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DECISIVE CONVERGENCE SPACES, FRÉCHET SPACES AND SEQUENTIAL SPACES

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Decisive convergence spaces were introduced in [9], where it is shown that they provide a solution to Question 1 of [3]. In this paper we show that the S-spaces of Čech [4] and the Fréchet spaces of Arhangel'skiĭ [1] can be identified either the pretopological modifications of decisive convergence spaces or the pretopological quotients of decisive topologies, and that the sequential spaces of Franklin [6] can be characterized as either the topological modifications of decisive convergence spaces or the topological quotients of decisive topologies. The well-known characterizations of Fréchet and sequential spaces as certain quotients of metrizable spaces are obtained (in a somewhat generalized form) as a by-product. Indeed, using the convergence space approach to quotient maps developed in [8], we are able to display a natural transition from first-countable spaces to almost first-countable spaces to Fréchet spaces to sequential spaces as increasingly more general quotients of pseudo-metric spaces.

1. Our notation and terminology agree with that of [3], [8], and [9], and the reader is asked to refer to one of these sources for the basic information about convergence spaces. Starting with a convergence space (S, q), we shall refer to $\pi(q)$, the finest pretopology on S coarser than q, as the *pretopological modification* of q and to $\lambda(q)$, the finest topology on S coarser than q, as the *topological modification* of q. As in [9], we use the abbreviation "npuf" for "nonprincipal ultrafilter". Given a convergence space (S, q), let $\mathcal{G}_q(x)$ be the filter obtained by intersecting all nonprincipal ultrafilters which fail to q-converge to x; (S, q) is said to be *decisive* if a npuf $\mathcal{F}$ fails to q-converge to x whenever $\mathcal{F} \geqq \mathcal{G}_q(x)$. Another characterization of decisive convergence spaces can be stated as follows: a subset B of S is said to be *q-decisive for x* if every npuf $\mathcal{F}$ which contains B q-converges to x; then (S, q) is decisive if and only if, for each x in S, every npuf which q-converges to x contains a q-decisive set for x (Theorem 1, [9]). A convergence space (S, q) is *first-countable* (or, more formally, satisfies the first axiom of countability) if each filter which q-converges to x contains a filter with a countable filter base which q-converges to x. The real line with its usual topology is

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a first-countable space which is not decisive; the cofinite topology (consisting of complements of finite sets) on an uncountable set is a decisive space which is not first-countable.

Let $C(S)$ denote the lattice of convergence structures on S; recall that the discrete space is the largest and the indiscrete space the smallest element of $C(S)$. If $M \subset C(S)$, then the infimum q of M in $C(S)$ can be characterized as follows: $\mathcal{F}$ q-converges to x if and only if $\mathcal{F}$ p-converges to x for some p in M. The straightforward proof of the first proposition will be omitted.

PROPOSITION 1. *The infimum in $C(S)$ of a set of decisive (first-countable convergence spaces is again a decisive (first-countable) convergence space.*

2. Let $\mathcal{F}$ be a filter on S and $x \in S$. We denote by $(x, \mathcal{F})$ the finest topology on S relative to which $\mathcal{F}$ converges to x. This topology is discrete at all points y other than x, and has $\mathcal{F} \cap \dot{x}$ for its neighborhood filter at x, where $\dot{x}$ is the principal ultrafilter generated by $\{x\}$.

PROPOSITION 2. *A topology of the form $(x, \mathcal{F})$ is pseudo-metrizable if and only if $\mathcal{F}$ has a countable filter base.*

PROOF. Assume that $\mathcal{F}$ has a countable base; then $\mathcal{F} \cap \dot{x}$ also has a countable base $\{F_0, F_1, F_2, \cdots\}$, and we can assume with no loss of generality that $S = F_0 \supset F_1 \supset F_2 \supset \cdots$. Define a real-valued function d on $S \times S$ as follows: $d(x, y) = d(y, x) = 1/n$ if $y \in F_{n-1}$ and $y \notin F_n$ for $n = 1, 2, 3, \cdots$, and $d(x, y) = d(y, x) = 0$ if $y \in F_n$ for all positive integers n; for arbitrary points y, z in S, $d(y, y) = 0$ and $d(y, z) = d(z, y) = d(x, y) + d(x, z)$ if $y \neq z$. It is a simple matter to verify that d is a pseudo-metric compatible with the topology $(x, \mathcal{F})$. The converse is obvious.

