ON SOME CLASSES OF ELLIPTIC SYSTEMS
WITH FRACTIONAL BOUNDARY RELAXATION

JAN PRÜSS

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ABSTRACT. Classes of second order, one- or two phase-elliptic systems with time-fractional boundary conditions are studied. It is shown that such problems are well posed in an L_q-setting, and stability is considered. The tools employed are sharp results for elliptic boundary and transmission problems and for the resulting Dirichlet-Neumann operators, as well as maximal L_p-regularity of evolutionary integral equations, based on modern functional analytic tools like $\mathcal{R}$-boundedness and the operator-valued $\mathcal{H}^\infty$-functional calculus.

1. Introduction. In applied fields, in particular, in mathematical physics, there are many problems leading to time-fractional equations on surfaces or interfaces; some are also nonlocal in space. Such problems occur, for example, in elasticity and fluid dynamics through fractional boundary dissipation or in electrodynamics as impedance boundary conditions. In this paper, we want to present general classes of such problems and study their well-posedness as well as their stability properties. To describe the classes of problems we have in mind, we consider the following general framework.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary of class C^{3-}, i.e., of class C^2 with Lipschitz curvatures, and set $\Gamma := \partial\Omega$; the outer normal of Ω is denoted by ν^Γ. The domain Ω consists of two disjoint open subsets Ω_j, $j = 1, 2$, such that Ω_1 is not in contact with Γ, i.e., we assume $\overline{\Omega}_1 \cap \Gamma = \varnothing$. Then, $\Sigma := \partial\Omega_1 \subset \Omega$ forms a closed, compact hyper-surface in $\mathbb{R}^d$, which we also assume to be of class C^{3-}. The outer

normal of Ω_1 on Σ is denoted by ν^Σ. In applications, Ω_2 typically is connected; we assume this, below, but Ω_1 is allowed to be disconnected.

Next, we let E be a finite-dimensional Hilbert space and consider the second order differential operator (using Einstein's sum convention)

$$\mathcal{A}(x, \nabla) = -\partial_{x_i}(a^{ij}(x)\partial_{x_j}), \quad x \in \Omega,$$

where the coefficients $a^{ij} \in BUC^1(\Omega \setminus \Sigma; \mathcal{B}(E))$ are self-adjoint in E in the sense that $a^{ij}(x)^* = a^{ji}(x)$, which are *uniformly normally strongly elliptic*, see Section 2 for the definition. Typical examples for $\mathcal{A}(x, \nabla)$ will be the diagonal negative Laplacian or the Lamé operator from elasticity theory. Then, we define the conormal boundary operator $\mathcal{B}_\Sigma(x, \nabla)$ on Σ by means of

$$\mathcal{B}_\Sigma(x, \nabla) = \nu_i^\Sigma(x)a^{ij}(x)\partial_{x_j}, \quad x \in \Sigma;$$

$\mathcal{B}_\Gamma(x, \nabla)$ is defined on Γ in the same way. Note that the coefficients $a^{ij}(x)$ may jump across the interface Σ. Furthermore, we assume that $\mathcal{P}_\Sigma \in C^{2-}(\Sigma; \mathcal{B}(E))$ is a given family of orthogonal projections in E, and we let $\mathcal{Q}_\Sigma = I - \mathcal{P}_\Sigma$ be its complementary projection; $\mathcal{P}_\Gamma$ and $\mathcal{Q}_\Gamma$ are defined similarly. One should think, e.g., of $\mathcal{P}_\Sigma$ as the identity I, a projection onto a fixed subspace of E, or of the projection to the tangent bundle of Σ.

Then, with $\mu \geq 0$, in the one-phase case, we consider the two problems

$$\begin{aligned}
\mu v + \mathcal{A}(x, \nabla)v &= 0 \quad \text{in } \Omega_1, \\
\mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v = 0, \quad \mathcal{P}_\Sigma(x)v &= u \quad \text{on } \Sigma,
\end{aligned}$$
(1.1)

where here, without loss of generality, we may assume that Ω_1 is connected, and

$$\begin{aligned}
\mu v + \mathcal{A}(x, \nabla)v &= 0 \quad \text{in } \Omega_2, \\
\mathcal{Q}_\Gamma(x)\mathcal{B}_\Gamma(x, \nabla)v = 0, \quad \mathcal{P}_\Gamma(x)v &= 0 \quad \text{on } \Gamma, \\
\mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v = 0, \quad \mathcal{P}_\Sigma(x)v &= u \quad \text{on } \Sigma.
\end{aligned}$$
(1.2)

The condition $\mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v = 0$ can be replaced by $\mathcal{Q}_\Sigma(x)v = 0$. In

the two-phase case, we have the problems

$$
\begin{aligned}
\mu v + \mathcal{A}(x, \nabla)v &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
\mathcal{Q}_\Gamma(x)\mathcal{B}_\Gamma(x, \nabla)v = 0, \qquad \mathcal{P}_\Gamma(x)v &= 0 \quad \text{on } \Gamma, \\
[\![\mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v]\!] = \mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v &= 0 \quad \text{on } \Sigma, \\
[\![\mathcal{P}_\Sigma(x)v]\!] = 0, \qquad \mathcal{P}_\Sigma(x)v &= u \quad \text{on } \Sigma,
\end{aligned}
$$

(1.3)

and

$$
\begin{aligned}
\mu v + \mathcal{A}(x, \nabla)v &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
\mathcal{Q}_\Gamma(x)\mathcal{B}_\Gamma(x, \nabla)v = 0, \quad \mathcal{P}_\Gamma(x)v &= 0 \quad \text{on } \Gamma, \\
[\![\mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v]\!] = \mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v &= 0 \quad \text{on } \Sigma, \\
[\![\mathcal{P}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v]\!] = 0, \quad [\![\mathcal{P}_\Sigma(x)v]\!] &= u \quad \text{on } \Sigma.
\end{aligned}
$$

(1.4)

Here, $[\![w]\!](x) = \lim_{\epsilon \to 0_+}(w(x + \epsilon\nu_\Sigma(x)) - w(x - \epsilon\nu_\Sigma(x)))$ denotes the jump of a quantity w across the interface Σ. In the two-phase problems, we may exchange the conditions of the $\mathcal{Q}$-part on Σ by

$$
[\![\mathcal{Q}_\Sigma(x)v]\!] = \mathcal{Q}_\Sigma(x)v = 0 \quad \text{or by} \quad [\![\mathcal{Q}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v]\!] = [\![\mathcal{Q}_\Sigma(x)v]\!] = 0.
$$

Thus, the main variable u lives on the hyper-surface Σ, and it serves as a boundary input for various elliptic problems. The outputs will yield the *Dirichlet-Neumann operators* which are defined by

$$
\begin{aligned}
A_{j,\mu}u(x) &= (-1)^{j+1}\mathcal{P}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v(x), \quad x \in \Sigma, \; j = 1, 2, 4, \\
A_{3,\mu}u(x) &= -[\![\mathcal{P}_\Sigma(x)\mathcal{B}_\Sigma(x, \nabla)v(x)]\!], \quad x \in \Sigma.
\end{aligned}
$$

(1.5)

