ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 2, Spring 1971

THE NILPOTENCE CLASS OF CORE-FREE QUASINORMAL SUBGROUPS

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1. **Introduction.** If H is a subgroup of the group G, H is said to be quasinormal in G if $HK = KH$ for each subgroup K of G. H is core-free in G if H contains no nonidentity normal subgroup of G. Itô and Szép [2] proved that a core-free quasinormal subgroup of a finite group must be nilpotent. The question raised by Deskins [1] of whether such a subgroup must be abelian was resolved first by Thompson [4] and later by Nakamura [3] who gave examples where the subgroup had class 2. In the present paper, it is shown that the nilpotence class is unbounded. More specifically, if n is a positive integer and p is a prime $> n$, there is a finite p-group containing a core-free quasinormal subgroup of class n. Using these examples, we show that the theorem of Itô and Szép is false for infinite groups. It is true, however, that a core-free quasinormal subgroup of an infinite group is residually finite nilpotent.

Our final result generalizes a theorem of Nakamura [3]. Suppose H is a core-free quasinormal subgroup of the finite p-group G. Nakamura proved that if H has exponent p, then H is abelian. Our generalization of this is that if H has exponent p^n, then H has a normal series of length n in which the factor groups all are elementary abelian.

2. **Notation and assumed results.** If H is a subgroup of G, H_G, the core of H in G, is the largest normal subgroup of G contained in H. Equivalently, $H_G = \bigcap_{x \in G} x^{-1}Hx$. $\phi(G)$ is the Frattini subgroup of G and $\phi^n(G)$ is defined inductively by $\phi^0(G) = G$, $\phi^{n+1}(G) = \phi(\phi^n(G))$. If G is a finite group, $f(G)$, the Frattini length of G, is the smallest integer n such that $\phi^n(G) = 1$. If G is a p-group, $\Omega_n(G)$ is the subgroup of G generated by all elements of order at most p^n and $\mho^n(G)$ is the subgroup generated by all p^nth powers of elements of G. Commutators are defined inductively by $[x, y] = x^{-1}y^{-1}xy$ and $[x_1, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$.

The following results are well known and easily proved. Hence we merely state them here.

Received by the editors July 22, 1969 and, in revised form, October 23, 1969.

AMS 1970 *subject classifications*. Primary 20D15, 20D40; Secondary 20F30, 20E25.

Key words and phrases. Quasinormal subgroups, nilpotence class, p-groups.

2.1. If $N \leqq H \leqq G$ and N is normal in G, then H is quasinormal in G if, and only if, H/N is quasinormal in G/N.

2.2. If $G = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1} \rangle$, where p is an odd prime, then $\Omega_1(G) = \langle x^p, y \rangle$, $\mho^1(G) = \langle x^p \rangle$, and $\langle y \rangle$ is quasinormal in G.

2.3. The class of finite nilpotent groups G satisfying $f(G) \leqq n$ for some fixed integer n is closed under the operations of taking homomorphic images, subgroups, and finite direct products.

3. Construction of the examples.

3.1. THEOREM. *Let n be a positive integer and let p be a prime $> n$. Then there is a finite p-group G which contains a core-free quasinormal subgroup H such that H has class n.*

PROOF. If $n = 1$ and $p \neq 2$, this follows from 2.2. If $n = 1$ and $p = 2$, let $G = \langle x, y \mid x^8 = y^2 = 1, \ y^{-1}xy = x^5 \rangle$ and $H = \langle y \rangle$. We now assume that $n \geqq 2$ and so p is odd. Set $m = (n-1)p + 1$ and let V be a vector space of dimension m over $GF(p)$ with basis $v_1, v_2, \cdots, v_m$. We adopt the convention that $v_i = 0$ if $i \leqq 0$. V_k will denote the subspace of V spanned by the v_i for $i \leqq k$. Next let X be the linear transformation of V defined by $v_iX = v_i + v_{i-1}$ for all $i \leqq m$. $(X-1)^m$ is both the minimal and characteristic polynomial of X. It follows from this that $(X-1)^{p^2} = 0 \neq (X-1)^p$ and so X is an element of order p^2 in the group $L = GL(V)$.

