. By Theorem 4.1, there is a countable S_i simple group R/B such that $Q/B \subset R/B$. Since $s(\Sigma) = \Sigma$, $R/B \notin \Sigma$. By Lemma 3.2, there is a countable subgroup M of R such that $M/M \cap B \simeq R/B$. Since $q(S_i\text{-}\Sigma_1) = S_i\text{-}\Sigma_1$, $M \notin S_i\text{-}\Sigma_1$ and $G \notin C(S_i\text{-}\Sigma_1)$. This completes the proof.

We observe that there are countably recognizable classes Σ such that $S_i\text{-}\Sigma_1$ does not satisfy the local theorem. Let $\mathfrak{F}$ be the class of finite groups. $\mathfrak{F}$ is countably recognizable. Alt $(\aleph_0)$ is locally finite and consequently is in both the classes $L(S_1\text{-}\mathfrak{F}_1)$ and $L(S_2\text{-}\mathfrak{F}_1)$. But Alt $(\aleph_0)$ is infinite and simple, so it is in neither $S_1\text{-}\mathfrak{F}_1$ nor $S_2\text{-}\mathfrak{F}_1$.

In the paper [3, p. 58], Černikov calls a group G an H-group if every infinite factor of every infinite subgroup of G is not simple.

It is not hard to show that the class of H groups coincides with the class $S_1\text{-}\mathfrak{F}_1$, where $\mathfrak{F}$ is the class of finite groups. It is then a consequence of Theorem 4.2 that the class of H groups is countably recognizable.

V. S_i composition factors.

DEFINITION 5.1. Let G be a group and $i \in \{1, 2\}$. An S_i composition factor of G is a group A/B where A and B are $S_i(G)$ invariant subgroups of G and B is a maximal proper $S_i(G)$ invariant subgroup of A.

Note that an S_1 composition factor of a group G is usually called a principal or chief factor of G.

THEOREM 5.1. *Let G be a group, $i \in \{1, 2\}$, and A be a minimal $S_i(G)$ invariant subgroup of G. Let Q be a countable subgroup of A. Then there exists a countable subgroup H of G and an $S_i(H)$ composition factor X/Y of H such that $Q \underset{\sim}{\subset} X/Y$.*

PROOF. Since A is a minimal $S_i(G)$ invariant subgroup of G, $e \neq x \in A$ implies $x^{S_i(G)} = A$. If $e \neq y \in Q$, there is a countable group $H(y) \subset S_i(G)$ such that $y^{H(y)} \supset Q$. Let $H = \langle H(y) \mid y \in Q \rangle$. Then H is countable and $e \neq y \in Q$ implies $y^H \supset Q$.

Case 1: $(i = 1)$. For each $h \in H$, there exists $f(h) \in G$ such that $yh = y^{f(h)}$ for all $y \in G$. Let $H_1 = \langle \{f(h) \mid h \in H\}, Q \rangle$. Then H_1 is countable and $e \neq y \in Q$ implies $y^{H_1} \supset Q$.

Now, let $e \neq x \in Q$. Let P be maximal with respect to the following properties:

(a) $P \subset x^{H_1}$,

(b) $x \notin P$,

(c) $P \lhd H_1$.

We now have x^{H_1}/P is an S_1 composition factor of H_1. Further $x^{H_1}/P \supset QP/P$. We show that $Q \cap P = E$.

Suppose $y \in Q \cap P$. Since $P \lhd H_1$, $y^{H_1} \subset P$ and it follows that $Q \subset P$. This is contrary to $x \notin P$. Thus, $Q \cap P = E$ and $QP/P \simeq Q$. This completes the proof of Case 1.

Case 2: ($i = 2$). Let $R = Q^H$. Then R is countable and H invariant. We may assume then that $H \subset \text{Aut}(R)$.

Let $e \neq x \in Q$. Let P be maximal with respect to the following properties:

(a) $P \subset x^{\text{Aut}(R)}$,

(b) $x \notin P$,

(c) P is a characteristic subgroup of R.