In Section 3, we prove that all of these operators are $\mathcal{R}$-sectorial with the $\mathcal{R}$-angle equal to zero, in a proper L_q-setting. In all cases, the operators $A_{j,\mu}$ are self-adjoint in $L_2(\Sigma, \mathcal{P}_\Sigma E)$, and we have

$$
(A_{j,\mu}u|u)_2 = \mu|v|_2^2 + (a\nabla v|\nabla v)_2 \geq 0.
$$

(1.6)

The dynamics lives on the hyper-surface Σ. It reads

$$
u(t) + b * Au(t) = f(t), \quad t > 0,
$$

(1.7)

where the forcing function $f(t)$ is given, and A denotes any of the Dirichlet-Neumann operators introduced above. Here, $*$ denotes convolution in time,

$$
b * w(t) := \int_0^t b(t - s)w(s)\, ds, \quad t > 0.
$$

The kernel $b(t)$ is a scalar function $b \in L_{1,loc}[0, \infty)$ of subexponential growth, which is 1-*regular* and *sectorial*, see Section 2 for the definitions. Typical examples for kernels $b(t)$ are the *standard kernels* $g_{\alpha,\eta}(t)$, defined by

$$g_{\alpha,\eta}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\eta t}, \quad t > 0,$$

where $\alpha \in (0, 2)$ and $\eta \geq 0$. Then, for $\alpha \in (0, 1]$, $\eta = 0$ and $f = u_0 + b * h$, equation (1.7) is equivalent to the time-fractional equation of order $\alpha \leq 1$

$$\partial_t^\alpha (u - u_0) + Au = h, \quad t > 0, \ u(0) = u_0.$$

On the other hand, for $\alpha \in (1, 2)$, $\eta = 0$ and $f = u_0 + t u_1 + b * h$, we obtain the equivalent time-fractional evolution equation of order $\alpha \in (1, 2)$

$$\partial_t^{\alpha-1} (\partial_t u - u_1) + Au = h, \quad t > 0, \ u(0) = u_0, \ \partial_t u(0) = u_1.$$

However, of course, the kernels $b(t)$ in (1.7) can be much more general among others; also fashionable limiting cases, like the so-called *ultra-slow diffusion* are allowed. An example for this is the kernel:

$$b(t) = e^{-t} \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} \, d\alpha, \quad t > 0.$$

In the simplest case, (1.1) with $E = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{P}_\Sigma = 1$ and $\mathcal{B}_\Sigma = \partial_\nu$ becomes the problem

$$
\begin{aligned}
\mu v - \Delta v &= 0 \quad \text{in } \Omega_1, \\
v &= u \quad \text{on } \Sigma, \\
u + b * \partial_\nu v &= f \quad \text{on } \Sigma.
\end{aligned}
$$

(1.8)

Here, $A_{1,\mu}$ is the standard Dirichlet-Neumann operator. For this problem, we refer to Section 4, Example 4.1.

In the general case, the analysis is based on the strong L_q-theory for elliptic problems. Thus, for each time $t > 0$, the bulk variable v will belong to the standard space H_q^2. Trace theory then yields the *regularity space* $X_1 = W_q^{2-1/q}(\Sigma; \mathcal{P}_\Sigma E)$ for u, and, as the Dirichlet-Neumann operators are all of order 1, the *base space* for the dynamic equation will be $X_0 = W_q^{1-1/q}(\Sigma; \mathcal{P}_\Sigma E)$.

For the elliptic theory, we refer to the monograph by Prüss and Simonett [8, Chapter 6], in particular, for the construction and

properties of Dirichlet-Neumann operators. On the other hand, the analysis of the dynamic equation (1.7) is based on vector-valued harmonic analysis, in particular, on maximal L_p-regularity. For this, we refer to the monographs [**6,** Section 8] and [**8,** Chapter 4].

The organization for this paper is as follows. In the next section, we state the main results which are proven in Section 3. Section 4 deals with several applications to problems in mathematical physics, and in the last two sections, we consider extensions of our theory to Stokes problems with fractional boundary damping, and to fractional impedance boundary conditions in electrodynamics.

2. Main results. As previously mentioned, we let $q \in (1, \infty)$, and set
$$X_0 = W_q^{1-1/q}(\Sigma; \mathcal{P}_\Sigma E), \qquad X_1 = W_q^{2-1/q}(\Sigma; \mathcal{P}_\Sigma E).$$

These spaces are well known to be of class $\mathcal{HT}$ (equivalently UMD). The first result concerns the Dirichlet-Neumann operators $A_{j,\mu}$.

Theorem 2.1. *Let $1 < q < \infty$, E a finite-dimensional Hilbert space and Ω, Γ and Σ as above. Assume*

(H1) $a^{ij} \in BUC^{1-}(\Omega \setminus \Sigma; \mathcal{B}(E))$ *are self-adjoint in E and $\mathcal{A}(x, \nabla)$ is uniformly, normally strongly elliptic.*

(H2) $\mathcal{P}_\Gamma \in C^{2-}(\Gamma; \mathcal{B}(E))$ *and $\mathcal{P}_\Sigma \in C^{2-}(\Sigma; \mathcal{B}(E))$ are families of orthogonal projections in E.*

Then, the Dirichlet-Neumann operators $A_{j,\mu}$ introduced above are well-defined $\mathcal{R}$-sectorial operators in X_0 with $\mathcal{R}$-angle zero. These operators have compact resolvent, their spectra consist only of nonnegative eigenvalues, their domain is $\mathsf{D}(A_{j,\mu}) = X_1$, and $A_{j,\mu}$ is invertible if and only if the kernel $\mathsf{N}(A_{j,\mu})$ is trivial. In particular, $A_{j,\mu}$ is invertible for all j, in case $\mu > 0$.

This result is proven in Section 3. Recall from [**8,** Chapter 6] that $\mathcal{A}(x, \nabla)$ is *strongly elliptic*, if there is a constant $c(x) > 0$, such that
$$\mathrm{Re}(\mathcal{A}(x, i\xi)v|v)_E \geq c(x)|v|_E^2, \quad \text{for all } v \in E, \ \xi \in \mathbb{R}^d, \ |\xi| = 1.$$

$\mathcal{A}(x, \nabla)$ is called *normally strongly elliptic* if in addition
$$\mathrm{Re}(a^{ij}(x)(\xi_j v + \nu_j w)|\xi_i v + \nu_i w)_E \geq c(x)|\mathrm{Im}(v|w)|$$

holds, for all $\xi, \nu \in \mathbb{R}^d$, $|\xi| = |\nu| = 1$, $(\xi|\nu) = 0$ and $v, w \in E$. The attribute *uniformly* means that the constant $c(x)$ can be chosen independently of $x \in \Omega \setminus \Sigma$. Note that, by self-adjointness, of the coefficients a^{ij}, the terms $(\mathcal{A}(x, \nabla)v|v)_E$ and $(a^{ij}(x)(\xi_j v + \nu_j w)|\xi_i v + \nu_i w)_E$ are actually real.

Next, we consider the kernel $b \in L_{1,loc}[0, \infty)$. It is of *subexponential growth* if

$$\int_0^\infty e^{-\eta t} |b(t)| \, dt < \infty, \quad \text{for each } \eta > 0.$$

Then, the Laplace transform

$$\mathcal{L}b(z) := \widehat{b}(z) := \int_0^\infty e^{-zt} b(t) \, dt, \quad \operatorname{Re} z > 0,$$

is well-defined and holomorphic on the right half-plane $\mathbb{C}_+ = \Sigma_{\pi/2}$, where we use the notation for sectors

$$\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}, \quad \theta \in (0, \pi).$$

We say that a kernel $b(t)$ of subexponential growth is *sectorial*, if $\widehat{b}(z) \neq 0$ in $\mathbb{C}_+$, and

$$\sup\{|\arg(\widehat{b}(z))| : z \in \mathbb{C}_+\} =: \theta_b \in (0, \pi);$$

then, we call θ_b the *angle* of b.