Now $C_V(X) = C_V(X^{p+1}) = V_1$. It follows from this that X and X^{p+1} have the same Jordan normal form. Hence X and X^{p+1} are conjugate in L. If $Y^{-1}XY = X^{p+1}$, then $Y^p \in C_L(X)$. This implies that X and X^{p+1} are conjugate under a p-element of L. Next, since $V_k/V_{k-1} = C_{V/V_{k-1}}(X)$ for $1 \leqq k \leqq m$, an inductive argument yields that V_k, $1 \leqq k \leqq m$, is invariant under $N_L(\langle X \rangle)$. It follows from this that $N_L(\langle X \rangle)$ consists entirely of lower triangular matrices.

Let S be the subgroup of L consisting of those lower triangular matrices whose eigenvalues all equal one and let P be the subgroup of S consisting of those elements of S which leave invariant the subspace spanned by $v_2, v_3, \cdots, v_m$. Then S is a Sylow p-subgroup of L and S contains all lower triangular matrices that are p-elements. The previous discussion implies that X and X^{p+1} are conjugate in S.

An easy calculation yields that the $m \times m$ matrix (a_{ij}) commutes with X if, and only if, $a_{ij} = 0$ for $1 \leqq i < j \leqq m$ and $a_{ij} = a_{i+1,j+1}$ for $1 \leqq i, j \leqq m-1$. It follows readily from this that $|C_S(X)| = p^{m-1}$ and $C_S(X) \cap P = 1$. Since $[S : P] = p^{m-1}$, this implies that $S = PC_S(X)$. Thus P contains a unique element U such that $U^{-1}XU = X^{p+1}$. Since $U^p \in P \cap C_S(X)$, we have $U^p = 1$. Thus the minimal polynomial for U is $(U-1)^r$ where $r \leqq p$. To obtain further information on r we need a lemma.

LEMMA. $v_k U \equiv v_k - k v_{k-p+1} \pmod{V_{k-p}}$.

PROOF. This is certainly true for $k=1$. Assume now that $k>1$ and that $v_k U \equiv v_k + \sum_{i=1}^{p} a_i v_{k-i} \pmod{V_{k-p-1}}$. Using induction on k, we obtain

$$v_k X U \equiv v_k + (a_1+1)v_{k-1} + \sum_{i=2}^{p-1} a_i v_{k-i} + (a_p - k + 1)v_{k-p} \pmod{V_{k-p-1}}.$$

On the other hand,

$$v_k U X^{p+1} \equiv v_k + (a_1+1)v_{k-1} + \sum_{i=2}^{p-1} (a_i + a_{i-1})v_{k-i} + (1 + a_p + a_{p-1})v_{k-p} \pmod{V_{k-p-1}}.$$

But $XU = UX^{p+1}$. Thus we obtain $a_1 = a_2 = \cdots = a_{k-2} = 0$ if $k \leqq p$, while $a_1 = a_2 = \cdots = a_{p-2} = 0$ and $a_{p-1} = -k$ if $k > p$. In either case, this implies $v_k U \equiv v_k - k v_{k-p+1} \pmod{V_{k-p}}$ and the lemma is proved.

A consequence of the lemma is

$$v_k(U-1)^r \equiv (-1)^r k(k+1)\cdots(k+r-1)v_{k-r(p-1)} \pmod{V_{k-r(p-1)-1}}.$$

It immediately follows that the minimal polynomial of U is $(U-1)^n$.

