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SMOOTH SOLUTIONS TO MIXED-ORDER FRACTIONAL DIFFERENTIAL SYSTEMS WITH APPLICATIONS TO STABILITY ANALYSIS

JAVIER A. GALLEGOS, NORELYS AGUILA-CAMACHO
AND MANUEL A. DUARTE-MERMOUD

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ABSTRACT. Conditions for existence, uniqueness and smoothness of solutions for systems of fractional differential equations of Caputo and/or Riemann-Liouville type having all of them in general and not of the same derivation order are established in this paper. It includes mixed-order, multi-order or non-commensurate fractional systems. The smooth property is shown to be relevant for drawing consequences on the global behavior of solutions for such systems. In particular, we obtain sufficient conditions for global boundedness of solutions to mixed-order nonlinear systems and asymptotic stability of nonlinear fractional systems using backstepping control.

1. Introduction. This paper deals with the existence and asymptotic behavior of smooth solutions for a system of fractional equations defined by

$$D^{\alpha_i} y_i(t) = f_i(t, y(t)), \tag{1.1}$$

where $y : [0,\infty) \to \mathbb{R}^n$ has components y_i, $\alpha_i > 0$ for $i = 1, \ldots, n$, and D stands for either the Caputo or Riemann-Liouville fractional derivative operator.

When $\alpha_i = \alpha$ for $i = 1, \ldots, n$, many authors established conditions for continuous solutions; among them, the reader may consult [**10,** Sections

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5, 6], since it is an illustrative reference of the reasoning involved. The system (1.1) is generally studied in its strong solutions, namely, solutions of an equivalent (under mild assumptions) integral system, which is a Volterra equation with a weakly singular kernel. Mathematical results in regards to the theory of integral equations are used to establish higher-order smoothness properties of the solution for the scalar case (see, e.g., [**11, 23**]). The study of weak solutions requires sophisticated tools and, even so, restricted answers were obtained [**20**].

Also, for the case $\alpha_i = \alpha$ for $i = 1, \dots, n$, further properties have been studied, such as an estimate of the growth [**6**], boundedeness properties [**15**] and stability of the solutions [**8**]. The smoothness is taken as an assumption in [**1**] in order to derive its asymptotic results, but no conditions for this were given. Our contributions to the problem are detailed below.

In Section 2, we set conditions for existence and uniqueness of continuous solutions to (1.1) in the general case, where y is a vector and α_i are not necessarily all the same. This problem is relevant for several reasons. First, it is necessary to simulate fractional systems in a software to know what conditions must be imposed at the initial time to have a well-defined continuous solution. In applications of fractional order controllers to integer order systems, mixed order equations like (1.1) appear [**17**]. On the other hand, some real process models are most precise if a system like (4.1) is used in the general case [**7, 12**]. Second, many authors still discuss the initialization problem [**24**], but, from a mathematical point of view, it reduces to determining what kind of initial conditions must be imposed on the relevant variable x. And, third, our method can be applied in generalizations for Dirichlet problems following our ideas, together with those exposed in [**3**]. In [**9**], this problem was analyzed, but with a sequential fractional derivative, which is neither Caputo nor Riemann-Liouville. After this paper was submitted for publication, an `ArXiv` publication [**13**], using a different approach, considered the existence problem for Caputo systems with continuous assumptions in closed intervals. We presented, in addition to this, results for Riemann-Liouville systems and two corollaries for continuity at open intervals.

In Section 3, we set conditions for existence and uniqueness of smooth solutions to (1.1). This is done in the general case, but, even for the same order of derivation, our results generalize the classical result of

[23] formulated for an scalar integral equation. The relevance of this problem derives from a fundamental quadratic inequality for fractional derivatives **[2, 4, 5]**, which allows for Lyapunov functions to be obtained for fractional order systems by requiring a smooth condition on the solution. In this way, weaker conditions for theorems involving Lyapunov functions for fractional systems than **[16,** Theorem 1] can be obtained.

In Section 4, we illustrate the relevance of smooth properties for the solutions of (1.1) to determine their asymptotic behavior by showing how to deduce boundedness properties for fractional differential equations. The method proposed represents a contribution since the usual techniques, comparison principles and Lyapunov functions **[16, 22]**, have no immediate generalization for mixed order systems. We also give conditions to assure the asymptotic stability for a nonlinear fractional system using backstepping control.

We conclude this section by introducing some notation which is used throughout the paper. $\mathcal{C}^n(I \subset \mathbb{R}, \mathbb{R}^s)$, or, in short, $\mathcal{C}^n(I \subset \mathbb{R})$, denotes the set of functions $f : I \to \mathbb{R}^s$ such that f has its n first derivatives continuous on I. We denote $\sum_i a_i := \sum_{i=1}^n a_i$ whenever the index limits are clear from the context. We will use the following norm for functions:

$$||y||_\infty := \sup_{t\in I} \sum_i |y_i(t)|$$

and, for vectors:

$$||y(t)||_1 := \sum_i |y_i(t)|.$$

Note that $||y||_\infty := \sup_{t\in I} ||y(t)||_1$ and that $||\cdot||_1$ is equivalent to the standard norm on $\mathbb{R}^n$. $\lceil \cdot \rceil$ denotes the ceiling function.

2. Continuity of solutions. In this section, conditions for $\mathcal{C}^0$-smooth solutions are established. It is shown that asking for $\mathcal{C}^0$-smooth solutions to (1.1) is equivalent to asking for its strong solutions, namely, solutions to an integral equation related to equation (1.1).

We recall a framework of fractional calculus. The fractional integral of order $\alpha \in \mathbb{R}_{>0}$ of a function $f : \mathbb{R} \to \mathbb{R}$ is given by (see **[10,** Section 2]):

$$I^\alpha[f(\cdot)](t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function and, without loss of generality, we have fixed the initial time at $t=0$. We define $K_{\alpha_i}(t) := 1/(\Gamma(\alpha_i))t^{\alpha_i-1}$ as the kernel of the fractional integral of order α_i.

The Riemann-Liouville fractional derivative of order α is given by ${}^R D^\alpha f := D^m I^{m-\alpha} f$, and the Caputo derivative is ${}^C D^\alpha f := I^{m-\alpha} D^m f$ [**10,** Sections 2, 3], where $m = \lceil \alpha \rceil$.

Consider the system of integral equations

$$y_i(t) = p_i(t) + I^{\alpha_i}[f_i(\cdot, y(\cdot))](t), \tag{2.2}$$

where $\alpha_i > 0$, $y_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $f_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$. Equation (2.3) will be compactly written as

$$y(t) = p(t) + I^\alpha[f(\cdot, y(\cdot))](t), \tag{2.3}$$

where α is to be seen as a vector with components α_i, and y, f, p are vectors of components y_i, f_i, p_i respectively, for $i = 1, \ldots, n$.

Theorem 2.1. *Consider system* (2.3) *with* $p : [0, T] \to \mathbb{R}^n$ *a continuous function and* $f_i(\cdot, \cdot)$ *continuous functions in their first variables and Lipschitz continuous functions in their second variables for* $i = 1, \ldots, n$. *Then*

(i) *There exists a unique continuous solution* $y \in \mathcal{C}[0, T]$ *to system* (2.3).

(ii) $y \in \mathcal{C}[0, T]$ *is a solution to system* (2.3) *for*

$$p_i(t) := \sum_{k=0}^{\lceil \alpha_i \rceil - 1} \frac{t^k}{k!} y_{i_0}^{(k)} \tag{2.4}$$

if and only if each of its components y_i *is a solution to* ${}^C D^{\alpha_i} y_i = f_i(t, y)$ *with initial condition* $y_i^{(k)}(0) = y_{i_0}^{(k)}$ *for* $k = 1, \ldots, \lceil \alpha_i \rceil - 1$ *and* $i = 1, \ldots, n$.

Proof. (i) Define the operator A on $\mathcal{C}[0, T']$ for any $0 \leq T' \leq T$ by

$$Ay(t) := p(t) + I^\alpha[f(\cdot, y(\cdot))](t), \tag{2.5}$$

that is,

$$(Ay)_i(t) := p_i(t) + I^{\alpha_i}[f_i(\cdot, y(\cdot))](t)$$

for $i = 1, \ldots, n$.