Given $k \in \mathbb{N}$, the kernel b of subexponential growth is called *k-regular*, if there is a constant $c > 0$ such that

$$|z^n \widehat{b}^{(n)}(z)| \leq c|\widehat{b}(z)|, \quad z \in \mathbb{C}_+, \quad 0 \leq n \leq k.$$

For properties of k-regular kernels, see [**6**, Sections 3, 8]. In particular, we note that the Laplace transform of a 1-regular kernel b continuously extends in $\mathbb{C}$ to $\overline{\mathbb{C}}_+ \setminus \{0\}$, $\widehat{b}(z)$ does not vanish there, and the function $\widehat{b}(i\cdot)$ belongs to $W_{\infty, loc}^1(\mathbb{R} \setminus \{0\})$.

Typical examples are the standard kernels previously introduced, with $\alpha \in (0, 2)$ and $\eta \geq 0$. We have

$$\widehat{g}_{\alpha,\eta}(z) = (z + \eta)^{-\alpha}, \quad z \in \mathbb{C}_+;$$

therefore, these kernels are k-regular for any $k \in \mathbb{N}$, and they are sectorial with angle $\theta_{g_{\alpha,\eta}} = \alpha\pi/2 < \pi$.

Another class of typical examples are kernels, which are nonnegative, nonincreasing and convex; such kernels are 1-regular and sectorial with angle $\theta_b \leq \pi/2$.

Let b be a 1-regular sectorial kernel. Then, we may define an operator B in $L_p(\mathbb{R}; Y)$ by means of

$$(2.1) \qquad \mathcal{F}(Bu)(\xi) := \frac{1}{\widehat{b}(i\xi)}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}, \ \mathcal{F}(u) \in \mathcal{D}(\mathbb{R} \setminus \{0\}; Y).$$

Here Y denotes a UMD-space and $\mathcal{F}$ means the Fourier transform. Since the set $\mathcal{D}(\mathbb{R} \setminus \{0\}; Y)$ is dense in $L_p(\mathbb{R}; Y)$ for $1 < p < \infty$, B is well defined with dense domain. It was shown in [6, subsection 8.4] that B is sectorial, causal and admits bounded imaginary powers. The proof given there also actually shows $B \in \mathcal{H}^\infty(L_p(\mathbb{R}; Y))$ with $\mathcal{H}^\infty$-angle $\phi_B^\infty = \theta_b$, which means that, for any $\phi > \theta_b$, there is a constant $C_\phi > 0$, such that

$$|h(B)| \leq C_\phi |h|_\infty, \quad \text{for all } h \in H^\infty(\Sigma_\phi).$$

By causality, its restriction to $L_p(\mathbb{R}_+; Y)$ has the same properties.

In [6, Section 8], it is also shown that the spectrum of B in $L_p(\mathbb{R}_+; Y)$ is given by

$$\sigma(B) = \overline{1/\widehat{b}(\mathbb{C}_+)};$$

in particular, B is invertible if and only if $\widehat{b}(z)$ is bounded on $\mathbb{C}_+$. This is trivially the case if $b \in L_1(\mathbb{R}_+)$.

Of interest are also the weighted spaces $L_{p,\mu}(\mathbb{R}_+; Y)$, defined by

$$u \in L_{p,\mu}(\mathbb{R}_+; Y) \iff t^{1-\mu}u \in L_p(\mathbb{R}_+; Y),$$

where $1 < p < \infty$ and $1/p < \mu \leq 1$. It turns out that the above properties of B are also valid in the weighted spaces $L_{p,\mu}(\mathbb{R}_+; Y)$. The reader is referred to [7] for an elementary proof of this fact.

Obviously, the operators $A = A_{j,\mu}$ and B commute in the resolvent sense; hence, we may apply a variant of the Dore-Venni theorem, see [8, Chapter 4, Corollary 4.5.9] to obtain closedness of $A + B$ with domain $\mathsf{D}(A) \cap \mathsf{D}(B)$, and $A + B$ is invertible if either A or B is invertible. This yields the following, global result.

Theorem 2.2. *Let $1 < p, q < \infty$, $1/p < \mu \leq 1$, assume that $b \in L_{1,\mathrm{loc}}[0, \infty)$ is of subexponential growth, 1-regular and sectorial with angle $\theta_b < \pi$,*

and let A denote any of the Dirichlet-Neumann operators introduced above. Furthermore, assume

$$\mathsf{N}(A) = 0 \quad or \quad b \in L_1(\mathbb{R}_+).$$

Then, for each $g \in L_{p,\mu}(\mathbb{R}_+; X_0)$, the equation

(2.2) $u(t) + b * Au(t) = b * g(t), \quad t > 0,$

admits a unique solution $u \in L_{p,\mu}(\mathbb{R}_+; X_1)$. Moreover, there is a constant $C > 0$ such that

$$|u|_{L_{p,\mu}(\mathbb{R}_+;X_1)} \le C |g|_{L_{p,\mu}(\mathbb{R}_+;X_0)}.$$

There is also a local version of this result, which is obtained by an exponential shift. For this purpose, multiply (2.2) with $e^{-\omega t}$, set $u_\omega(t) = e^{-\omega t} u(t)$, $g_\omega(t) = e^{-\omega t} g(t)$ and $b_\omega(t) = e^{-\omega t} b(t)$. Then, (2.2) is equivalent to

$$u_\omega(t) + b_\omega * Au_\omega(t) = b_\omega * g_\omega(t), \quad t > 0.$$

Since b_ω is in $L_1(\mathbb{R}_+)$ and is also 1-regular and sectorial with the same angle θ_b, we may apply Theorem 2.2 to this equation to obtain the following corollary.

Corollary 2.3. *Let $1 < p$, $q < \infty$, $1/p < \mu \le 1$, $J = (0, a)$, assume that $b \in L_{1,\mathrm{loc}}[0, \infty)$ is of subexponential growth, 1-regular and sectorial with angle $\theta_b < \pi$, and let A denote any of the Dirichlet-Neumann operators introduced above. Then, for each $g \in L_{p,\mu}(J; X_0)$, the equation*

(2.3) $u(t) + b * Au(t) = b * g(t), \quad t \in J,$

admits a unique solution $u \in L_{p,\mu}(J; X_1)$. Moreover, there is a constant $C(a) > 0$ such that

$$|u|_{L_{p,\mu}(J;X_1)} \le C(a) |g|_{L_{p,\mu}(J;X_0)}.$$

A natural question which arises is that of the precise time-regularity of the solution of (2.2). Of course, this will depend on the kernel, and, for the standard kernels, it is easily seen that $u \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; X_0)$. Based on the characterization of $\mathsf{D}(B)$ obtained in [**6**, Section 8] for L_p, which carries over to $L_{p,\mu}$, we have the following corollary.