Next let A be the group of order p^5 with generators x and u and relations $x^{p^3} = u^{p^2} = 1$, $u^{-1}xu = x^{p+1}$. Then the mapping $x \rightarrow X$, $u \rightarrow U$ determines a homomorphism of A onto $\langle X, U\rangle$. Let B be the semidirect product AV where A operates on V as indicated. We now use multiplicative notation for V. Let $C = \langle u, v_2, \cdots, v_m\rangle$. It is easily verified that $\langle x^{p^2} v_1^{-1}, u^p v_2\rangle$ is a normal subgroup of order p^2 in B. Finally, let G be the factor group B modulo this subgroup and let H be the image of C in G. We assert that G and H satisfy the conclusion of 3.1.

Let W, x_1 and u_1 be the images of V, x, and u, respectively, in G. If H is not core-free in G, then H contains an element z of order p in $Z(G)$. But $C_{W \cap H}(x_1) = 1$. Thus $z \notin W \cap H$. Since $[H : H \cap W] = p$, this would imply that $H = (H \cap W) \times \langle z\rangle$ and so H would be abelian. Since u_1 does not centralize $H \cap W$, this is impossible. Therefore H is core-free in G. The fact that H has class n follows from the fact that the minimal polynomial of U is $(U-1)^n$. It only remains to show that H is quasinormal in G. This is a consequence of the next theorem.

3.2. THEOREM. *Assume the following*:
(a) $G = \langle x \rangle H$ *is a finite p-group*, $p > 2$.
(b) W *is a normal elementary abelian subgroup of* G.
(c) H *is a subgroup of* G, $u \in H$.
(d) $W = (W \cap \langle x \rangle) \times (W \cap H)$, $H = (W \cap H)\langle u \rangle$.
(e) $x^{p^2} \in W$, $u^p \in W$, $u^{-1}xu = x^{p+1}$.
(f) *If* y *is an element of order* p^2 *in* $\langle x \rangle$ *and* $v \in W$, *then*

$$[v, \underbrace{y, y, \cdots, y}_{(p-1)\ times}] = 1.$$

Then H *is quasinormal in* G.

PROOF. All of the above conditions are satisfied by the group G and its subgroup H which were constructed above. (The assumption in Theorem 3.1 that $p > n$ is necessary to ensure that condition (f) is satisfied.) Thus Theorem 3.1 will be proved once the above theorem is proved.

We now assume that G is a minimal counterexample to Theorem 3.2. HW/W is quasinormal in G/W from 2.2. If $\langle x \rangle \cap W = 1$, then $W \leqq H$, and then 2.1 would imply that H is quasinormal in G. Hence $\langle x \rangle \cap W \neq 1$. Also $x^p \neq 1$ since otherwise $[G:H] = p$ which would imply that H is normal in G.

Next suppose H contains a nontrivial normal subgroup N of G. Then, replacing G, H, W, x, and u, respectively, by G/N, H/N, WN/N, xN, and uN, we find that the hypothesis of the theorem is satisfied. Hence, by induction, H/N is quasinormal in G/N. But this implies that H is quasinormal in G. Thus $H_G = 1$. Since $G = H\langle x \rangle$, this implies $\bigcap_i (x^{-i}Hx^i) = 1$. Since $C_H(x) \leqq x^{-i}Hx^i$ for all i, we have $C_H(x) = 1$. It now follows that $C_G(x) = \langle x \rangle$.

Now let K be a subgroup of G. We will finish the proof of the theorem by showing that HK is always a subgroup of G. We consider four distinct cases.

Case 1. $K \leqq \langle x^p \rangle H$.

It is easy to see that $\langle x^p \rangle H$ is a proper subgroup of G. Then, replacing x by x^p, $\langle x^p \rangle H$ satisfies the hypothesis of the theorem. Hence, by induction, H is quasinormal in $\langle x^p \rangle H$. Therefore HK is a subgroup of $\langle x^p \rangle H$.

Case 2. $K \cap W \nleqq H \cap W$.

In this case we must have $W = (K \cap W)(H \cap W)$. But then

$HK = H(H \cap W)(K \cap W)K = HWK$. Since HW/W is quasinormal in G/W, HWK must be a subgroup of G.