We will first prove that $A : \mathcal{C}[0,T'] \to \mathcal{C}[0,T']$. To prove that $Ay \in \mathcal{C}[0,T']$ for any $y \in \mathcal{C}[0,T']$ whenever $p \in \mathcal{C}[0,T']$, we must show that $I^\alpha[f(\cdot, y(\cdot))](t)$ is continuous in $[0,T']$; thus, we can drop p to prove this fact. For any $0 \le t_1 \le t_2 \le T'$, we have

$$\begin{aligned}||Ay(t_1) - Ay(t_2)||_1 &= \sum_i \bigg| \int_0^{t_1} K_{\alpha_i}(t_1 - \tau) f_i(\tau, y(\tau))\, d\tau \\ &\qquad - \int_0^{t_2} K_{\alpha_i}(t_2 - \tau) f_i(\tau, y(\tau))\, d\tau \bigg| \\ &= \sum_i \bigg| \int_0^{t_1} [K_{\alpha_i}(t_1 - \tau) - K_{\alpha_i}(t_2 - \tau)] f_i(\tau, y(\tau))\, d\tau \\ &\qquad - \int_{t_1}^{t_2} K_{\alpha_i}(t_2 - \tau) f_i(\tau, y(\tau))\, d\tau \bigg|.\end{aligned}$$

By using that $|f_i(t)| \le ||f||_\infty$ for any $t \ge 0$ and any $i \in \{1, \dots, n\}$ and inequalities in equation ([**10,** (6.7)]), we have

$$\begin{aligned}||Ay(t_1) - Ay(t_2)||_1 &\le 2||f||_\infty \sum_{i\in J^c} \frac{(t_2 - t_1)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\ &\qquad + ||f||_\infty \sum_{i \in J} \frac{(t_2 - t_1)^{\alpha_i} + t_2^{\alpha_i} - t_1^{\alpha_i}}{\Gamma(\alpha_i + 1)},\end{aligned}$$

where J denotes the set of indices i such that $\alpha_i > 1$. By using the mean value theorem, there exists a $0 \le t_1 \le \xi_i \le t_2 \le T'$ for all $i \in \{1, \dots, n\}$ such that

$$\begin{aligned}||Ay(t_1) - Ay(t_2)||_1 &\le 2||f||_\infty \sum_{i\in J^c} \frac{(t_2 - t_1)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\ &\qquad + ||f||_\infty \sum_{i \in J} \frac{(t_2 - t_1)^{\alpha_i} + \alpha_i (t_2 - t_1)\xi_i^{\alpha_i - 1}}{\Gamma(\alpha_i + 1)}.\end{aligned}$$

Thus,

$$\begin{aligned}||Ay(t_1) - Ay(t_2)||_1 &\le 2||f||_\infty \sum_{i\in J^c} \frac{(t_2 - t_1)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\ &\qquad + ||f||_\infty \sum_{i \in J} \frac{(t_2 - t_1)^{\alpha_i} + \alpha_i (t_2 - t_1)T^{\alpha_i - 1}}{\Gamma(\alpha_i + 1)},\end{aligned}$$

yielding that $||Ay(t_1) - Ay(t_2)||_1$ converges to zero whenever so does $|t_1 - t_2|$. Hence, $A:\mathcal{C}[0,T'] \to \mathcal{C}[0,T']$. We now prove that, for a particular $T_0 \leq T$, A is a contraction map. For any $y, \widetilde{y} \in \mathcal{C}[0,T]$, we have

$$||Ay - A\widetilde{y}||_\infty = \sup_{t\in[0,T]} \sum_i |I^{\alpha_i}[f_i(\cdot, y(\cdot))](t) - I^{\alpha_i}[f_i(\cdot, \widetilde{y}(\cdot))](t)|.$$

By Lipschitz assumption (without loss of generality, we assume a common Lipschitz constant L for each f_i, $i = 1, \ldots, n$),

$$||Ay - A\widetilde{y}||_\infty \leq L \sup_{t\in[0,T]} \sum_i I^{\alpha_i}[||y(\cdot) - \widetilde{y}(\cdot)||_1](t),$$

and hence,

$$\begin{aligned} ||Ay - A\widetilde{y}||_\infty &\leq L(\sup_{t\in[0,T]} ||y(t) - \widetilde{y}(t)||_1) \sum_i T^{\alpha_i}/\Gamma(\alpha_i) \\ &= L||y - \widetilde{y}||_\infty \sum_i T^{\alpha_i}/\Gamma(\alpha_i). \end{aligned}$$

Choosing a natural number N such that $T_0 = T/N$ implies

$$\sum_i T_0^{\alpha_i}/\Gamma(\alpha_i) < 1/L,$$

A becomes a contraction auto map on $\mathcal{C}[0,T_0]$, whereby existence and uniqueness of a fixed point of A follows by **[25,** Theorem 1.A]. Furthermore, we can write system (2.3) for a time greater than T_0 as

$$\begin{aligned} y_i(t+T_0) = p(t+T_0) &+ \int_0^t K_{\alpha_i}(t+T_0-\tau)[f(\tau, y(\tau))]\, d\tau \\ &+ \int_0^t K_{\alpha_i}(t-\tau)[f(t+T_0, y(t+T_0))]\, d\tau, \end{aligned}$$

and, by continuity of the solution in $[0,T_0]$, function

$$\widetilde{p}(t) = p(t+T_0) + \int_0^t K_{\alpha_i}(t+T_0-\tau)[f(\tau, y(\tau))]\, d\tau$$

is continuous. By repeating the arguments, we conclude continuity and uniqueness of this solution in $[0,T_0]$, and hence, continuity and uniqueness of the solution to the original equation in $[0,2T_0]$. Recursively, and since the fixed point of A is a solution to system (2.3), it follows that there exists a unique solution to (2.3) in $\mathcal{C}[0,T]$.

(ii) The proof is similar to **[10,** Lemma 6.2].

Necessity. Suppose that $y(t) = p(t) + I^{\alpha}[f(\cdot, y(\cdot))](t)$ is a continuous function. We first will see that this function holds the initial conditions associated to a fractional derivative. By continuity of the solution $I^{\alpha}[f(\cdot, y(\cdot))](0) = 0$, thereby, $y(0) = y_0$. By differentiating,

$$y_i^{(k)}(0) = p_i^{(k)}(0) + D^k I^k I^{\alpha_i - k}[f(\cdot, y(\cdot))](t)$$

for $k = 1, \ldots, \lceil \alpha_i \rceil - 1$, where it was the semi group property of integral **[10,** Theorem 2]. Then, using **[10,** Theorem 3.7],

$$y_i^{(k)}(0) = y_{i_0}^{(k)}(0) + I^{\alpha_i - k}[f(\cdot, y(\cdot))](t)$$

for $k = 1, \ldots, \lceil \alpha_i \rceil - 1$ and, by continuity of the solution, the fractional integral vanishes at zero, whereby $y_i^{(k)}(0) = y_{i_0}^{(k)}(0)$. Next, we will see that the function holds for the Caputo fractional equation. By α_i-differentiating the equality $y_i(t) = p_i(t) + I^{\alpha_i}[f_i(\cdot, y(\cdot))](t)$, which holds for all $t \in [0, T]$, we have $D^{\alpha_i} y_i = f(t, y)$, where we have used on the right hand side, **[10,** Theorem 3.7] to obtain $D^{\alpha_i} I^{\alpha_i}[f_i(\cdot, y(\cdot))](t) = f(t, y(t))$, since the function y is continuous by hypothesis. Hence, as a composition of continuous functions, $f(\cdot, y(\cdot))$ is continuous and $D^{\alpha_i} p_i \equiv 0$ since p_i is a polynomial of order at most $\lceil \alpha_i \rceil - 1$. In particular, we prove that y is continuous and α_i-differentiable in $[0, T]$ since the α-derivative of $p_i(t) + I^{\alpha_i}[f_i(\cdot, y(\cdot))](t)$ is continuous in $[0, T]$.