$x^{\text{Aut}(R)}/P$ is an S_2 composition factor of R, and $x^{\text{Aut}(R)}/P \supset x^H P/P \supset QP/P$. Now if $y \in Q \cap P$, then $y^H \subset P$ since P is characteristic in R. Hence, $Q \subset y^H \subset P$ which is contrary to the choice of P. We have then $Q \cap P = E$ and $x^H/P \supsetneq Q$. This completes the proof of Case 2.

Definition 5.2. Let Σ be a class of groups. $P\Sigma$ is the class of all groups G such that every principal factor (S_1 composition factor) of every subgroup of G is a Σ group.

It is easy to verify that for any class Σ, $\{s, q\}P\Sigma = P\Sigma$.

Theorem 5.2. *Let Σ be a class of groups. If $\{s, C\}\Sigma = \Sigma$, the $C(P\Sigma) = P\Sigma$.*

Proof. Since $s(P\Sigma) = P\Sigma$, $P\Sigma \subset C(P\Sigma)$.

Suppose $G \notin P\Sigma$. Then there is a subgroup L of G and a principal factor A/B of L such that $A/B \notin \Sigma$. Since $C(\Sigma) = \Sigma$, there is a countable subgroup Q of A/B such that $Q \notin \Sigma$. By Theorem 5.1, there is a countable subgroup H/B and a principal factor X/Y of H/B such that $Q \subsetneq X/Y$. Since $s(\Sigma) = \Sigma$, $X/Y \notin \Sigma$. By Lemma 3.2 there is a countable subgroup M of H such that $M/M \cap B \simeq H/B$. Thus, M has a principal factor that is not in Σ, so that $M \notin P\Sigma$. It follows that $G \notin C(P\Sigma)$ and the proof is complete.

Again, we observe that if $\mathfrak{F}$ is the class of finite groups, $P\mathfrak{F}$ does not satisfy the local theorem: Alt ($\aleph_0$) is again the example.

We also point out that the method of proof used in Theorem 5.1 was indicated in [8, p. 105].

VI. Maximal subgroups.

Theorem 6.1. *Let G be a group and $i \in \{1, 2\}$. Suppose A is a maximal subgroup of G such that $S_i(G)$-$\text{Core}_G(A) = E$ and that Q is a countable subgroup of G. Then there exists a countable subgroup H of G and a maximal subgroup L of H such that $Q \subsetneq H/S_i(H)$-$\text{Core}_H(L)$.*

Proof. Let $x \in G \backslash A$. Then $G = \langle x, A \rangle$. So, for every element

$y \in G$ there exists a word $w(y, x)$ such that $y = w(y, x)(\bar{a}(y, x), x)$ where $\bar{a}(y, x)$ is some k-tuple of elements of A. We define a function g_x on the elements of G by $g_x(y) = \{p_k(\bar{a}(y, x)) \mid k = 1, 2, \cdots\}$ where the p_k's are the usual projection functions.

Since $S_i(G)\text{-Core}_G(A) = E$, for each $e \neq y \in Q$ there exists $T(y) \in S_i(G)$ such that $yT(y) \notin A$. Let $T = \langle T(y) \mid y \in Q \rangle$. Then T is a countable subgroup of $S_i(G)$.

Let $Q_1 = \langle \bigcup \{g_x(y) \mid y \in Q\} \rangle$. Let $e \neq y \in Q$. Since $yT(y) \notin A$, $g_{yT(y)}(x)$ is defined. Let $Q_2 = \langle \bigcup \{g_{yT(y)}(x) \mid y \in Q\} \rangle$. Let $Q_3 = \langle Q_1, Q_2 \rangle$. Q_3 is a countable subgroup of A.