Corollary 2.4. *In addition to the assumptions of Theorem* 2.2, *suppose that*:

$$(2.4) \qquad\qquad \liminf_{z \to 0+} |\widehat{b}(z)| > 0,$$

and, for some $\alpha \in (0, 2)$,

$$(2.5) \qquad 0 < \liminf_{r \to \infty} r^{\alpha} |\widehat{b}(r)| \le \limsup_{r \to \infty} r^{\alpha} |\widehat{b}(r)| < \infty.$$

Then, $u \in {}_0 H^{\alpha}_{p,\mu}(\mathbb{R}_+; X_0)$. *Moreover, we have*

$$u \in H^{\alpha}_p((\delta, \infty); X_0) \cap L_p((\delta, \infty); X_1),$$

for any $\delta > 0$.

Note that (2.4) is only a mild restriction; it holds as soon as $\lim_{t \to \infty} \int_0^t b(s) \, ds \ne 0$ exists in $\mathbb{C} \cup \{\infty\}$. In particular, this condition is satisfied if either $b \in L_1(\mathbb{R}_+)$ with $\widehat{b}(0) \ne 0$, or if $b(t)$ is nonnegative on $\mathbb{R}_+$.

By [**8,** Theorem 3.4.8], for $\alpha > 1 - \mu + 1/p$, we have the embedding

$$H^{\alpha}_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1) \hookrightarrow C([0, \infty); (X_0, X_1)_{(1-(1-\mu+1/p)/\alpha, p)}).$$

This implies the following result, which in particular, yields parabolic regularization of the solution of (2.2).

Corollary 2.5. *Let the assumptions of Corollary* 2.4 *hold, and let* $\alpha > 1 - \mu + 1/p$. *Then,*

$$u \in C([0, \infty); W^{s_0}_q(\Sigma; \mathcal{P}_\Sigma E)) \cap C((0, \infty); W^{s_1}_q(\Sigma; \mathcal{P}_\Sigma E)),$$

where $s_0 = 2 - 1/q - (1 - \mu + 1/p)/\alpha$ *and* $s_1 = 2 - 1/q - 1/\alpha p$.

For the solution v of the bulk-part, we have

Corollary 2.6. *Let the assumptions of Corollary* 2.4 *hold. Then, the solution* v *of the elliptic problem satisfies*

$$v \in {}_0 H^{\alpha}_{p,\mu}(\mathbb{R}_+; H^1_q(\Omega \setminus \Sigma; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\Omega \setminus \Sigma; E)).$$

Since either $A = A_{j,\mu}$ is invertible or the eigenvalue 0 is semisimple, we have the decomposition $X_0 = \mathsf{N}(A) \oplus \mathsf{R}(A)$. Let P denote the projection of X_0 to $\mathsf{N}(A)$ along $\mathsf{R}(A)$, and $Q = 1 - P$. Then, we may split the dynamic equation $u + b * Au = f$ into two parts, namely,

$$Qu + b * AQu = Qf, \qquad Pu = Pf.$$

This shows that, in the half-line case, the conditions for well-posedness are $Pf \in L_{p,\mu}(\mathbb{R}_+; X_1)$ and $Qf = b * Qg$ with $g \in L_{p,\mu}(\mathbb{R}_+; X_0)$. Then, the solution belongs to $L_{p,\mu}(\mathbb{R}_+; X_1)$ as $\mathsf{N}(A) \subset \mathsf{D}(A) = X_1$.

3. Proof of Theorem 2.1.

Step I. *The Dirichlet-Neumann operators.* Here, we show that the Dirichlet-Neumann operators $A_{j,\mu}$, introduced in Section 1, are well defined and bounded from X_1 to X_0. The exposition will be brief since all essential ingredients are available in the literature, see e.g., [**8,** Chapter 6].

(i) Since, by assumption, $\mathcal{A}(x, \nabla)$ is uniformly normally strongly elliptic, by [**8,** subsection 6.2.5], we see that, in all four problems under consideration, the *Lopatinskii-Shapiro condition* is satisfied on Γ as well as on Σ. Therefore, by localization and perturbation arguments, given $u \in X_1$, problems $(1.1) \sim (1.4)$ are uniquely solvable with solution $v \in H_q^2$, provided $\mu > 0$ is large enough. Therefore, the Dirichlet-Neumann operators $A_{j,\mu}$ are well defined and bounded from X_1 to X_0 for large $\mu > 0$.

(ii) Next, we set $\mathbb{X}_{0,j} = L_q(\Omega_j; E)$ for $j = 1, 2$ and $\mathbb{X}_{0,j} = L_q(\Omega; E)$ for $j = 3, 4$, and define operators $\mathbb{A}_j$ in $\mathbb{X}_{0,j}$ by means of

$$\mathbb{A}_j v(x) = \mathcal{A}(x, \nabla)v(x), \quad v \in \mathsf{D}(\mathbb{A}_j),$$

with domains $\mathbb{X}_{1,j} := \mathsf{D}(\mathbb{A}_j)$, defined by

$$\mathsf{D}(\mathbb{A}_1) = \{v \in H_q^2(\Omega_1) : \mathcal{Q}_\Sigma \mathcal{B}_\Sigma(x, \nabla)v = \mathcal{P}_\Sigma v = 0 \quad \text{on } \Sigma\},$$
$$\mathsf{D}(\mathbb{A}_2) = \{v \in H_q^2(\Omega_2) : \mathcal{Q}_K \mathcal{B}_K(x, \nabla)v = \mathcal{P}_K v = 0 \ \text{on } K,$$
$$K = \Sigma, \Gamma\},$$
$$\mathsf{D}(\mathbb{A}_3) = \{v \in H_q^2(\Omega \setminus \Sigma) : \mathcal{Q}_\Gamma \mathcal{B}_\Gamma(x, \nabla)v = \mathcal{P}_\Gamma v = 0 \ \text{on } \Gamma,$$
$$[\![\mathcal{Q}_\Sigma \mathcal{B}_\Sigma(x, \nabla)v]\!] = \mathcal{Q}_\Sigma \mathcal{B}_\Sigma(x, \nabla)v = [\![\mathcal{P}_\Sigma v]\!]$$
$$= \mathcal{P}_\Sigma v = 0 \quad \text{on } \Sigma\},$$

$$\mathsf{D}(\mathbb{A}_4) = \{v \in H_q^2(\Omega \setminus \Sigma) : \mathcal{Q}_\Gamma \mathcal{B}_\Gamma(x, \nabla)v = \mathcal{P}_\Gamma v = 0 \quad \text{on } \Gamma,$$
$$[\![\mathcal{B}_\Sigma(x, \nabla)v]\!] = \mathcal{Q}_\Sigma \mathcal{B}_\Sigma(x, \nabla)v$$
$$= [\![\mathcal{P}_\Sigma v]\!] = 0 \text{ on } \Sigma\}.$$

Then, the results in [**8,** Chapter 6] show that $\mathbb{A}_j + \mu$ are $\mathcal{R}$-sectorial in $\mathbb{X}_{0,j}$, with $\mathcal{R}$-angle $< \pi/2$, provided $\mu \geq 0$ is sufficiently large. Moreover, $\mathbb{A}_j$ is self-adjoint for $q = 2$, as the coefficients a^{ij} are self-adjoint in E. Since Ω is bounded, the embeddings $\mathbb{X}_{1,j} \hookrightarrow \mathbb{X}_{0,j}$ are compact, $\mathbb{A}_j$ has compact resolvent, and thus, the spectrum $\sigma(\mathbb{A}_j)$ consists only of countably many eigenvalues of finite multiplicity, which by elliptic regularity are independent of q. Hence, to study these eigenvalues, it is sufficient to consider the case $q = 2$.