Sufficiency. Assume that y is a continuous function such that $D^{\alpha_i} y_i(t) = f(t, y)$. Since $f(t, y(t))$ is a continuous function and by definition of the Caputo derivative **[10,** Definition 3.2], we have, equivalently, ${}^{R}D^{\alpha_i}[y_i - p_i](t) = f(t, y)$, where, in this case, p_i is an order $\lceil \alpha_i \rceil - 1$ Taylor expansion of y and ${}^{R}D^{\alpha_i}$ denotes the Riemann-Liouville derivative. By definition of the Riemann-Liouville derivative, we have, equivalently,

$$D^{\lceil \alpha_i \rceil} I^{\lceil \alpha_i \rceil - \alpha_i}[y_i - p_i](t) = f(t, y).$$

By $\lceil \alpha_i \rceil$-integration, we have

$$I^{\lceil \alpha_i \rceil - \alpha_i}[y_i - p_i](t) = I^{\lceil \alpha_i \rceil} f(\cdot, y(\cdot))(t) + q(t), \tag{2.6}$$

where $q(t)$ is a polynomial of order at most $\lceil \alpha_i \rceil - 1$. From continuity of functions $f(\cdot, y(\cdot))$ and $y(t) - p_i(t)$, we deduce that both fractional integrals of (2.6) are zero order $\lceil \alpha_i \rceil - 1$ at $t = 0$, that is, integer differentiation up to $\lceil \alpha_i \rceil - 1$ of both integrals vanish at $t = 0$. Hence, q is zero order $\lceil\alpha_i\rceil - 1$, and therefore, $q \equiv 0$. By differentiating and applying [**10,** Theorem 2.14],

$$y_i(t) - p_i(t) = DI^{1-\lceil\alpha_i\rceil+\alpha_i} I^{\lceil\alpha_i\rceil} f(\cdot, y(\cdot))(t).$$

By [**10,** Theorem 2.14], $y_i(t) = p_i(t) + I^{\alpha_i} f(\cdot, y(\cdot))(t)$, which concludes the proof. $\square$

Remark 2.2. The uniqueness of the solution on the set $\mathcal{C}[0,T]$ was asserted, but not in general. This drawback, common in integral equation analysis, is due to the fact that the fixed point tool applies only on a particular Banach space. Many authors using a fixed point argument forget this fact claiming uniqueness without specifying with respect to which space (see, e.g., [**6,** Theorem 2]).

Corollary 2.3. *Consider the system of integral equations*

$$y(t) = g(t) + I^{\alpha}[h(\cdot)y(\cdot)](t), \tag{2.7}$$

where $\alpha, y(t), g(t)$ *are vectors in* $\mathbb{R}^n$ *and* $h(t) \in \mathbb{R}^n \times \mathbb{R}^n$ *for every* $t \geq 0$. *Suppose* $g \in \mathcal{C}(0,T] \cap L^1(0,T)$, *and* h *is a bounded matrix function* ($||h||_\infty \leq h_0$) *continuous on* $[0,T]$. *Then, there exists a unique solution* $y \in \mathcal{C}(0,T] \cap L^1(0,T)$.

Proof. Defining the operator $Ay := g + I^{\alpha} h(\cdot)y(\cdot)$ on $L^1(0,T)$ and, considering the L^1-norm,

$$||f||_{L^1(0,T)} := \left(\int_0^T ||f(\tau)||_1 d\tau\right) = \left(\int_0^T \sum_i |f_i(\tau)|\, d\tau\right),$$

and we note that

$$\begin{aligned} ||Ay(t) - A\widetilde{y}(t)||_1 &= \left\| \sum_i I^{\alpha_i} h_{ij}(\cdot)[y_i(\cdot) - \widetilde{y}_j(\cdot)](t) \right\|_1 \\ &\leq h_0 \left\| \int_0^t \sum_i K_{\alpha_i}(\tau) |y_i(t-\tau) - \widetilde{y}_i(t-\tau)|\, d\tau \right\|_1 \\ &\leq h_0 \int_0^t \sum_i K_{\alpha_i}(\tau) \sum_i |y_i(t-\tau) - \widetilde{y}_i(t-\tau)|\, d\tau. \end{aligned}$$

Choosing a sufficiently small time s so that $h_0(\int_0^s \sum_i K_{\alpha_i}(\tau)\,d\tau) < \eta$ for $\eta < 1$ a fixed number, we have

$$\|Ay(t) - A\widetilde{y}(t)\|_{L^1(0,s)} \leq \int_0^s \int_0^t \sum_i K_{\alpha_i}(\tau) \sum_i |y_i(t-\tau) - \widetilde{y}_i(t-\tau)|\,d\tau\,dt,$$

and, since y_i, K_{α_i} are in $L^1(0,s)$, we get

$$||Ay(t) - A\widetilde{y}(t)||_{L^1(0,s)} \leq \int_0^t \sum_i K_{\alpha_i}(\tau)\left(\int_0^s \sum_i |y_i(t-\tau) - \widetilde{y}_i(t-\tau)|\,dt\right)d\tau$$

whereby $||Ay(t) - A\widetilde{y}(t)||_{L^1(0,s)} \leq \eta||y(t) - \widetilde{y}(t)||_{L^1(0,s)}$, i.e., A is a contraction map in $L^1(0,s)$. Moreover, since $g \in L^1(0,T)$, A is an auto map in $L^1(0,s)$. The fixed point theorem ([**25,** Theorem 1.A]) can be applied on the Banach space $L^1(0,s)$, whereby there exists a unique solution $y \in L^1(0,s)$.

We choose $\delta \in (0,s)$ and consider the following system of integral equations

$$y_i(t+\delta) = g_i(t+\delta) + I^{\alpha_i}[f_i(\cdot, y(\cdot))](t+\delta).$$

for $i = 1, \ldots, n$. Note that $g_i(t+\delta) \in \mathcal{C}[0, T-\delta]$ and

$$I^{\alpha_i}[f_i(\cdot, y(\cdot))](t) = \int_0^\delta \frac{(t+\delta-\tau)^{\alpha_i-1}}{\Gamma(\alpha_i)} \sum_i h_{ij}(\tau) y_j(\tau)\,d\tau + \int_0^t \frac{(t-\tau)^{\alpha_i-1}}{\Gamma(\alpha_i)} \sum_i h_{ij}(\tau+\delta) y_j(\tau+\delta)\,d\tau.$$

The term $q_i(t) := \int_0^\delta (t+\delta-\tau)^{\alpha_i-1} \sum_j h_{ij}(\tau) y_j(\tau)\,d\tau \in \mathcal{C}[0,T]$ since $hy \in L^1(0,s)$ (by the Holder inequality), whereby its fractional integral exists ([**10,** Theorem 2.1]), $t^{\alpha_i-1} \in L^1(0,s)$ is a non negative monotone function so that, for a sequence t_n tending monotonically to $t \in [0,T]$, we have

$$(t_n+\delta-\tau)^{\alpha_i-1} \sum_j h_{ij}(\tau) y_j(\tau) \longrightarrow (t+\delta-\tau)^{\alpha_i-1} \sum_j h_{ij}(\tau) y_j(\tau)$$

monotonically and, by dominated convergence, $q_i(t_n) \to q_i(t)$.

Therefore, calling $\widetilde{y}_i(t) := q_i(t+\delta)$, $\widetilde{h}_{ij}(t) := h_{ij}(t+\delta)$, we get

$$\widetilde{y}_i(t) = q_i(t) + I^{\alpha_i}\widetilde{h}\widetilde{y}(t).$$

By Theorem 2.1, $\widetilde{y}_i(t) \in \mathcal{C}[0, T-\delta]$. Since $0 < \delta < s$ is arbitrary, $y_i(t) \in \mathcal{C}(0, T]$. □

Remark 2.4. Theorem 2.1 can be improved to obtain uniqueness on the set $\mathcal{C}(0, T] \cap L^1(0, T)$, with similar translation and contracting mapping arguments used in the proof of Corollary 2.3.