Case 3. $K \not\leqq \langle x^p \rangle H$, $K \cap W \leqq H \cap W$, $x^{p^2} \neq 1$.

From 2.2, $\Omega_1(G/W) = H\langle x^p \rangle/W$ and $\mho^1(G/W) = \langle x^p \rangle W/W$. It now follows that K must contain an element y such that $y^p \equiv x^p \pmod W$. Hence $y^p = x^p v$ for some $v \in W$. Then $(x^p v)^p \in K \cap W \leqq H \cap W$. But an easy calculation shows that

$$(x^p v)^p = x^{p^2}[v, \overbrace{x^p, x^p, \cdots, x^p}^{(p-1)\text{ times}}] = x^{p^2}$$

since x^p is an element of order p^2 in $\langle x \rangle$. Since x^{p^2} does not belong to $H \cap W$, we have a contradiction. Thus this case cannot occur.

Case 4. $K \not\leqq \langle x^p \rangle H$, $K \cap W \leqq H \cap W$, $x^{p^2} = 1$.

In this case x has order p^2 and so, since $\langle x \rangle \cap W \neq 1$, $x^p \in W$. Then $\langle x^p \rangle H = WH$. Since $u^{-p}xu^p = x^{(p+1)^p} = x$, $u^p \in C_H(x)$. Thus $u^p = 1$.

Now since $|C_W(x)| = p$, the Jordan normal form of $\bar{x}$, the transformation of W_+ (W written additively) induced by x, must be

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Since we must have $(\bar{x} - 1)^{p-1} = 0$, this implies that W_+ has dimension at most $p - 1$. In particular, then, any transformation of W_+ has minimal polynomial of degree at most $p - 1$. Reverting to multiplicative notation, this implies that

$$[v, \overbrace{y, y, \cdots, y}^{(p-1)\text{ times}}] = 1$$

for all $y \in G$, $v \in W$. An immediate consequence of this is that $(yv)^p = y^p$ for all $y \in G$, $v \in W$.

Since $K \not\leqq HW$, K must contain an element of the form $y = u^i x^j v$ where $v \in W$ and p does not divide j. Then $y^p = (u^i x^j)^p = u^{ip} x^{je} = x^{je}$ where $e = [(p+1)^{ip} - 1]/[(p+1)^i - 1]$. Since p does not divide j, p^2 does not divide je. This implies that $x^p \in K \cap W$ which is a contradiction. This finishes the proof of Theorems 3.1 and 3.2.

Before showing that the theorem of Itô and Szép is false for infinite groups, we require a preliminary result about the direct product of quasinormal subgroups. In general if H_i is a quasinormal subgroup of G_i for $i = 1, 2$, $H_1 \times H_2$ is not necessarily a quasinormal subgroup of $G_1 \times G_2$. For example, let G_1 and G_2 both be nonabelian of order p^3 and exponent p^2 where p is an odd prime. Let H_i, $i = 1, 2$, be a nonnormal subgroup of order p in G_i. Then H_i is quasinormal in G_i, but $H_1 \times H_2$ is not quasinormal in $G_1 \times G_2$.

3.3. THEOREM. *Let $\{G_i \mid i \in I\}$ be a collection of periodic groups such that if $i \neq j$, $x \in G_i$, and $y \in G_j$, then x and y have relatively prime orders. Assume that H_i is a quasinormal subgroup of G_i for each $i \in I$. Then $\sum_{i \in I} H_i$ is quasinormal in $\sum_{i \in I} G_i$.*