PROPOSITION 3. *Each first-countable convergence structure q is the infimum in $C(S)$ of a set of pseudo-metrizable topologies.*

PROOF. If q is first-countable, then q is the infimum of the set of all pseudo-metrizable topologies of the form $(x, \mathcal{F})$ that are over q in $C(S)$.

The next result is an easy consequence of Proposition 2 and Theorem 4, [**9**]. Recall that a *Fréchet filter* is one which is generated by the sections of some sequence [**10**].

PROPOSITION 4. *A topology of the form $(x, \mathcal{F})$ is decisive and pseudo-metrizable if and only if $\mathcal{F}$ is a Fréchet filter.*

We will henceforth identify topologies of the type described in Proposition 4 as *fundamental pseudo-metrizable topologies.*

PROPOSITION 5. *A first-countable convergence structure q is decisive if and only if it is the infimum in $C(S)$ of a set of fundamental pseudo-metrizable topologies.*

3. Let (S, q) and (T, p) be convergence spaces and f a continuous mapping from (S, q) onto (T, p). If p is the finest convergence structure on T relative to which f is continuous, then f is said to be a *convergence quotient map*. If (S, q) and (T, p) are pretopological spaces (topological spaces) and p the finest pretopology (topology) on T relative to which f is continuous, then f is called a *pretopological (topological) quotient map*. The next proposition is an immediate consequence of these definitions.

PROPOSITION 6. *If (S, q) is a topological space and $f: (S, q) \to (T, p)$ a convergence quotient map, then $f: (S, q) \to (T, \pi(p))$ is a pretopological quotient map and $f: (S, q) \to (T, \lambda(p))$ is a topological quotient map.*

PROPOSITION 7. *The image of a first-countable (decisive) convergence space under a convergence quotient map is again first-countable (decisive).*

PROOF. For decisive spaces, the statement is Theorem 9, [9]. Let f be a convergence quotient map from a first-countable space (S, q) onto a space (T, p). Let $\mathcal{F}$ be a filter on T which p-converges to a point y; then (by Proposition 3, [8]) there is a filter $\mathcal{G}$ on S which maps on $\mathcal{F}$ and q-converges to some point x in $f^{-1}(y)$. Since (S, q) is first-countable, there is a filter $\mathcal{H}$ on S which is coarser than $\mathcal{G}$, has a countable filter base, and q-converges to x. Then the image filter $f(\mathcal{H})$ is coarser than $\mathcal{F}$, has a countable base, and p-converges to y. Thus (T, p) is first-countable, and the proof is complete.

PROPOSITION 8. *If a convergence structure q is the infimum in $C(S)$ of a set of pseudo-metrizable topologies on S, then q is the convergence quotient of a pseudo-metrizable topological space.*

PROOF. Let $\{q_\lambda\}$ be a set of pseudo-metrizable topologies whose infimum is q, and let $\{d_\lambda\}$ be a set of compatible pseudo-metrics, each of which is bounded above by 1. Let (T, p) be the (topological) direct union of the spaces (S_λ, q_λ), where each S_λ is a copy of S, and S_λ is taken to be disjoint from S_μ for $\lambda \neq \mu$. Let d be the real-valued function on $T \times T$ defined as follows: $d(x, y) = d_\lambda(x, y)$ if x and y both belong to the same component S_λ; $d(x, y) = 1$ otherwise. Then d is a pseudo-metric compatible with p, and the canonical mapping of (T, p) onto (S, q) is a convergence quotient map.