Corollary 2.5. *Consider the system of integral equations*

$$y(t) = p(t) + I^{\alpha} f(\cdot, y(\cdot))(t), \tag{2.8}$$

where α is a vector of positive components, and $y(t)$, $p(t)$, $f(t, y(t))$ are vectors in $\mathbb{R}^n$ for every $t \geq 0$. Suppose $p_i \in \mathcal{C}(0, T] \cap L^1(0, T)$ and each f_i is continuous and Lipschitz continuous on the second variable for $i = 1, \dots, n$. Then, there exists a unique solution $y \in \mathcal{C}(0, T] \cap L^1(0, T)$ for (2.8)*.*

In particular, $y \in \mathcal{C}(0, T] \cap L^1(0, T)$ is a solution to system (2.8) *for p_i, given by*

$$p_i(t) := \sum_{k=1}^{\lceil \alpha_i \rceil} \frac{t^{\alpha_i - k}}{\Gamma(\alpha_i - k + 1)} \lim_{\tau \to 0^+} I^{\lceil \alpha_i \rceil - k} y_i(\tau), \tag{2.9}$$

if and only if it is a solution to ${}^R D^{\alpha_i} y_i = f_i(t, y)$ with initial conditions given by $\lim_{\tau \to 0^+} I^{\lceil \alpha_i \rceil - \alpha_i} y_i(\tau)$. For null initial conditions, $y \in \mathcal{C}[0, T] \cap L^1(0, T)$.

Proof. Defining the operator $Ay := p + I^{\alpha} f(\cdot, y(\cdot))$ on $L^1(0, T)$, the proof of the first part follows the same procedure as the proof of Corollary 2.3, since, by Lipschitz assumption:

$$||Ay(t) - A\widetilde{y}(t)||_1 \leq L|| \sum_i I^{\alpha_i} [y_i(\cdot) - \widetilde{y}_j(\cdot)](t)||_1$$

where L is a common Lipschitz constant (see the proof of Theorem 2.1). The second part is proved by a similar procedure as the proof of Theorem 2.1 (ii). □

Remark 2.6. (i) Note that, by a suitable choice of p, Corollary 2.5 includes a system defined with Caputo and Riemann-Liouville derivatives. For instance, for index i odd, the Caputo derivative is used and all the rest are Riemann-Liouville derivatives.

(ii) Corollary 2.5 provides specific conditions for application of the inequality [**5,** Lemma 1] for Riemann-Liouville systems as it requires continuity of the solution.

3. Differentiability of solutions. In this section, we establish conditions for $\mathcal{C}^p-$smooth solutions (for p a natural number greater than 1) to the equation

$$D^{\alpha}y(t) = f(t,y), \tag{3.1}$$

where α is a vector of positive components, $y(t)$ is a vector for all $t>0$ and D refers to the Caputo or Riemann-Liouville derivative. Since at least the solution of (3.1) is required to be $\mathcal{C}^1$-continuous, we must study the integral version of (3.1), according to Theorem 2.1 or Corollary 2.5.

The main motivation of this problem is that Caputo fractional differentiation of the quadratic expression on y obeys a simple inequality when the function y is continuously differentiable (or, more weakly, absolutely continuous) and $0<\alpha\leq 1$, from which an asymptotic analysis can be performed [**4, 14**]. In its simplest form [**4,** Lemma 1], this inequality establishes that, for an absolutely continuous function y,

$$^{C}D^{\alpha}[y^2](t) \leq 2y(t)D^{\alpha}[y](t), \quad \text{for all } t\geq 0. \tag{3.2}$$

It was also proven that the Riemann-Liouville derivative holds the same inequality [**5,** Lemma 1]. Note that this inequality does not hold for $\alpha>1$ (just take $\alpha=3/2$ and $x(t)=t$). But, as suggested in [**11,** Remark 2.2], the solution of (3.1) with $\alpha<1$ in $\mathcal{C}^1[0,T]$ only occurs under null initial conditions. Analytically, it follows from the following reasoning: if $f(0,y(0))\neq 0$, then y is not continuously differentiable at $t=0$. Indeed, by contradiction, if $\dot{y}$ is continuous at zero, then it is bounded in $[0,t]$. Therefore, $^{C}D^{\alpha}y(0) = \lim_{t\to 0^+} {}_0I_t^{1-\alpha}\dot{y}(t) = 0$. However, $^{C}D^{\alpha}y(0) = f(0,y(0))\neq 0$, which is a contradiction.

The type of singularity of $\dot{y}$ that occurs in the origin in the scalar case is $O(t^{\alpha-1})$, as follows from [**23,** Theorem 1]. In fact, one can write (up to constants) $\dot{y}(t)=t^{\alpha-1}+\psi(t)$ for any $t>0$, with $\psi(t)$ a continuous function in $[0,T]$. Since $\lim_{t\to 0^+}\int_0^t[\tau^{\alpha-1}+\psi(\tau)](t)=0$, we can express $y(t)=y(0)+\int_0^t[\tau^{\alpha-1}+\psi(\tau)]\,d\tau$ for any $t\geq 0$, and thus, $y\in\mathcal{A}^1[0,T]$, i.e., y is absolutely continuous in $[0,T]$ (**[10,** Definition 1.5]), and thus, **[4,** Lemma 1] can be applied.

Next, we generalize **[23,** Theorem 1] to establish differentiability for vector mixed order differential equations, mainly following the reasoning of the proof of **[23,** Theorem 1]. This will require some previous results.

We recall that, for the scalar equation,

$$y(t)=g(t)+I^\alpha y(t)=g(t)+\int_0^t K_\alpha(t-\tau)y(\tau)\,d\tau,$$

with the kernel function K_α in $L^1(0,T)$, there exists a resolvent function R such that

$$y(t)=g(t)+\int_0^t R(t-\tau)g(\tau)\,d\tau,$$

with $R(t)=t^{\alpha-1}+I^\alpha R(t)$ (see **[23,** Section 2]). Moreover, using **[23,** Lemma 1], $R\geq 0$ and $R\in L^1(0,T)$, whenever $g\in L^1(0,T)$.

The next result, which generalizes **[23,** Lemma 3], will be instrumental in the proof of the main result of this section.

Lemma 3.1. *Consider the following pair of nonlinear equations*:

$$X_j(t)=g_j(t)+I^\alpha f_j(\cdot,X_j(\cdot))(t), \tag{3.3}$$

where $\alpha\in\mathbb{R}^n$, $X_j(t)\in\mathbb{R}^n$ *for any* $t\geq 0$, $f_j(\cdot,\cdot)$ *is a continuous function,* $g_j\in L^1(0,T)$ *for any* $j=1,2$, *and* f_1 *is a Lipschitz continuous function in its second argument of Lipschitz constant* L. *Then*

$$||X_1(t)-X_2(t)||_1\leq ||Q(t)||_1+\int_0^t R_\alpha(t-\tau))||Q(\tau)||_1\,d\tau,$$

where R_α *is the resolvent of* $\sum_{i=1}^n LK_{\alpha_i}(t)$ *and*

$$Q(t):=g_1(t)-g_2(t)+I^\alpha[f_1(\cdot,X_2(\cdot))-f_2(\cdot,X_2(\cdot))](t).$$

Proof. Defining $Z:=X_1-X_2$, $G:=g_1-g_2$, we have

$$Z=G+I^\alpha[f_1(\cdot,X_2(\cdot))-f_2(\cdot,X_2(\cdot))]+I^\alpha[f_1(\cdot,X_1(\cdot))-f_1(\cdot,X_2(\cdot))].$$

Setting $Q := G + I^{\alpha}[f_1(\cdot, X_2(\cdot)) - f_2(\cdot, X_2(\cdot))]$, it follows from the Lipschitz assumption that $|Z_i| \leq |Q_i| + LI^{\alpha_i}||Z||_1$ for $i = 1, \dots, n$. Hence,

$$\begin{aligned}||Z(t)||_1 &\leq ||Q(t)||_1 + \sum_i LI^{\alpha_i}||Z(t)||_1 \\ &\leq ||Q(t)||_1 + \int_0^t \sum_i LK_{\alpha_i}(t-\tau)||Z(\tau)||_1\, d\tau.\end{aligned}$$

Defining $K_{\alpha}(t) := \sum_i LK_{\alpha_i}(t)$, we obtain

$$||Z(t)||_1 \leq ||Q(t)||_1 + \int_0^t K_{\alpha}(t-\tau)||Z(\tau)||_1\, d\tau.$$

It follows that there exists a nonnegative function p such that $||Z(t)||_1 = ||Q(t)||_1 - p(t) + \int_0^t K_{\alpha}(t-\tau))||Z(\tau)||_1\, d\tau$. Since $K_{\alpha} \in L^1(0,T)$ because $K_{\alpha_i} \in L^1(0,T)$ for $i = 1, \dots, n$, we can express $||Z(\tau)||_1$ in terms of a resolvent, namely,

$$||Z(t)||_1 = ||Q(t)||_1 - p(t) + \int_0^t R_{\alpha}(t-\tau)\big(||Q(\tau)||_1 - p(\tau)\big)\, d\tau.$$

Since p is nonnegative and R_{α} is continuous and positive, and $K_{\alpha_i} \geq 0$ [**23,** Lemma 1], we conclude that

$$||Z||_1(t) \leq ||Q||_1(t) + \int_0^t R_{\alpha}(t-\tau))||Q(\tau)||_1\, d\tau. \qquad \square$$

We have all the ingredients to prove the main theorem.