(iii) For each $j = 1, \ldots, 4$, by integration by parts, we have

$$(\mathbb{A}_j v|v)_{L_2} = (a\nabla v|\nabla v)_{L_2}, \quad v \in \mathsf{D}(\mathbb{A}_j),$$

as the boundary terms disappear due to the imposed boundary conditions. The coefficients a^{ij} are, by assumption, self-adjoint, and also positive definite by strong ellipticity; therefore, $(\mathbb{A}_j v|v)_{L_2}$ is real and nonnegative, which shows that $\sigma(\mathbb{A}_j) \subset [0, \infty)$ for all j, and thus, by a perturbation argument, $\mathbb{A}_j$ is also $\mathcal{R}$-sectorial and invertible if and only if its kernel $\mathsf{N}(\mathbb{A}_j)$ is trivial. More generally, we have $\mathsf{N}(\mathbb{A}_j) \oplus \mathsf{R}(\mathbb{A}_j) = \mathbb{X}_{0,j}$ for each j.

(iv) To solve problems $(1.1) \sim (1.4)$ for a given $u \in X_1$ and any $\mu \geq 0$, we proceed as follows. Decompose the solution as $v = v_0 + w$, where v_0 solves the corresponding problem with large enough μ_0. Then, w must solve the problem

$$(\mu + \mathcal{A}(x, \nabla))w = (\mu + \mathcal{A}(x, \nabla))v - (\mu + \mathcal{A}(x, \nabla))v_0 = (\mu_0 - \mu)v_0,$$

and w must have homogeneous boundary data. This means

$$(\mu + \mathbb{A}_j)w = (\mu_0 - \mu)v_0;$$

hence, $w = (\mu + \mathbb{A}_j)^{-1}(\mu_0 - \mu)v_0$. This shows that $(1.1) \sim (1.4)$ is uniquely solvable for each $\mu > 0$ as well as for $\mu = 0$ in the case $\mathsf{N}(\mathbb{A}_j) = 0$.

From (iii), we see that, if $e \in \mathsf{N}(\mathbb{A}_j)$, then $\nabla e = 0$. Hence, e is constant in Ω_2 (as Ω_2 is connected by assumption), and e is constant in the components of Ω_1. This implies, in particular, that $\mathcal{B}_\Sigma(x, \nabla)v = 0$; thus, $A_{j,0}e = 0$. On the other hand, if $\mathsf{N}(\mathbb{A}_j)$ is nontrivial, then the problem

$\mathbb{A}_j v = f$ admits a solution if and only $(f|e)_{L_2} = 0$ for all $e \in \mathsf{N}(\mathbb{A}_j)$. From integration by parts, we obtain

$$0 = (a\nabla v_0|\nabla e)_{L_2} = (\mathcal{A}(x, \nabla)v|e)_{L_2} = -\mu_0(v|e),$$

which shows that (1.1) $\sim$ (1.4) admits a solution $v \in H_q^2$ even if $\mu = 0$. Therefore, the Dirichlet-Neumann operators $A_{j,\mu} \in \mathcal{B}(X_1, X_0)$ are well defined, for all $\mu \geq 0$.

Step II. $\mathcal{R}$-*sectoriality of* $A_{j,\mu}$.

(i) To obtain the resolvent of $A_{j,\mu}$, i.e., to solve the problem $\lambda u + A_{j,\mu} u = f$, we must replace the boundary condition $\mathcal{P}_\Sigma v = u$ by the condition

$$\lambda \mathcal{P}_\Sigma v + (-1)^{j+1}\mathcal{P}_\Sigma \mathcal{B}_\Sigma(x, \nabla)v = f$$

and set $u = \mathcal{P}_\Sigma v$, for the cases $j = 1, 2, 3$. For $j = 4$, in a similar manner, we must replace the condition $[\![\mathcal{P}_\Sigma v]\!] = u$ by

$$\lambda[\![P_\Sigma v]\!] - [\![\mathcal{P}_\Sigma \mathcal{B}_\Sigma(x, \nabla)v]\!] = f,$$

and set $u = [\![\mathcal{P}_\Sigma v]\!]$. Since $X_1 \hookrightarrow X_0$ compactly, we see that the operators $A_{j,\mu}$ also have compact resolvents; hence, their spectra are discrete and consist only of countably many eigenvalues of finite algebraic multiplicity, which are also independent of q provided the resolvent sets are nonempty. Furthermore, if this is the case, by (1.6), we see $\sigma(A_{j,\mu}) \subset [0, \infty)$, for all $\mu \geq 0$ and j.

(ii) To prove that $A_{j,\mu}$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle zero, we observe that it is sufficient to prove this for some large $\mu_0 > 0$. In fact, $A_{j,\mu}$ is a relatively compact perturbation of A_{j,μ_0}. In order to see this, let $u \in X_1$ be given, and let $T_{j,\mu}u$ denote the solution of problem (1.j). Then, $v := T_{j,\mu}u - T_{j,\mu_0}$ satisfies the problem

$$(\mu + \mathbb{A}_j)v = (\mu_0 - \mu)T_{j,\mu_0},$$

since the boundary conditions for v are homogeneous. Hence, we obtain the identity

$$T_{j,\mu}u = T_{j,\mu_0}u + (\mu_0 - \mu)(\mu + \mathbb{A}_j)^{-1}T_{j,\mu_0}u,$$

which yields, for $j = 1, 2, 3$,

$$A_{j,\mu}u = A_{j,\mu_0}u + (\mu_0 - \mu)(-1)^{j+1}\mathcal{P}_\Sigma \mathcal{B}_\Sigma(x, \nabla)(\mu + \mathbb{A}_j)^{-1}T_{j,\mu_0}u.$$

Obviously, the second term on the right hand side is compact in $\mathcal{B}(X_1, X_0)$.

As a consequence, well-known perturbation theory, see [**8,** Chapters 3, 4], shows that, with A_{j,μ_0}, $A_{j,\mu}$ is also $\mathcal{R}$-sectorial with the same $\mathcal{R}$-angle zero.

(iii) To show that $A_{j,\mu}$ is $\mathcal{R}$ sectorial with $\mathcal{R}$-angle zero for $\mu > 0$ large, we apply the standard techniques of localization and perturbation, as in [**8,** Chapter 6], to reduce to the following three model problems at points on Σ.

$$(3.1) \qquad \mu v + \mathcal{A}(\nabla)v = 0 \quad \text{in} \quad \mathbb{R}^{d-1} \times \mathbb{R}_+,$$
$$\mathcal{Q}\mathcal{B}(\nabla)v = 0, \quad \lambda \mathcal{P}v - \mathcal{P}\mathcal{B}(\nabla)v = f \quad \text{on} \quad \mathbb{R}^{d-1};$$

$$\mu v + \mathcal{A}(\nabla)v = 0 \quad \text{in} \quad \mathbb{R}^{d-1} \times \dot{\mathbb{R}},$$
$$(3.2) \qquad [\![\mathcal{Q}\mathcal{B}(\nabla)v]\!] = \mathcal{Q}\mathcal{B}(\nabla)v = [\![\mathcal{P}v]\!] = 0, \quad \text{on} \quad \mathbb{R}^{d-1},$$
$$\lambda \mathcal{P}v + [\![\mathcal{P}\mathcal{B}(\nabla)v]\!] = f \quad \text{on} \quad \mathbb{R}^{d-1};$$

$$\mu v + \mathcal{A}(\nabla)v = 0 \quad \text{in} \quad \mathbb{R}^{d-1} \times \dot{\mathbb{R}},$$
$$(3.3) \qquad [\![\mathcal{B}(\nabla)v]\!] = \mathcal{Q}\mathcal{B}(\nabla)v = 0 \quad \text{on} \quad \mathbb{R}^{d-1},$$
$$\lambda [\![\mathcal{P}v]\!] + \mathcal{P}\mathcal{B}(\nabla)v = f \quad \text{on} \quad \mathbb{R}^{d-1}.$$