PROOF. Here $\sum_{i \in I} G_i$ is the restricted direct product of the groups G_i. Let $G = \sum_{i \in I} G_i$, $H = \sum_{i \in I} H_i$. H is quasinormal in G if, and only if, for each pair $x \in H$, $y \in G$, there is an integer n such that $y^{-n}xy \in H$. Accordingly, assume $x \in H$, $y \in G$. Then $y = y_{i_1} y_{i_2} \cdots y_{i_m}$ for some m where $y_{i_k} \in G_{i_k}$. Let e_k be the order of y_{i_k} and let x_{i_k} be the i_k-component of x. Then, since H_{i_k} is quasinormal in G_{i_k}, there is an integer n_k such that $y_{i_k}^{-n_k} x_{i_k} y_{i_k} \in H_{i_k}$. By the Chinese Remainder Theorem, there is an integer n such that $n \equiv n_k \pmod{e_k}$ for $1 \leqq k \leqq m$. It now follows that $y^{-n}xy \in H$.

3.4. THEOREM. *There is a countable, solvable, locally finite, locally nilpotent group G containing a nonnilpotent, metabelian, core-free quasinormal subgroup H.*

PROOF. Let $p_1, p_2, \cdots$ be all the odd primes. By Theorem 3.1, there is a finite p_i-group G_i containing a core-free quasinormal subgroup H_i of class $p_i - 1$. Let $G = \sum_i G_i$ and $H = \sum_i H_i$. An examination of the groups constructed in the proof of 3.1 reveals that G_i and H_i can be chosen so that G_i has derived length 3 and H_i is metabelian. It now is verified easily that G and H satisfy the conditions in the theorem.

4. Residual nilpotence of core-free quasinormal subgroups.

4.1. THEOREM. *If H is quasinormal in $G = HC$, $H \cap C = 1$, and C is infinite cyclic, then H is normal in G.*

PROOF. Using 2.1, it may be assumed that H is core-free. We also may assume $H \neq 1$. Let $C = \langle x \rangle$. Since $H \cap C = 1$, if $h \in H$ and $n > 1$, then $hx \notin H\langle x^n \rangle = \langle x^n \rangle H$. Hence $hx \in xH$ or $x^{-1}H$. Similarly, $hx^{-1} \in xH$ or $x^{-1}H$, $xh \in Hx$ or Hx^{-1}, and $x^{-1}h \in Hx$ or Hx^{-1}. If $hx = xh_1$ and $hx^{-1} = xh_2$ with $h_i \in H$, then

$$x^2 = (hx^{-1})^{-1}(hx) = (xh_2)^{-1}(xh_1) \in H,$$

a contradiction. Therefore, if $hx \in xH$, then $hx^{-1} \in x^{-1}H$. By this and an analogous argument, we have

$$(1) \qquad \begin{array}{l} \text{either } hx \in xH \quad \text{and} \quad hx^{-1} \in x^{-1}H \\ \text{or} \quad hx \in x^{-1}H \quad \text{and} \quad hx^{-1} \in xH\,. \end{array}$$

Let $K = \{h \in H \mid hx \in xH\} = H \cap xHx^{-1}$. If follows from (1) that K also equals $H \cap x^{-1}Hx$. Thus $x^{-1}Kx = x^{-1}Hx \cap H = K$. This implies that $K \leqq \bigcap_{g \in G} g^{-1}Hg = H_G = 1$. Hence $hx \in x^{-1}H$ for all nonidentity elements of H. Then, if $h,\ h' \in H - \{1\}$, $h'hx \in h'x^{-1}H = xH$, which implies $h'h = 1$.

Thus $|H| = 2$. Therefore C is normal in G, and either $G = H \times C$ or G is the infinite dihedral group. In the latter case H is not quasinormal since it does not permute with other subgroups of order 2. Hence the first case holds and H is normal in G.

4.2. Theorem. *A core-free quasinormal subgroup H of a group G is residually finite nilpotent. If, in addition, H satisfies the minimum condition on normal subgroups, then H is nilpotent.*

Proof. If H is residually nilpotent, its lower central series reaches 1 in ω steps. Thus the first statement implies the second.