Theorem 3.2. *Consider the following system of integral equations*:

$$y_i(t) = p_i(t) + \int_0^t K_{\alpha_i}(\tau) f_i(t-\tau, y(t-\tau))\, d\tau, \quad i = 1, \dots, n, \tag{3.4}$$

where $\alpha_i > 0$ and $p_i \in \mathcal{C}^1[0,T]$. $f_i(t,x)$ is a $\mathcal{C}^1$ function for $0 \leq t \leq T$ and for all x, and a Lipschitz continuous function in the second variable, for every $i = 1, \dots, n$. Then $y \in \mathcal{C}^1(0,T]$. Moreover, if $\alpha_i \geq 1$ for every $i = 1, \dots, n$, then $y \in \mathcal{C}^1[0,T]$.

Proof. First part. The formal derivative of (3.4) obtained by applying the Leibniz rule without verifying its differentiability requirements (avoiding a vicious circle) is given by:

$$\begin{aligned} Dy_i(t) = Dp_i(t) + K_{\alpha_i}(t) f(0, y(0)) \\ + \int_0^t K_{\alpha_i}(t-\tau)[f_{i,1}(\tau, y(\tau)) \\ + f_{i,2}(\tau, y(\tau)) Dy(\tau)]\, d\tau, \end{aligned} \tag{3.5}$$

where $f_{i,1} = \partial f_i(x,y)/\partial x$, $f_{i,2} = \partial f(x,y)/\partial y$. Defining

$$F_i(t) := Dp_i(t) + K_{\alpha_i}(t) f(0, y(0)) + \int_0^t K_{\alpha_i}(t-\tau) f_{i,1}(\tau, y(\tau))\, d\tau$$

for all $i = 1, \ldots, n$, and since, by Theorem 2.1, $y \in \mathcal{C}[0,T]$, we have that $F \in \mathcal{C}(0,T]$. Thus, we can write

$$Dy_i(t) = F_i(t) + \int_0^t K_{\alpha_i}(t-\tau) f_{i,2}(\tau, y(\tau)) Dy(\tau)\, d\tau,$$

where $f_{i,2}(\tau, y(\tau))$ is bounded and continuous in $[0,T]$ since $y \in \mathcal{C}[0,T]$ and the hypothesis $f_i \in \mathcal{C}^1$. By Corollary 2.3, $y \in \mathcal{C}^1(0,T]$ and, by Theorem 2.1, if $\alpha_i \geq 1$ for every $i = 1, \ldots, n$, then $y \in \mathcal{C}^1[0,T]$.

Second part. To conclude, we must prove that the formal derivative (3.5) is indeed the derivative. Choose $\delta \in (0, T/2)$, any $h \in (0, \delta]$ and $t \in (0, T-\delta]$. Define $Z_i(t,h) := [y_i(t+h) - y_i(t)]/h$. By using the mean value theorem on function f_i, we get for some $\tau_i^* \in [t, t+h]$,

$$Z_i(t,h) = R_i(t,h) + \int_0^t K_{\alpha_i}(t-\tau) f_{i,2}(\tau, y(\tau_i^*)) Z(\tau, h)\, d\tau,$$

where

$$\begin{aligned} R_i(t,h) = \frac{p_i(t+h) - p_i(t)}{h} \\ + h^{-1} \int_t^{t+h} K_{\alpha_i}(\tau) f_i(t+h-\tau, y(t+h-\tau))\, d\tau \\ + \int_0^t K_{\alpha_i}(\tau) f_{i,1}(t+\theta(h)-\tau, y(t-\tau))\, d\tau, \end{aligned}$$

for $0 < \theta(h) < h$. By Lemma 3.1,

$$||Z(t,h) - Dy||_1 \leq ||Q(t,h)||_1 + \int_0^t R_\alpha(t-\tau) ||Q(\tau,h)||_1\, d\tau,$$

where

$$Q_i(t,h) := R_i(t,h) - F_i(t) + \int_0^t K_{\alpha_i}(t-\tau)[f_{i,2}(\tau, y(\tau_i^*)) - f_{i,2}(\tau, y(\tau))]\, d\tau.$$

Let C be a common bound of the norm of functions Dp_i, f_i, $f_{i,1}$, $f_{i,2}$ and F, which exists since they are continuous at $[0,T]$ due to the hypotheses and the fact that $y \in \mathcal{C}[0,T]$. We have the following bounds:

$$|R_i(t,h)| \leq C + C/h \int_t^{t+h} K_{\alpha_i}(\tau)\, d\tau + C \int_0^t K_{\alpha_i}(\tau)\, d\tau$$

and

$$|F_i(t)| \leq C + CK_{\alpha_i}(t) + C \int_0^t K_{\alpha_i}(\tau) d\tau.$$

Therefore,

$$\begin{aligned} |Q_i(t,h)| &\leq 2C + C/h \int_t^{t+h} K_{\alpha_i}(\tau)\, d\tau + 4C \int_0^t K_{\alpha_i}(\tau)\, d\tau \\ &\leq 2C \left[1 + K_{\alpha_i}(t) + 2 \int_0^T K_{\alpha_i}(\tau)\, d\tau \right] \\ &:= C_0 + C_1 K_{\alpha_i}(t). \end{aligned}$$

Let C_2 be the bound of R_α at $[\delta, T-\delta]$ (by the proof of Lemma 3.1, R_α is continuous) and $0 < s \leq \delta$ such that $\int_0^s C_2[C_0 + C_1 K_{\alpha_i}(\tau)]\, d\tau < \epsilon/3n$ for every $i = 1, \ldots, n$.

Let h_0 be small enough such that, for every $h \leq h_0$, $|Q_i(t,h)| \leq \epsilon[3n(1 + \int_0^T R_\alpha(\tau)\, d\tau)]^{-1}$ uniformly over $s \leq t \leq T - s$. We can always do this since $Q_i(t,h)$ converges to zero with h converging to zero, uniformly on $0 < t \leq T$. In fact, by a dominated convergence argument similar to that of the proof of Corollary 2.3 (since $y \in \mathcal{C}[0,T]$ and the hypothesis $f_i \in \mathcal{C}^1$), we have that

$$\int_0^t K_{\alpha_i}(t-\tau)[f_{i,2}(\tau, y(\tau_i^*)) - f_{i,2}(\tau, y(\tau))]\, d\tau$$

and

$$\int_0^t K_{\alpha_i}(\tau) f_{i,1}(t + \theta(h) - \tau, y(t-\tau))\, d\tau - \int_0^t K_{\alpha_i}(\tau) f_{i,1}(t-\tau, y(t-\tau))\, d\tau$$

converge to zero uniformly on $0 < t \leq T$ when h converges to zero (recall that τ_i^* goes to τ when h converges to zero and y is continuous); $[p_i(t+h) - p_i(t)/h] - Dp_i(t)$ converges to zero when h converges to zero, since, by hypothesis, $p_i \in \mathcal{C}^1[0,T]$, whereby it is uniform on $0 < t \leq T$; and, finally,

$$h^{-1} \int_t^{t+h} K_{\alpha_i}(\tau) f_i(t+h-\tau, y(t+h-\tau))\, d\tau - K_{\alpha_i}(t) f(0, y(0))$$

converges to zero uniformly on $0 < t \leq T$ when h converges to zero, using that $f_i(t,x)$ is a $\mathcal{C}^1$, by the fundamental theorem of calculus.