Here, $\mathcal{A}(\nabla)$ is normally strongly elliptic and self-adjoint, $\mathcal{P}$ is an orthogonal projection in E, $\mathcal{Q} = I - \mathcal{P}$, and $\lambda \in \Sigma_\pi$. The inhomogeneity $f \in W_q^{1-1/q}(\mathbb{R}^{d-1})$ is given.

The most important fact is that each of these model problems satisfies the corresponding *Lopatinskii-Shapiro condition*. This condition means that the ode-systems resulting from the Fourier transform in the tangential space directions are uniquely solvable. In order to prove this, we replace ∇ by $i\xi + \nu\partial_y$, with $y > 0$ for (3.1) and $y \in \dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$ for the others, and we set $f = 0$. With integration by parts, in each problem, we obtain

$$\lambda |\mathcal{P}v|_{L_2}^2 + \mu |v|_{L_2}^2 + (a(i\xi + \nu\partial_y)v|(i\xi + \nu\partial_y)v)_{L_2} = 0.$$

This implies the unique solvability in L_2 of the Fourier transformed problems.

In the sequel, we concentrate on problem (3.1) since the other two can be treated in a similar manner, reflecting the lower half-space to the

upper and doubling the variables, as explained in [**8,** subsection 6.5].

(iv) Thus, we must solve the Fourier transformed problem (3.1), which reads

$$
\begin{aligned}
(\mu + \mathcal{A}(i\xi + \nu\partial_y))\widetilde{v} &= 0, & y &> 0, \\
\mathcal{Q}\mathcal{B}(i\xi + \nu\partial_y)\widetilde{v} &= 0, & y &= 0, \\
\lambda\mathcal{P}\widetilde{v} - \mathcal{P}\mathcal{B}(i\xi + \nu\partial_y)\widetilde{v} &= \widetilde{f}, & y &= 0.
\end{aligned}
$$

(3.4)

Here, we have suppressed the dependence of $\widetilde{v}$ and $\widetilde{f}$ on ξ. The appropriate scaling is

$$
\rho = \lambda + (\mu + |\xi|^2)^{1/2}, \quad \sigma = \sqrt{\mu}/\rho, \ \ b = \xi/\rho, \ \lambda = 1 - (\sigma^2 + |b|^2)^{1/2}.
$$

Proceeding as in [**8,** subsection 6.2], we find a function M, bounded and holomorphic such that

$$
\widetilde{u} = \big((\lambda + A_{1,\mu})^{-1}f\big)\widetilde{} = \mathcal{P}\widetilde{v} = \frac{1}{\rho}M(\sigma, b)\widetilde{f}.
$$

(3.5)

Then, by [**8,** Theorem 4.3.9], the operator family $\{\lambda(\lambda + A_{1,\mu})^{-1} : \lambda \in \Sigma_\phi\}$ is $\mathcal{R}$-bounded in $L_q(\mathbb{R}^{d-1}; E)$, for any $\phi < \pi$. Since the tangential variables commute with $A_{1,\mu}$, this is also true in $H_q^1(\mathbb{R}^{d-1}; E)$, and then, by real interpolation, also in $X_0 = W_q^{1-1/q}(\mathbb{R}^{d-1}; E)$. This shows that $A_{1,\mu}$ is $\mathcal{R}$-sectorial in X_0, with $\mathcal{R}$-angle zero.

4. Examples and applications. The first example serves as the prototype example for our theory. It appears in several applications, for example, in eddy current models in electrodynamics, cf., [**1**].

Example 4.1. Let $E = \mathbb{C}$, $\mathcal{P}_\Gamma = \mathcal{P}_\Sigma = I$ and $a^{ij} = \delta_{ij}$, i.e., $\mathcal{A}(x, \nabla) = -\Delta$ and $\mathcal{B}(x, \nabla) = \partial_\nu$. Then, obviously, the assumptions of Theorem 2.1 are satisfied, and thus, the results in Section 2 apply. Therefore, it remains to compute the kernels $\mathsf{N}(A_{j,0})$.

Therefore, let $A_{j,\mu}u = 0$. From (1.6), then, we deduce that $\nabla v = 0$ in Ω; hence, v is constant in Ω_2 and in the components of Ω_1. This implies that $v = 0$ in Ω_2 for $j = 2, 3, 4$; hence, $u = 0$ for $j = 2, 3$. Thus, in these cases, $A_{j,0}$ is invertible. On the other hand, for $j = 1, 4$, we only see that u is constant on the components of Σ. Therefore, for $j = 2, 3$, Theorem 2.2 yields well-posedness on the half-line without any

further assumptions on the kernel $b(t)$, while, for $j = 1, 4$, we will need, in addition, $b \in L_1(\mathbb{R}_+)$.

In the second example, we consider an *elastic body with boundary relaxation*. Let v denote the displacement, and assume the linear homogeneous isotropic *stress-strain relation*

$$S = \mu_s(\nabla v + [\nabla v]^{\mathsf{T}}) + \mu_b(\operatorname{div} v)I,$$

where $\mu_s, \mu_b \in \mathbb{R}$ are constants.

Example 4.2. Let $E = \mathbb{C}^d$, $\mathcal{P}_\Gamma v = v \cdot \nu^\Gamma$ and $\mathcal{P}_\Sigma v = v \cdot \nu^\Sigma$, the orthogonal projections to the normal component of v, and $\mathcal{A}(x, \nabla)v = -\operatorname{div} S$, as well as $\mathcal{B}(x, \nabla) = \nu \cdot S$. We restrict attention here to the one-phase case $j = 2$. In this case, we have

$$A_{2,\mu} u = S\nu^\Sigma \cdot \nu^\Sigma = (2\mu_s + \mu_b)\partial_\nu v_\perp + \mu_b \operatorname{div}_\Sigma v_\|,$$

which is the normal component of the normal stress on Σ, where $v = v_\| + v_\perp \nu^\Sigma$ denotes the normal decomposition of v near Σ. It is well known that, in this case, $\mathcal{A}(x, \nabla)$ is strongly elliptic if and only if

$$\mu_s > 0 \quad \text{and} \quad 2\mu_s + \mu_b > 0,$$

while it is normally strongly elliptic if, in addition, $\mu_s + \mu_b > 0$, see, e.g., [**8**, Chapter 6.2.5]. Thus, Theorems 2.1 and 2.2 apply, and we only must compute the kernel of $A_{2,\mu}$. From the identity (1.6), we see that $N(A_{2,\mu}) = 0$ if $\mu > 0$. In the cases of $\mu = 0$ and $A_{2,0}u = 0$, we obtain $v = \text{const}$ in Ω_2, and, on Γ, we have the boundary condition $v_\perp = v \cdot \nu^\Gamma = 0$. Set $\varphi(p) = v \cdot p$; since Γ is compact, φ attains its maximum on Γ at some point p_0, and there we have, for the tangential derivatives,