Since H is core-free, $\bigcap_{x \in G} H^x = 1$. It will therefore suffice to prove the following statement:

(2) If $x \in G$, then there is a normal subgroup N_x of H such that $N_x \subseteq H^x$ and H/N_x is finite nilpotent.

If $x^n \notin H$ for all $n > 0$, then the preceding theorem shows that (2) holds with $N_x = H$. Suppose that some $x^n \in H$ with $n > 0$. Then $[K : H]$ is finite, where $K = H\langle x \rangle$. Therefore K/H_K is finite with quasinormal core-free subgroup H/H_K. By [1], H/H_K is nilpotent. Since H_K is normal in K, $H_K \subseteq H^x$, and (2) again holds.

5. Quasinormal subgroups of finite p-groups.

5.1. Theorem. *Suppose G is a finite p-group, $|G| > 1$, such that $G = HK$ where H, K are subgroups, K is cyclic, and $H_G = 1$. Let $p^m = |K|$ and let p^n be the exponent of H. Then*

(a) *$f(H) \leqq m - 1$, $f(G) = m$, and $n < m$.*

(b) *If H is quasinormal in G, then $f(H) = n$.*

Proof. If $m \leqq 1$, then H is normal in G which implies $H = H_G = 1$. We now assume $m > 1$ and proceed by induction on $|G|$. Since $1 = H_G = \bigcap_{x \in G} x^{-1}Hx = \bigcap_{x \in K} x^{-1}Hx$, we must have $C_H(K) = 1$. Hence $C_G(K) = K$ and $H \cap K = 1$. This implies that $Z(G) \leqq K$. A consequence of this is that $\Omega_1(K) \leqq Z(G)$.

Now let $L/\Omega_1(K)$ be the core of $H\Omega_1(K)/\Omega_1(K)$ in $G/\Omega_1(K)$. Then, by induction, $f(G/L) = m - 1$ and $f(H\Omega_1(K)/L) \leqq m - 2$. Since $\Omega_1(K) \leqq L \leqq H\Omega_1(K)$, we must have $L = \Omega_1(K)(H \cap L) = \Omega_1(K) \times (H \cap L)$. But this implies that $x^{-1}(H \cap L)x$ is a normal subgroup of index p in L for all $x \in G$. Since $\bigcap_{x \in G} x^{-1}(H \cap L)x = 1$, L is a subdirect product of groups of order p. Hence L is elementary abelian. From this follows $f(G) \leqq m$ and $f(H) \leqq m - 1$. Since G contains an element of order p^m, $f(G) \geqq m$. Similarly, $f(H) \geqq n$. (a) now follows.

Now assume H is quasinormal in G and let K_1 be the subgroup of K of order p^n. $K_1 \neq K$ since $n < m$. If x is an element of order $\leqq p^n$ in G, then, since $\langle x \rangle H$ is a group and $[\langle x \rangle H : H] \leqq p^n$, we have $\langle x \rangle H = (\langle x \rangle H \cap K)H \leqq K_1 H$. Therefore, $\Omega_n(G) = K_1 H$, and so $K_1 H$ is normal in G. Let M be the core of H in $K_1 H$. Then, by part (a), $f(K_1 H/M) = n$. Since $\bigcap_{x \in G} x^{-1} M x = 1$, $K_1 H$ is the subdirect product of p-groups of Frattini length n. It follows from this and 2.3 that $n \leqq f(H) \leqq f(K_1 H) \leqq n$. Hence (b) is proved.

COROLLARY. *Suppose H is a core-free quasinormal subgroup of the finite p-group G. Let p^n be the exponent of H. Then*

(a) $f(H) = n$.

(b) *The exponent of G is* $\geqq p^{n+1}$.

PROOF. By applying the theorem to $\langle x \rangle H$ where x runs through the elements of G, we obtain that H is the subdirect product of groups of Frattini length $\leqq \text{Min}\{n, m - 1\}$ where p^m is the exponent of G. Since $f(H) \geqq n$, the corollary follows.

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