The first theorem is an immediate consequence of Propositions 3, 7, and 8.

THEOREM 1. *The following statements about a convergence space* (S, q) *are equivalent.*

(1) (S, q) *is first-countable;*

(2) (S, q) *is the convergence quotient of a pseudo-metrizable space;*

(3) q *is the infimum in* $C(S)$ *of a set of pseudo-metrizable topologies.*

With each convergence structure q there is an associated closure operation (in the sense of Čech [4]) Γ_q, defined by $\Gamma_q(A) = \{x \in S$: there is a filter $\mathcal{F}$ which contains A and q-converges to $x\}$ for all $A \subset S$. If $p = \pi(q)$, then $\Gamma_p = \Gamma_q$ (see [7]). Thus there is a one-to-one correspondence between pretopologies and closure operations of this type (see [5]) so that what we call "pretopological spaces" are essentially the same as Čech's "closure spaces".

A pretopology p will be called a *Fréchet pretopology* if Γ_p is completely determined by the convergence of sequences; more precisely, this means that for each $A \subset S$, $\Gamma_p(A) = \{x \in S$: there is a sequence s in A whose Fréchet filter $\mathcal{F}_s$ p-converges to $x\}$. Čech [4] calls such spaces *S-spaces*, but we are adapting the terminology of Arhangel'skiĭ [1].

PROPOSITION 9. *Each Fréchet pretopology* p *on* S *is the pretopological modification of the infimum in* $C(S)$ *of a set of fundamental pseudo-metrizable topologies.*

PROOF. If a sequence s p-converges to a point x, then $(x, \mathcal{F}_s)$ is a fundamental pseudo-metrizable topology, where $\mathcal{F}_s$ is the Fréchet filter associated with s; let M be the set of all topologies of the form $(x, \mathcal{F}_s)$ which can be obtained in this way, and let q be the infimum in $C(S)$ of M. Then it is easy to see that $p = \pi(q)$.

THEOREM 2. *The following statements about a pretopology* p *are equivalent.*

(1) p *is a Fréchet pretopology.*

(2) p *is a pretopological modification of a first-countable convergence space.*

(3) p *is a pretopological modification of a decisive convergence space.*

PROOF. The infimum of a set of fundamental pseudo-metrizable spaces is decisive and first-countable by Propostion 1. Thus (1) implies both (2) and (3) by Proposition 9.

To establish that (2) implies (1), let q be a first-countable space such that $p = \pi(q)$. For some $A \subset S$, let $x \in \Gamma_q(A)$ then there is a filter

$\mathcal{F}$ on S which contains A and q-converges to x. By Theorem 1(c) there is a pseudo-metrizable topology t finer than q in $C(S)$ such that $\mathcal{F}$ t-converges to x. Thus $x \in \Gamma_t(A)$, and so there is a Fréchet filter $\mathcal{F}_s$ corresponding to some sequence s on A such that $\mathcal{F}_s$ t-converges to x. But $t \geqq q$, and so $\mathcal{F}_s$ q-converges to x. Thus the closure operation Γ_q is sequentially determined, and since $\Gamma_q = \Gamma_p$, p is a Fréchet pretopology.

To show that (3) implies (1), let q be a decisive convergence structure such that $p = \pi(q)$. For some $A \subset S$, let $x \in \Gamma_p(A)$. Then there is an ultrafilter $\mathcal{F}$ on S which contains A and q-converges to x. If $\mathcal{F}$ is a principal ultrafilter then $\mathcal{F}$ is already a Fréchet filter. If $\mathcal{F}$ is a npuf, then $\mathcal{F}$ contains a q-decisive set B for x, and $A \cap B$ is also an infinite q-decisive set for x. Let s be any sequence on $A \cap B$ consisting of infinitely many distinct terms; then the corresponding Fréchet filter $\mathcal{F}_s$ p-converges to x, since each ultrafilter finer than $\mathcal{F}_s$ contains $A \cap B$ and hence q-converges to x. Thus the closure operation Γ_p is sequentially determined, and p is a Fréchet pretopology.