$$\partial_i \varphi(p_0) = v \cdot \partial_i p_0 = v \cdot \tau_i^\Gamma(p_0) = 0,$$

i.e., $v \perp T_{p_0}\Gamma \oplus \operatorname{span}\{\nu^\Gamma(p_0)\}$, and thus, $v = 0$. This, in turn, yields $u = v \cdot \nu^\Sigma = 0$, and thus, $N(A_{2,0}) = 0$. As a consequence, we obtain, with Theorem 2.2, global existence in $L_{p,\mu}$ without any further restrictions on the involved kernel $b(t)$. Note that, in contrast to this, for the first one-phase problem, i.e., $j = 1$, the kernel $N(A_{1,0})$ is nontrivial, and it consists of functions of the form $u = v \cdot \nu^\Sigma$, with $v \in E$ arbitrary.

5. Stokes flows with the dynamic Navier condition. The results from Section 2 on problems (1.1) ∼ (1.4) can be extended for $E = \mathbb{C}^d$ to the case of Stokes problems, where the elliptic problem $\mu v + \mathcal{A}(x, \nabla)v = 0$ is replaced by

$$\mu v + \mathcal{A}(x, \nabla)v + \nabla \pi = 0, \quad \text{div } v = 0,$$

and the boundary operator $\mathcal{B}(x, \nabla)v = \nu_k a^{kl} \partial_{x_l} v$ by

$$\mathcal{B}(x, \nabla)v = \nu_k a^{kl}(x) \partial_{x_l} v - \pi \nu.$$

We will not carry out this extension in full generality here, but restrict attention to the Stokes analogue of the one-phase problem (1.1). Thus, we consider the problem

$$\begin{aligned} \mu v + \mathcal{A}(x, \nabla)v + \nabla \pi &= 0 \quad \text{in } \Omega_1, \\ \text{div } v &= 0 \quad \text{in } \Omega_1, \\ v \cdot \nu^{\Sigma} = 0, \quad \mathcal{P}_{\Sigma}(x)v &= u \quad \text{on } \Sigma, \end{aligned}$$

(5.1)

where $\mathcal{P}_{\Sigma}(x)$ denotes the orthogonal projection onto the tangent bundle of Σ, and with the dynamic equation

(5.2) $u(t) + b * Au(t) = f(t), \quad t > 0, \qquad Au := \mathcal{P}_{\Sigma} \nu_k^{\Sigma} a^{kl}(x) \partial_{x_l} v.$

This models a one-phase *Stokes flow with dynamic Navier condition* on the boundary.

Then, as in Section 3, we can prove that A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_A^R = 0$, employing the results in [**8,** Chapter 7]. In this case, the kernel of A is trivial. In fact, also in this situation, we have the identity

$$(Au|u)_2 = \mu|v|_2^2 + (a\nabla v|\nabla v)_2;$$

hence, $Au = 0$ and $\mu \geq 0$ imply $\nabla v = 0$ in Ω_1, by strong ellipticity. Therefore, v is constant, and $v \cdot \nu^{\Sigma} = 0$ on Σ. As in the previous section, the function $\varphi(p) = v \cdot p$ has a maximum at some point $p_0 \in \Sigma$ by compactness of Σ, which implies $v \cdot \tau_i^{\Sigma} = 0$ for all i at p_0, and thus, $v = 0$. Hence, $u = 0$.

As a consequence, Theorem 2.2 is also valid in the Stokes case, and (5.2) is globally well-posed in the $L_{p,\mu}$-setting. More precisely, we have

Corollary 5.1. *Let $1 < p, q < \infty$, $1/p < \mu \leq 1$, assume that $b \in L_{1,\mathrm{loc}}[0, \infty)$ is of subexponential growth, 1-regular and sectorial with*

angle $\theta_b < \pi$; suppose that $a^{ij} \in BUC^{1-}(\Omega_1; \mathcal{B}(E))$ is self-adjoint and uniformly normally strongly elliptic. Then, for each $f = b * g$ and $g \in L_{p,\mu}(\mathbb{R}_+; X_0)$, there is a unique solution $u \in L_{p,\mu}(\mathbb{R}_+; X_1)$ of (5.2). If, in addition, (2.4) and (2.5) hold for some $\alpha \in (0,2)$, then we also have $u \in {}_0 H^\alpha_{p,\mu}(\mathbb{R}_+; X_0)$.

6. Impedance boundary conditions.

(i) There is much physical and engineering literature on *eddy current models* and *surface impedance conditions*, also called *Leontovich conditions* in electro-magnetics, see e.g., [4, 9] for the physical background. A 3D-prototype example for this consists of the quasi-steady Maxwell equations

$$\text{rot rot } \mathsf{E} = \mathsf{F}, \quad \text{div } \mathsf{E} = 0 \quad \text{in } \Omega_2,$$
$$(6.1) \qquad\qquad\qquad\qquad \mathsf{E}_\| = 0 \quad \text{on } \Gamma,$$
$$\mathsf{E}_\| + b * (\nu^\Sigma \times \mathsf{E}) = 0 \quad \text{on } \Sigma.$$

In this problem, E denotes the electric field in Ω_2, consisting of a nonconducting material, Ω_1 is (not ideally) conducting, and the exterior of Ω is considered to be perfectly conducting. Here, $\mathsf{E}_\| := E - \nu(\mathsf{E}|\nu)$ means the parallel component of E near the boundary. We found in frequency domain expressions like

$$\widehat{b}(z) = z^{-1/2}, \; = z^{-1/2}(1+z)^{-1/2}, \; = z^{-1/2}(1+z)^{-1}, \; = z^{-1}(1+z)^{-1/2}$$

that the kernel $b(t)$ differs from case to case. These examples have been normalized, but, in each case, b is 1-regular and sectorial. In this last section, we want to show that our approach also works for such impedance boundary conditions.

(ii) For this purpose, observe that, with $\text{div } \mathsf{E} = 0$ and $R(\mathsf{E}) = \nabla \mathsf{E} - [\nabla \mathsf{E}]^\mathsf{T}$,

$$\text{rot rot } \mathsf{E} = -\Delta \mathsf{E} + \nabla \text{div } \mathsf{E} = -\text{div}(\nabla \mathsf{E} - [\nabla \mathsf{E}]^\mathsf{T}) = -\text{div } R(\mathsf{E})) = -\Delta \mathsf{E},$$

and
$$\nu^\Sigma \times \text{rot } \mathsf{E} = -\nu^\Sigma \cdot (\nabla \mathsf{E} - [\nabla \mathsf{E}]^\mathsf{T}]) = -\nu^\Sigma \cdot R(\mathsf{E}).$$