For $0 < h \leq h_0$ and $t \in [\delta, T-\delta]$, we have

$$\begin{aligned}
||Z(t,h) - Dy(t)||_1 &\leq ||Q(t,h)||_1 + \int_0^s R_\alpha(t-\tau)||Q(\tau,h)||_1\, d\tau \\
&\quad + \int_s^t R_\alpha(t-\tau)||Q(\tau,h)||_1\, d\tau \\
&\leq \epsilon/3 + \int_0^s C_2[C_0 + C_1 K_{\alpha_i}(\tau)]\, d\tau \\
&\quad + \epsilon \int_s^t R_\alpha(t-\tau) \left[3 \int_0^T R_\alpha(u)\, du\right]^{-1} d\tau \\
&< \epsilon.
\end{aligned}$$

Hence, $\lim_{h\to 0^+} Z(t,h) = Dy(t)$ uniformly on $[\delta, T-\delta]$. Since δ is arbitrary, it holds for any (δ, T). Since convergence is uniform in any interval $[\delta, T-\delta]$, the set of functions $\{Z(\cdot, h) : 0 < h < \delta\}$ is equicontinuous. Thus, $\lim_{h\to 0^+} Z(t,h) = \lim_{h\to 0^+} Z(t-h,h) = Dy(t)$ uniformly on $[\delta, T-\delta]$. Similarly, $\lim_{h\to 0^+} Z(T,h) = Dy(T)$, which concludes the proof. □

Remark 3.3. (i) Theorem 3.2 is also valid for any kernel in $L^1(0,T) \cap \mathcal{C}^1(0,T]$ instead of K_{α_i}, which makes possible to extend its claim to other fractional-like systems. In particular, it holds for Riemann-Liouville systems.

(ii) We note that, by the same reasons as in the scalar case, $y \in \mathcal{A}^1[0,T]$, namely, $Dy_i(t) = Ct^{\alpha_i - 1} + \psi_i(t)$ for any $t > 0$ with C a constant, $\psi_i(t)$ a continuous function in $[0,T]$ and $\lim_{t\to 0^+} \int_0^t [\tau^{\alpha_i - 1} + \psi_i(\tau)](t) = 0$.

Then, we can express

$$y_i(t) = y_i(0) + \int_0^t [\tau^{\alpha_i - 1} + \psi_i(\tau)]\, d\tau$$

for any $t \geq 0$, and thus, $y \in \mathcal{A}^1[0,T]$.

(iii) Since $(0,T]-$smooth solutions contain only a singularity at $t=0$, whenever the initial condition is non zero, $[0,T]$-smooth periodic solution cannot be expected for all $t \geq 0$.

Using the equivalence with the integral equation, we generalize [**11,** Theorems 2.4, 2.5] and [**23,** Theorem 3] to account for higher order differentiability in mixed or multi order systems.

Corollary 3.4. *Consider the system of integral equations* (3.4) *with* f *holding the same assumptions as in Theorem* 3.2. *Let* $k_i = \lceil \alpha_i \rceil - 1$ *and* $k = \min_i(\lceil \alpha_i \rceil) - 1$.

(i) *If* $p_i \in \mathcal{C}^{k_i}[0,T]$ *then* $y_i \in \mathcal{C}^{k_i}[0,T]$ *for any* $i = 1, \dots, n$ *and* $y \in \mathcal{C}^k[0,T]$.

(ii) *If* $p_i \in \mathcal{C}^{k_i+1}[0,T]$, *then* $y_i \in \mathcal{C}^{k_i+1}(0,T]$ *for any* $i = 1, \dots, n$ *and* $y \in \mathcal{C}^{k+1}(0,T]$, *and its derivative is given by the Leibniz rule.*

Proof. (i) By Theorem 2.1, the solution of (3.4) is continuous. Using the fact that the fractional integral of a continuous function is also continuous, that a continuous function x satisfies $I^{\alpha_i} x = I^{k_i} I^{\alpha_i - k_i} x$ ([**10,** Theorem 2.2]) and $D^{k_i} I^{k_i} x = x$ ([**10,** Theorem 3.7]), we arrive at

$$D^k y_i(t) = D^k p_i(t) + I^{\alpha_i - k}[f_i(\cdot, y(\cdot))](t). \tag{3.6}$$

By definition, $\alpha_i - k_i > 0$, and hence, both terms on the right-hand side of (3.6) are continuous. Then, $y_i \in \mathcal{C}^{k_i}[0,T]$, and therefore, $y \in \mathcal{C}^k[0,T]$.

(ii) By differentiating again (3.5), we obtain

$$D^2 y_i(t) = G_i(t) + \int_0^t K_{\alpha_i}(t-\tau)[f_{i,2,2}(\tau, y(\tau)) D^2 y(\tau)]\, d\tau,$$

where $f_{i,2,2} = \partial f_{i,2}(x,y)/\partial y$, $G_i := M_i + N_i$,

$$\begin{aligned} M_i(t) = D^2 p_i(t) &+ DK_{\alpha_i}(t) f(0, y(0)) \\ &+ K_{\alpha_i}(t) f_{i,1}(0, y(0)) \end{aligned}$$

$$+\int_0^t K_{\alpha_i}(t-\tau)[f_{i,1,1}(\tau, y(\tau))]\,d\tau,$$

where $f_{i,2,1} = \partial f_{i,2}(x,y)/\partial x$ and

$$N_i(t) = \int_0^t K_{\alpha_i}(t-\tau)[f_{i,1,2}(\tau, y(\tau)) + f_{i,2,1}(\tau, y(\tau)) \\ + f_{i,2,2}(\tau, y(\tau))Dy(\tau)]Dy(\tau)\,d\tau.$$

By Theorem 3.2, $G_i \in \mathcal{C}(0,T]$ and, by Corollary 2.3, $G \in \mathcal{C}(0,T]$. By using the second part of the proof of Theorem 3.2, D^2y is indeed the derivative. Recursively, the claims follow. □

Remark 3.5. (i) $p(t)$, chosen as (2.4), satisfies the hypotheses of Corollary 2.5, and so, for equation ${}^C D^{\alpha_i} y_i(t) = f_i(t, y(t))$, $y \in \mathcal{C}^{k+1}(0,T]$.

(ii) The proof of [**11,** Theorem 2.5], for the same order case, is slightly incomplete since it merely uses the formal derivative.

We exemplify the claim of Theorem 3.2 by showing an informal proof for a specific case. Consider the following system:

$$\begin{cases} {}^C D^\beta x(t) = -y(t) f(t) \\ {}^C Dy(t) = Ay(t) + f(t)^T x(t), \end{cases}$$

where $0 < \beta < 1$, A is a constant matrix and f is a bounded $\mathcal{C}^1[0,\infty)$ function. By Theorem 2.1, $x, y \in \mathcal{C}[0,T]$ for every $T > 0$. Hence, $y \in \mathcal{C}^1[0,T]$ and $yf \in \mathcal{C}^1[0,T]$.