Case 1: $(i = 1)$. For $e \neq y \in Q$ there exists $f(y) \in G$ such that $aT(y) = a^{f(y)}$ for all $a \in G$. Let $B = \langle f(y) \mid y \in Q \rangle$ and

$$B_1 = \langle \bigcup \{g_x(y) \mid y \in B\} \rangle .$$

Let $H = \langle Q_3, B_1, x \rangle$. Then

(i) H is countable,

(ii) B and Q are subgroups of H,

(iii) $yT(y) \in H$ for all $y \in Q$.

Let L be maximal with respect to

(a) $A \cap H \subset L \subset H$,

(b) $x \notin L$.

Then L is a maximal subgroup of H. Let $Y = \text{Core}_H(L)$. Then $H/Y \supset QY/Y \simeq Q/Q \cap Y$. Suppose $y \in Q \cap Y$. Then since $Y \lhd H$ and $B \subset H$, $y^{f(y)} = yT(y) \in Y$. Thus,

$$x = w(x, yT(y))(\bar{a}(x, yT(y)), yT(y)) \in (A \cap H)Y \subset L.$$

This is contrary to the choice of L. Hence $Q \cap Y = E$ and $Q \lesssim H/Y$. This completes the proof of Case 1.

Case 2: $(i = 2)$. Let $J = \langle Q_3, x \rangle$. Then $Q \subset J$ and $yT(y) \in J$ for every $y \in Q$. Let $J_1 = J$ and inductively define J_k by

$$J_k = \langle \bigcup \{g_x(y) \mid y \in J_{k-1}^T\}, x \rangle$$

for every positive integer $k \geqq 2$. Observe that $J_1 \subset J_2 \subset \cdots \subset J_k \subset \cdots$. Let $H = \bigcup_{k=1}^{\infty} J_k$. Then H is countable and $H^T = H$. Thus, we may view T as a subgroup of Aut (H).

Let L be maximal with respect to

(a) $A \cap H \subset L \subset H$,

(b) $x \notin L$.

L is a maximal subgroup of H.

Let $Y = S_2(H)\text{-Core}_H(L)$. As in Case 1, $H/Y \supset Q/Q \cap Y$. Suppose $y \in Q \cap Y$. Since Y is characteristic in H and $T(y) \in \text{Aut}(H)$, $yT(y) \in Y$. Then $x = w(x, yT(y))(\bar{a}(x, yT(y)), yT(y)) \in (A \cap H)Y \subset L$, which is contrary to the choice of L. Thus, $Q \cap Y = E$ and $Q \subsetneq H/Y$.

DEFINITION 6.1. Let Σ be a class of groups. $M\Sigma$ is the class of all groups G such that for every subgroup A of G and every maximal subgroup B of A, $A/\mathrm{Core}_A(B) \in \Sigma$ ($\mathrm{Core}_A(B) = S_1(A)\text{-}\mathrm{Core}_A(B)$).

THEOREM 6.2. *Let Σ be a class of groups. If $\{s, C\}\Sigma = \Sigma$, then $C(M\Sigma) = M\Sigma$.*

PROOF. Since $s(M\Sigma) = M\Sigma$, $M\Sigma \subset C(M\Sigma)$.

Suppose $G \notin M\Sigma$. Then there is a subgroup A of G and a maximal subgroup B of A such that $A/\mathrm{Core}_A(B) \notin \Sigma$. Since $C(\Sigma) = \Sigma$, there is a countable subgroup Q of $A/\mathrm{Core}_A(B)$ such that $Q \notin \Sigma$. By Theorem 6.1, there is a countable subgroup $H/\mathrm{Core}_A(B)$ of $A/\mathrm{Core}_A(B)$ and a maximal subgroup $L/\mathrm{Core}_A(B)$ of $H/\mathrm{Core}_A(B)$ such that

$$Q \subsetneq H/\mathrm{Core}_A(B)/\mathrm{Core}_{H/\mathrm{Core}_A(B)}(L/\mathrm{Core}_A(B)) = X\,.$$

Since $s(\Sigma) = \Sigma$, $X \notin \Sigma$.