Proposition 10. *Let $f: (S, q) \to (T, p)$ be a pretopological quotient map, where (S, q) and (T, p) are pretopological spaces. Then there is a convergence structure r on T such that $\pi(r) = p$ and such that $f: (S, q) \to (T, r)$ is a convergence quotient map.*

Proof. Let r be defined as follows: a filter $\mathcal{F}$ on T p-converges to y if and only if there is a filter $\mathcal{G}$ on S such that $\mathcal{G}$ maps onto $\mathcal{F}$ and, for some $x \in f^{-1}(y)$, $\mathcal{G}$ q-converges to x. It is clear that r is a well-defined convergence structure, and it follows from Proposition 3, [8] that $f: (S, q) \to (T, r)$ is a convergence quotient map. It remains to show that $\pi(r) = p$. Let V be a subset of T belonging to every filter $\mathcal{F}$ that r-converges to y; that is, $V \in \mathcal{V}_r(y)$, the r-neighborhood filter at y. If $x \in f^{-1}(y)$, then $f(\mathcal{V}_q(x))$ r-converges to y by construction, and so $V \in f(\mathcal{V}_q(x))$, and $f^{-1}(V) \in \mathcal{V}_q(x)$. Since this is true for all x in $f^{-1}(y)$, and $\mathcal{V}_r(y) = \mathcal{V}_{\pi(r)}(y)$, it follows from Proposition 6, [8] that $f: (S, q) \to (T, \pi(r))$ is a pretopological quotient map. This implies that $\pi(r) = p$, and the proof is complete.

Theorem 3. *The following statements about a pretopological space (S, p) are equivalent.*

(1) *(S, p) is a Fréchet space.*

(2) *(S, p) is the pretopological quotient of a pseudo-metrizable space.*

(3) *(S, p) is the pretopological quotient of a first-countable topological space.*

(4) *(S, p) is the pretopological quotient of a decisive topological space.*

(5) *(S, p) is the pretopological quotient of a decisive pseudo-metrizable space.*

PROOF. First note that a direct union of decisive topological spaces is again decisive. Thus it follows from Propositions 6, 7, 8, and 9 that (1) implies (5). Clearly (5) implies (2) and (4), and (2) implies (3). Thus it remains to show that (3) and (4) each implies (1).

To establish that (4) implies (1), let $f: (T, q) \to (S, p)$ be a pretopological quotient map, where (T, q) is a decisive topological space and (S, p) is a pretopological space. By Proposition 10, there is a convergence structure r on S such that $f: (T, q) \to (S, r)$ is a convergence quotient map and $\pi(r) = p$. But (S, r) is decisive by Proposition 7, and so it follows from Theorem 2 that (S, p) is a Fréchet space. The proof that (3) implies (1) is exactly the same argument with "decisive" replaced by "first-countable". Thus the proof of Theorem 3 is complete.

Following Franklin [6], we say that a topological space is *sequential* if an open set is characterized by the fact that it eventually contains every sequence that converges to a point in the set, or, equivalently, if every closed set is characterized by the fact that it contains all of its sequential limits.

PROPOSITION 11. *A topological space (S, t) is sequential if and only if it is the topological modification of a Fréchet pretopological space (S, p).*

PROOF. Assume that $t = \lambda(p)$, where p is a Fréchet pretopology. A set $A \subset S$ is t-closed if and only if $\Gamma_p(A) = A$ (Theorem 2, [7]). Since Γ_p is sequentially determined, saying that $\Gamma_p(A) = A$ is equivalent to saying that A contains the p-limits of every Fréchet filter on A. If A contains the t-limits of every sequence on A, then A certainly contains the p-limits of every Fréchet filter on A, and hence A is t-closed. Thus t is sequential if p is a Fréchet pretopology.