On the other hand, for any $\alpha, \beta \leq 1$ and $\alpha + \beta \leq 1$, by applying the Laplace transform to any (locally integrable) function z, we have:

$$\begin{aligned} \mathcal{L}[{}^C D^\alpha \rangle {}^C D^\beta z] &= s^\alpha(\mathcal{L}[{}^C D^\beta z]) - s^{\alpha-1} {}^C D^\beta z(0) \\ &= s^\alpha(s^\beta z - s^{\beta-1} z(0)) - s^{\alpha-1} {}^C D^\beta z(0) \\ &= s^{\alpha+\beta} z - s^{\alpha+\beta-1} z(0) - s^{\alpha-1} {}^C D^\beta z(0) \end{aligned}$$

Applying the Laplace inverse transform, we get

$$\begin{aligned} \mathcal{L}^{-1}\mathcal{L}[{}^C D^\alpha {}^C D^\beta z] &= \mathcal{L}^{-1}[s^{\alpha+\beta} z - s^{\alpha+\beta-1} z(0))] - \mathcal{L}^{-1}[s^{\alpha-1} {}^C D^\beta z(0)] \\ &= {}^C D^{\alpha+\beta} z - \mathcal{L}^{-1}[s^{\alpha-1} {}^C D^\beta z(0)], \end{aligned}$$

and thus, ${}^{C}D^{\alpha C}D^{\beta}z = {}^{C}D^{\alpha+\beta}z - \Gamma(1-\alpha)^{-1}t^{-\alpha C}D^{\beta}z(0)$. Since ${}^{C}D^{1-\beta C}D^{\beta}x(t) = -{}^{C}D^{1-\beta}[yf](t)$, we can write

$$\begin{aligned} Dx(t) &= -{}^{C}D^{1-\beta}[yf](t) + t^{\beta-1C}D^{\beta}x(0) \\ &= -{}^{C}D^{1-\beta}[yf](t) - \Gamma(1-\alpha)^{-1}t^{\beta-1}y(0)f(0), \end{aligned}$$

and, since $D^{1-\beta}[yf](t)$ is the fractional integral of a continuous function, it is continuous. Hence, $x \in \mathcal{C}^1(0,T]$, which directly follows from applying Theorem 3.2, and from Corollary 2.5, $y \in \mathcal{C}^1[0,T]$.

4. Stability of solutions.

In this section, we provide examples as to how the smoothness property of the solutions to systems of type (1.1) is used to study to their asymptotic properties. It is assumed that the fractional derivative is of Caputo type. We deal with continuous solutions to nonlinear mixed order (or multi-order) systems. For the linear case, in **[13]**, necessary and sufficient conditions for asymptotic stability were recently provided and, in **[18,** Theorem 2.6], sufficient conditions for robustness were obtained.

First, we establish a condition which guarantees bounded solutions of the system of equations, given by:

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(x,y,t) \\ {}^{C}D^{\beta}y(t) = g(x,y,t), \end{cases} \tag{4.1}$$

where $x:[0,\infty)\to\mathbb{R}^n$, $y:[0,\infty)\to\mathbb{R}^m$, f,g are Lipschitz continuous functions in their first two arguments and C^1. Assume $0<\alpha\le\beta\le 1$, and suppose that

$$x^T f(x,y,t) + y^T g(x,y,t) = (x,y)^T(f(x,y,t), g(x,y,t)) \le 0 \tag{4.2}$$

for any $x,y\in\mathbb{R}^n$ and for any $t\ge 0$. In the following, C denotes various constant numbers, and $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^p$ for any $p\in\mathbb{N}$.

Theorem 4.1. *Consider a solution (x,y) to system* (4.1), *where f and g are Lipschitz continuous functions in their first two arguments, C^1 in its arguments and $0<\alpha\le\beta\le 1$. Suppose that condition* (4.2) *holds.*

(a)

(i) *If $\alpha=\beta$ then (x,y) is a bounded function.*

(ii) *If* $\alpha < \beta$, $||f(x(t), y(t), t)|| \leq C||x(t)||$ *for all* $t > 0$ *with* C *a constant number, and the system in* y *is* BIBO *stable with respect to the input* x, *then* (x, y) *remains bounded.*

(iii) *If* $\alpha < \beta$, $||f(x(t), y(t), t)|| \leq C||x(t)||$ *for all* $t > 0$ *with* C *a constant independent of* t, *and* $y^T g(\cdot, y, \cdot)$ *diverges to* $+\infty$ *whenever* $||y||$ *diverges to* $+\infty$, *then* (x, y) *remains bounded.*

(b) *Suppose that there exists a constant number* $\lambda > 0$ *such that*

$$x^T f(x, y, t) + y^T g(x, y, t) \leq -\lambda y^T y \tag{4.3}$$

for every x, y, t *in the domain. Suppose there exists a* C *such that* $||f(x(t), y(t), t)|| \leq C||x(t)||$ *and* $||g(x(t), y(t), t)|| \leq C||y(t)||$. *Then,* (x, y) *is a bounded function.*

Part (a). We will prove by contradiction that a number $L > ||x(0)||$ cannot exist such that $||x(t)|| > L$ for all $t \geq T$. Using (4.2), we have

$$x^{T\,C} D^\alpha x + y^{T\,C} D^\beta y = x^T f(x, y, t) + y^T g(x, y, t) \leq 0,$$

and hence, by using Theorem 3.2 (which can be applied by the smoothness assumptions of functions f and g) together with Remark 2.6 (ii) and the inequality (3.2) ([**4,** Lemma 1] requiring absolute continuity, and directly extended to the vector case [**14**]), we have

$$^C D^\alpha [x^T x] + {}^C D^\beta [y^T y] \leq 0. \tag{4.4}$$

Applying again [**10,** Theorem 3.8] and [**10,** Theorem 2.2], since by Theorem 2.1, x and y are continuous functions and by Theorem 3.2, $\dot{x}$ is locally integrable, we obtain by β-integration of (4.4):

$$I^{\beta-\alpha}[x^T x(\cdot) - x^T x(0)](t) + y^T y(t) - y^T y(0) \leq 0, \quad \text{for all } t > 0. \tag{4.5}$$

(i) For $\alpha = \beta$, inequality (4.5) can be written as $x^T x(t) + y^T y(t) \leq x^T x(0) + y^T y(0)$, from which follows boundedness of $(x(t), y(t))$. Moreover, Lyapunov stability of the origin $(x, y) = (0, 0)$ follows from standard ϵ-δ arguments.

(ii) If there exists a number $L > x^T x(0)$ such that $x^T x(t) > L$ for all $t \geq T$, then the integral $I^{\beta-\alpha}[x^T x(\cdot) - x^T x(0)]$ diverges to $+\infty$ and, since $y^T y(t) - y^T y(0)$ is bounded from below, the inequality (4.5) does not hold. Hence, we have proved that a number $L > ||x(0)||$ cannot exist such that $||x(t)|| > L$ for an unbounded increasing period of time.

Note that we have not yet proved that x is bounded, since it could occur that, in an unbounded sequence of intervals of finite large, x grows unbounded. A simple, but restrictive, condition to avoid this kind of divergence is that $||f(x,y,t)|| < C$ for every t with C a constant independent of t,x,y. Indeed, this condition implies $||D^\alpha x|| < C$ and, by [**15,** Proposition 1], we conclude that x is α-Holder continuous function and, in particular, uniformly continuous, whereby its increments or decrements in finite time intervals cannot be unbounded. Hence, since x cannot diverge on either a finite or on an unbounded interval, x is bounded.

We have proved that a number $L > ||x(0)||$ cannot exist such that $||x(t)|| > L$ for an unbounded increasing period of time. Thus, the only alternative is that there exist a number L and a sequence $(T_n)_{n\in\mathbb{N}}$ with bounded separation ($|T_n - T_{n+1}| < C$ for C, a constant independent of n) such that $||x(T_n)|| = L$. Writing

$$\begin{aligned} x(t+T_n) = x(0) &+ \int_0^{T_n} K_\alpha(t+T_n-\tau) f(x(\tau),y(\tau),\tau)\,d\tau \\ &+ \int_{T_n}^{t+T_n} K_\alpha(t+T_n-\tau) f(x(\tau),y(\tau),\tau)\,d\tau \end{aligned}$$

and, putting $\widetilde{x}(t) := x(t+T_n)$, we have

$$\widetilde{x}(t) = R(t) + \int_0^t K_\alpha(t-\tau) f(\widetilde{x}(\tau),\widetilde{y}(\tau),\tau+T_n)\,d\tau,$$

where $R(t) := x(0) + \int_0^{T_n} K_\alpha(t+T_n-\tau) f(x(\tau),y(\tau),\tau)\,d\tau$. Note that $||R(0)|| = L$ and that $\int_0^{T_n} K_\alpha(t+T_n-\tau)||f(x(\tau),y(\tau),\tau)||\,d\tau$ is decreasing and converges to zero as $t \to +\infty$ ([**19,** Property 17]). Thus, $||R(t)|| \le L$, and therefore,

$$||\widetilde{x}(t)|| \le L + \left|\left| \int_0^t K_\alpha(t-\tau) f(\widetilde{x}(\tau),\widetilde{y}(\tau),\tau+T_n)\,d\tau \right|\right|.$$

By using the hypothesis $||f(x(t),y(t),t)|| \le C||x(t)||$ for all $t > 0$, we have

$$||\widetilde{x}(t)|| \le L + C\int_0^t K_\alpha(t-\tau)||\widetilde{x}(\tau))||\,d\tau.$$

Using the generalized Gronwall inequality (see, e.g., [**10,** Lemma 6.19]), $||\widetilde{x}(t)|| \le LE_\alpha(Ct^\alpha)$, where E_α is the Mittag-Leffler function.