By Lemma 3.2, there is a countable subgroup R of H such that $R/R \cap \mathrm{Core}_A(B) \simeq H/\mathrm{Core}_A(B)$. An easy argument shows that $R \notin M\Sigma$. Hence, $G \notin C(M\Sigma)$ and the proof is complete.

VII. Residual properties.

DEFINITION 7.1. Let Σ be a class of groups. $R(\Sigma)$ is the class of all groups G such that $e \neq x \in G$ implies there exists a normal subgroup N of G such that $x \notin N$ and $G/N \in \Sigma$.

A group in $R(\Sigma)$ is said to be residually a Σ group. If $s(\Sigma) = \Sigma$, then $s(R(\Sigma)) = R(\Sigma)$.

The central question here is the following: if $\{s, C\}\Sigma = \Sigma$, is $C(R(\Sigma)) = R(\Sigma)$? We have no complete solution, but the question can be answered affirmatively for some special classes.

LEMMA 7.1. *Let $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \cdots$ be an ascending sequence of varieties of groups and $W_1, W_2, \cdots, W_n, \cdots$ the associated sets of laws. Let $\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n$. Then $G \in R(\Sigma)$ if and only if $\bigcap_{n=1}^{\infty} W_n(G) = E$ ($W_n(G)$ is the verbal subgroup of G generated by the set of words W_n).*

Lemma 7.1 is easily proved using techniques of [9, p. 30].

THEOREM 7.1. *Let Σ be defined as in Lemma 7.1. Then $C(R(\Sigma)) = R(\Sigma)$.*

PROOF. Obviously, $R(\Sigma) \subset C(R(\Sigma))$. Suppose $G \notin R(\Sigma)$. Then by Lemma 7.1, $\bigcap_{n=1}^{\infty} W_n(G) \neq E$. Let $e \neq x \in \bigcap_{n=1}^{\infty} W_n(G)$. For each positive integer n, there exists

(i) elements $w_{n,1}, \cdots, w_{n,j_n}$ of W_n, and

(ii) $k_{n,i}$-tuples $a_{n,1}, \cdots, a_{n,j_n}$ of elements of G such that $x = \prod_{i=1}^{j_n}(w_{n,i}(a_{n,i}))^{s_i}$ where $s_i \in \{1, -1\}$.

Let $A = \bigcup\{a_{n,i} \mid n = 1, 2, \cdots : 1 \leqq i \leqq j_n\}$. Define a function g on A by $g(a_{n,i}) = \{p_r(a_{n,i}) \mid 1 \leqq r \leqq k_{n,i}\}$, where the p_r's are projections.

Let $H = \langle g(y) \mid y \in A\rangle$. H is countable, and for every positive integer n, $x \in W_n(H)$. Thus, H is not residually Σ and $G \notin C(R(\Sigma))$.

Corollary 7.1. *The classes residually solvable and residually nilpotent are countably recognizable.*

Neither of the classes residually solvable nor residually nilpotent satisfy the local theorem. This is easily seen from the characteristically simple locally finite p-group of McLain [7].

VIII. We briefly note some classes of groups that are not countably recognizable. A group G is an SN^* group if G has an ascending normal series with abelian factors [6, p. 183]. In [5], a group G is constructed with the property that every countable subgroup of G is an SN^* group, but G is not an SN^* group.

A group G is an $F(\aleph_0)$ group if G has a complete ascending series of subgroups $E \subset G_1 \subset \cdots \subset G_\alpha \subset \cdots \subset G$ such that for all α, $[G_{\alpha+1} : G_\alpha] < \infty$. Using techniques similar to those of [5], the author and Mr. K. Hickin in [11] have constructed a group G such that every countable subgroup of G is an $F(\aleph_0)$ group, but G is not an $F(\aleph_0)$ group.

If in the above two classes, SN^* and $F(\aleph_0)$, one insists that the ascending series be invariant series, then both of the resulting classes are countably recognizable [1, pp. 360–362].

Finally, in the book [4, p. 168] it is shown that the class of free abelian groups is not countably recognizable.

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