Conversely, let t be a sequential topology, and let p be the pretopological modification of the infimum in $C(S)$ of the set of all pseudo-metrizable topologies over t. It follows from Proposition 9 and Theorem 2 that p is a Fréchet pretopology, and it can be shown by a direct argument that $\lambda(p) = t$.

The last theorem is a corollary to Theorems 2 and 3 and Proposition 11.

THEOREM 4. *The following statements about a topological space (S, t) are equivalent.*

(1) *(S, t) is a sequential space.*

(2) *t is the topological modification of a decisive convergence structure on* S.

(3) *t is the topological modification of a first-countable convergence structure on* S.

(4) *t is the topological modification of a Fréchet pretopology on* S.

(5) *(S, t) is the topological quotient of a decisive topological space.*

(6) *(S, t) is the topological quotient of a first-countable topological space.*

(7) *(S, t) is the topological quotient of a pseudo-metrizable topological space.*

(8) *(S, t) is the topological quotient of a decisive pseudo-metrizable space.*

Concluding remarks. It was proved in [8] that, when their ranges are restricted to topological spaces, convergence quotient maps coincide with almost open maps [2] and pretopological quotient maps with pseudo-open maps [1]. Thus Theorem 1 establishes that a topological space is first-countable if and only if it is the almost open image of a pseudo-metrizable space, and Theorem 3 reaffirms the fact, stated in [1] and proved in [6] that Fréchet topological spaces are precisely the pseudo-open images of pseudo-metrizable spaces. From Proposition 9 and Theorem 2 one can easily deduce, as in [10], that a pretopological space (S, p) is a Fréchet space if and only if each neighborhood filter is an intersection of Fréchet filters.

A *pseudo-topology* (see [5], [7], [8]) is defined to be a convergence structure with the property that a filter $\mathcal{G}$ converges to x whenever each ultrafilter finer than $\mathcal{G}$ converges to x. Since every pretopology is a pseudo-topology, we see that the notion of a pseudo-topological quotient map [8] is intermediate in generality between a convergence quotient map and a pretopological quotient map; and indeed it is shown in [8] that the *bi-quotient maps* of Michael [11] are precisely the pseudo-topological quotient maps with ranges in the class of topological spaces. We shall ignore this distinction and use the term "bi-quotient map" in place of "pseudo-topological quotient map".

It is natural to seek a characterization of those spaces which are bi-quotient images of pseudo-metrizable spaces, and with the help of Proposition 9, [8] the following characterization can be obtained: *The bi-quotient images of pseudo-metrizable spaces are precisely those pseudo-topological spaces with the property that each ultrafilter which converges to a point x contains a filter with a countable base that converges to x.* We will say that such spaces are *almost first-countable.*

One encounters some rather fundamental difficulties when trying to find examples of almost first-countable spaces which are not first-countable. Consider, for instance, the cofinite topology on an uncountable set; in order to satisfy the condition for being almost first-countable, each npuf must contain a countable subcollection whose intersection is the empty set. Such ultrafilters are said to be *normal,* and so an uncountable cofinite space is almost first-countable if and only if all ultrafilters on the space are normal. Čech [4] asserts that the existence of nonnormal ultrafilters can be neither proved nor disproved on the basis of mathematical axiom systems currently in use. If the existence of nonnormal ultrafilters is assumed, then it remains an open question whether there are almost first-countable pretopologies which are not first-countable.

Finally it is worth noting that (in contrast with the results of Theorems 3 and 4) the class of bi-quotient images of pseudo-metrizable spaces does not coincide with the class of bi-quotient images of decisive topological spaces. Indeed, it can be shown that the bi-quotient image of a decisive pseudo-topological space is again decisive. If one assumes that all ultrafilters are normal, then one can show that a decisive pseudo-topology is almost first-countable, so that the class of bi-quotient images of decisive spaces is (under this assumption) a subclass of the class of bi-quotient images of pseudo-metrizable spaces.

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