The bound on the right hand side of this inequality is independent of n, whereby x cannot diverge on a sequence of finite intervals. Then, x is a bounded function.

Now, we will consider the variable y. By α-integration of (4.4), we obtain

$$x^T x(t) - x^T x(0) + I^{\alpha+1-\beta C} D[y^T y](t) \leq 0,$$

and the same reasoning as above shows that $D[y^T y]$ cannot be unbounded for an unbounded increasing period of time. On the other hand, if $D[y^T y]$ diverges to $-\infty$ at an unbounded increasing period of time, integer order integration of it yields $y^T y(t) = y^T y(0) + ID[y^T y] \geq 0$, arriving at a similar contradiction as above. Hence, $D[y^T y]$ is bounded from below at an unbounded increasing period of time. If $D^\alpha y = g(x, y, t)$ is BIBO stable with respect to the input x and output y, then y is also bounded.

(iii) Using that x is bounded and $||f(x, y, t)|| \leq C||x||$ is also bounded, if g is radially unbounded respect to its first coordinate, $x^T f(x, y, t) + y^T g(x, y, t) \leq 0$ does not hold if y is unbounded.

Part (b). By the same reasons as above, we get

$$I^{\beta-\alpha}[x^T x(\cdot) - x^T x(0)] + y^T y(t) - y^T y(0) + I^\beta y^T y \leq 0. \tag{4.6}$$

From the condition $||f(x(t), y(t), t)|| \leq C||x(t)||$ we conclude that, if x is unbounded, inequality (4.6) does not hold. Hence, x is bounded. From this inequality, we have

$$y^T y(t) - y^T y(0) + I^\beta y^T y \leq I^{\beta-\alpha}[x^T x(0)]. \tag{4.7}$$

From the condition $||g(x(t), y(t), t)|| \leq C||y(t)||$, if y is unbounded, $I^\beta y^T y$ will grow in an order larger than t^β, contradicting the right hand side of (4.6) that grows in an order $t^{\beta-\alpha}$. Hence, y is bounded. □

A trivial application of Theorem 4.1 is given by the following system

$$\begin{cases} {}^C D^{\alpha_1} x_1(t) = \lambda_1 x_1(t) \\ \cdots \\ {}^C D^{\alpha_n} x_n(t) = \lambda_n x_n(t), \end{cases}$$

where $\lambda_i < 0$, $0 < \alpha_i < 1$ for $i = 1, \ldots, n$, so that condition (4.2) holds. The interesting fact is that the solution verifies Theorem 3.2, as we expected, since $x_i(t) = x_i(0)E_{\alpha_i}(\lambda_i t^{\alpha_i}) \in \mathcal{C}^1(0,\infty]$ due to the fact that $DE_{\alpha_i}(t)$ is singular at zero (see, e.g., [**21,** 1.8.20]), but $x_i(t) = x_i(0)E_{\alpha_i}(\lambda_i t^{\alpha_i}) \in \mathcal{C}[0,\infty]$ as Theorem 2.1 indicates.

A slight modification of the above system gives another example

$$\begin{cases} {}^C D^{\alpha_1} x_1(t) = \lambda_1 x_1(t) + x_1 x_2^2 (1 + x_1^2 + x_2^2)^{-1} \\ {}^C D^{\alpha_2} x_2(t) = \lambda_2 x_2(t) - x_1^2 x_2 (1 + x_1^2 + x_2^2)^{-1}, \end{cases}$$

where $\lambda_1, \lambda_2 < 0$, whereby equations in x_1 and x_2 are BIBO stable with respect to x_2 and x_1, respectively. Since condition (4.2) holds and $|{}^C D^{\alpha_1} x_1(t)| \le (1 + |\lambda_1|)|x_1|$ and $|{}^C D^{\alpha_2} x_2(t)| \le (1 + |\lambda_2|)|x_2|$, we conclude from Theorem 4.1 (ii) that the solutions are bounded.

In the second example, we consider the following system:

$$\begin{cases} {}^C D^{\alpha} x = f(x) + g(x) z \\ {}^C D^{\alpha} z = f_1(x,z) + g_1(x,z) v, \end{cases} \tag{4.8}$$

where $\alpha \le 1$, $x : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$, $z : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and $f(0) = 0$. The functions $f, g : \mathbb{R}^n \to \mathbb{R}^n$a and $f_1, g_1 : \mathbb{R}^{n+1} \to \mathbb{R}$ are smooth Lipschitz continuous functions. It is assumed that, for all $(x^T, z)^T \in \mathbb{R}^{n+1}$, $g_1(x,z) \ne 0$. The problem is to find a function v such that x converges to the equilibrium point $x = 0$ of the autonomous system and z converges to zero. In that case, we say that the system is asymptotically stabilizable by feedback input. A backstepping control is proposed in the next result.

Theorem 4.2. *Consider that for system* (4.8) *there exists a smooth function u such that $F(x) := f(x) + g(x)u(x)$ is Lipschitz continuous and $x^T F(x) \le -w(x)$ for all x in a neighborhood of $x = 0$, with w a positive definite function greater than or equal to $x^T x$ and $u(0) = 0$. Then* (4.8) *is asymptotically stabilizable by feedback input.*

Proof. For simplicity, we first prove the statement for $f_1 \equiv 0$ and $g_1 \equiv 1$. By defining $e = z - u$, system (4.8) can be rewritten as

$$\begin{cases} {}^C D^{\alpha} x = F(x) + g(x)(z - u) =: F(x) + g(x) e \\ {}^C D^{\alpha} e = v - \rangle {}^C D^{\alpha} u, \end{cases} \tag{4.9}$$

where, by assumption, $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, Lipschitz continuous and $F(0) = 0$. By defining $2V(x,e) = x^T x + e^2$, using Theorem 3.2 and [**4,** Lemma 1], we have

$$^{C}D^{\alpha}V \leq x^{T}F(x) + x^{T}g(x)e + ev - e\rangle^{C}D^{\alpha}u.$$

By choosing $v = -x^{T}g(x) + {}^{C}D^{\alpha}u - ke$, where $k > 0$, we obtain

$$^{C}D^{\alpha}V(t) \leq -w(x) - ke^{2}$$

whereby $(x, e) = (0, 0)$ is an asymptotically stable point [**16,** Theorem 1]. In particular, $x \to 0$ and $z(t) \to u(x(t)) \to 0$ as $t \to \infty$ since u is continuous. Note, for the application of Theorem 3.2 that equation (4.8) takes the form:

$$\begin{cases} ^{C}D^{\alpha}x = F(x) + g(x)e \\ ^{C}D^{\alpha}e = -x^{T}g(x) - ke, \end{cases}$$

and the function $G(x, e) := (F(x) + g(x)e, -x^{T}g(x) - ke,)$ is smooth and Lipschitz continuous around $(x, e) = (0, 0)$.

For the general case, it is enough to take the control law $v_1 := (1/g_1(x, z))(-f_1(x, z) + v)$ with v defined as above. $\square$

As an example holding the assumption of Theorem 2.1, take $f_1 \equiv 0$, $g_1 \equiv 1$, $g(x) = 1$ and $u(x) = -x - f(x)$.

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University of Chile, Department of Electrical Engineering, Av. Tupper 2007, Santiago, Chile
Email address: **jgallego@ing.uchile.cl**

Universidad Tecnológica Metropolitana, Department of Electricity, Av. José Pedro Alessandri 1242, Santiago, Chile
Email address: **norelys.aguila@utem.cl**

Universidad Tecnológica Metropolitana, Department of Electricity, Av. José Pedro Alessandri 1242, Santiago, Chile
Email address: **m.duartem@utem.cl**