Therefore, the generalization of (6.1) to arbitrary dimensions $d \geq 2$

(without loss of generality, with $\mathsf{F} = 0$) reads

$$
\begin{aligned}
(6.2) \quad & \mu v + \mathcal{A}(\nabla)v := \mu v - \operatorname{div} R(v) = 0, \qquad \operatorname{div} v = 0 \quad \text{in } \Omega_2, \\
& \quad v_\parallel = 0 \quad \text{on } \Gamma, \qquad v_\parallel = u \quad \text{on } \Sigma,
\end{aligned}
$$

where $v_\parallel = \mathcal{P}_K v$, $K = \Gamma, \Sigma$, with $\mathcal{P}_\Gamma$ and $\mathcal{P}_\Sigma$ the orthogonal projections to the tangent bundles of Γ and Σ, respectively. The dynamic boundary equation reads

$$
(6.3) \qquad u + b * A_\mu u = f, \quad A_\mu u = -\nu^\Sigma \cdot R(v).
$$

For this problem, we have in analogy to (1.6),

$$
(6.4) \qquad (A_\mu u | u)_2 = \mu |v|_2^2 + \frac{1}{2} |R(v)|_2^2 \geq 0,
$$

as $\nu \cdot R(v)\nu = 0$. In order to remove the divergence condition for v, we observe that (6.2) is equivalent to

$$
\begin{aligned}
(6.5) \quad & \mu v - \Delta v = 0 & & \text{in } \Omega_2, \\
& \quad v_\parallel = \partial_\nu v_\perp = 0 & & \text{on } \Gamma, \\
& \quad \partial_\nu v_\perp = -\operatorname{div}_\Sigma u, \quad v_\parallel = u & & \text{on } \Sigma.
\end{aligned}
$$

Here, we used the decomposition $v = v_\parallel + v_\perp \nu^K$ near $K = \Sigma, \Gamma$. In fact, on Γ, we have $\operatorname{div} v = 0$ if and only if $\partial_\nu v_\perp = 0$, and on Σ, accordingly, if and only if $\operatorname{div}_\Sigma v_\parallel + \partial_\nu v_\perp = 0$. Thus, if v is a solution of (6.5) for given u, then, necessarily, $\operatorname{div} v = 0$; hence, v solves (6.2), and vice versa.

(iii) The equivalent problems (6.2) and (6.5) are not quite of the form (1.2) since $\mathcal{A}(\nabla) = -\operatorname{div} R$ is not strongly elliptic, and, in (6.5), both boundary conditions on Σ are non-homogenous. In addition, the resulting Dirichlet-Neumann operator is not of the form $A_{2,\mu}$. Instead, it reads

$$
(6.6) \qquad A_\mu u = -\nu^\Sigma \cdot R(v) = -\partial_\nu v_\parallel + \nabla_\Sigma v_\perp + L_\Sigma u,
$$

where $L_\Sigma = -\nabla_\Sigma \nu^\Sigma$ denotes the Weingarten tensor on Σ. Nevertheless, by means of similar arguments as in Section 3, Step I, we show that $A_\mu \in \mathcal{B}(X_0, X_1)$ is well defined.

For this purpose, define $\mathbb{A} = -\Delta$ in $\mathbb{X}_0 = L_q(\Omega_2; \mathbb{C}^d)$ with domain

$$
\mathbb{X}_1 = \mathsf{D}(\mathbb{A}) = \{v \in H_q^2(\Omega_2; \mathbb{C}^d) : v_\parallel = \partial_\nu v_\perp = 0 \text{ on } \Gamma \cup \Sigma\}.
$$

This operator is $\mathcal{R}$-sectorial in $\mathbb{X}_0$ with $\mathcal{R}$-angle zero, it has compact resolvent and is self-adjoint in L_2. It is also invertible, as $\mathbb{A}v = 0$ implies that v is constant in Ω_2, and

$$0 = \int_\Omega \operatorname{div} v dx = \int_\Gamma \nu \cdot v d\Gamma = \int_\Gamma v_\perp \, d\Gamma$$

shows that $v_\perp(x_0) = 0$ for some $x_0 \in \Gamma$; the boundary condition $v_\| = 0$ on Γ yields $v = 0$. As in Step I of Section 3, this implies that the Dirichlet-Neumann operator $A_\mu \in \mathcal{B}(X_1, X_0)$ is well defined, for each $\mu \geq 0$.

(iv) Next, we show that A_μ is $\mathcal{R}$-sectorial in X_0 with $\mathcal{R}$-angle zero, for all $\mu \geq 0$. Following the strategy in Section 3, Step II, we consider the problem

$$
\begin{aligned}
\mu v - \Delta v &= 0 && \text{in } \Omega_2, \\
v_\| = \partial_\nu v_\perp &= 0 && \text{on } \Gamma, \\
\operatorname{div}_\Sigma v_\| + \partial_\nu v_\perp &= 0 && \text{on } \Sigma, \\
\lambda v_\| - \nu^\Sigma R(v) &= f && \text{on } \Sigma.
\end{aligned}
$$

(6.7)

It is not difficult to show that this problem satisfies the Lopatinskii-Shapiro condition for all $\mu > 0$ and all $\lambda \in \Sigma_\pi$. As in Step II of Section 3, we deduce that A_μ is $\mathcal{R}$-sectorial in X_0 with $\mathcal{R}$-angle zero, provided $\mu > 0$ is sufficiently large.

Further, we use (6.4) to extend this result to all values of $\mu \geq 0$. Therefore, Theorem 2.1 and so also Theorem 2.2 are valid in the case of surface impedance conditions as well. Summarizing, we have

Corollary 6.1. *Let $1 < p, q < \infty$ and $1/p < \mu \leq 1$, and assume that $b \in L_1[0, \infty)$ is of subexponential growth, 1-regular and sectorial with angle $\theta_b < \pi$. Then, for each $f = b * g$ and $g \in L_{p,\mu}(\mathbb{R}_+; X_0)$, there is a unique solution $u \in L_{p,\mu}(\mathbb{R}_+; X_1)$ of (6.3). If, in addition, (2.4) and (2.5) hold for some $\alpha \in (0, 2)$, then we also have $u \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; X_0)$.*

(v) It is of interest to determine the kernel $\mathsf{N}(A_0)$. Here, we will do this only if Ω_2 is simply connected. Then, if $A_0 u = 0$, by (6.4), we obtain $R(v) = 0$ in Ω_2. If Ω_2 is simply connected, there is a potential $\phi \in H^3_q(\Omega_2)$ such that $v = \nabla \phi$ in Ω_2; hence, $u = v_\| = \nabla_\Sigma \phi$ on Σ.

Conversely, suppose $u = \nabla_\Sigma \psi$ for some $\psi \in W_q^{3-1/q}(\Sigma)$. Then, we solve the Laplace problem

$$\Delta\phi = 0 \quad \text{in } \Omega_2, \qquad \phi = 0 \quad \text{on } \Gamma, \qquad \phi = \psi \quad \text{on } \Sigma.$$

ϕ is well defined, unique and belongs to $H_q^3(\Omega_2)$. Then, $v = \nabla\phi$ satisfies $R(v) = \operatorname{div} v = 0$ in Ω_2, as well as

$$u = v_\parallel = \nabla_\Sigma \phi = \nabla_\Sigma \psi.$$

This shows that

$$\mathsf{N}(A_0) = \nabla_\Sigma W_q^{3-1/q}(\Sigma),$$

provided Ω_2 is simply connected.

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MARTIN-LUTHER-UNIVERSITÄT HALLE-WITTENBERG, INSTITUT FÜR MATHEMATIK, THEODOR-LIESER-STRASSE 5, D-06120 HALLE, GERMANY
Email address: **jan.pruess@mathematik.uni-halle.de**