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ON THE UNIQUE CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS BY SINGLE REGRESSION OF NON-ADJACENT GENERALIZED ORDER STATISTICS

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Dedicated to Professor Dominik Szynal
on the occasion of his 80*th birthday*

ABSTRACT. We show a new and unexpected application of integral equations and their systems to the problem of the unique identification of continuous probability distributions based on the knowledge of exactly one regression function of ordered statistical data. The most popular example of such data are the order statistics which are obtained by non-decreasing ordering of elements of the sample according to their magnitude. However, our considerations are conducted in the abstract setting of so-called generalized order statistics. This model includes order statistics and other interesting models of ordered random variables. We prove that the uniqueness of characterization is equivalent to the uniqueness of the solution to the appropriate system of integral equations with non-classical initial conditions. This criterion for uniqueness is then applied to give new examples of characterizations.

1. Introduction.

We begin with introducing the notation and definitions used throughout this paper. Our basic objects of study are various models of ordered statistical data. The simplest example of such a model are the order statistics $X_{1:n} \le \cdots \le X_{n:n}$ obtained by putting elements of the random sample $X_1, \ldots, X_n$ of size n in increasing order. The less standard example is comprised of record values $\{R_n,\, n \ge 1\}$ of the sequence $\{X_n,\, n \ge 1\}$ of independent identically distributed ran-

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dom variables. They are defined recursively as $R_1 = X_1$, $U(1) = 1$ and $R_n = X_{U(n)}$, $n \geq 1$, where $U(n+1) = \min\{j > U(n) : X_j > R_n\}$. Other examples include progressively censored type II order statistics or kth record values (see [**1, 6**]).

Kamps [**16**] demonstrated that research concerning various models of ordered random variables can be unified with the aid of the model of generalized order statistics (GOSs), defined as follows. Fix $n \in \mathbb{N}$, parameters $\gamma_1, \ldots, \gamma_n > 0$, and let $B_1, \ldots, B_n$ be independent random variables such that B_i has Beta$(\gamma_i, 1)$ distribution. Then, GOSs based on uniform distribution on $[0,1]$ are defined as the random vector $(U_*^{(1)}, \ldots, U_*^{(n)})$, where

$$U_*^{(r)} = 1 - \prod_{i=1}^{r} B_i \quad \text{for } 1 \leq r \leq n.$$

Furthermore, if F is a continuous distribution function with the quantile function

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}, \quad y \in [0,1),$$

then GOSs $X_*^{(1)}, \ldots, X_*^{(n)}$, with parameters $\gamma_1, \ldots, \gamma_n$, based on F, are defined by $X_*^{(r)} = F^{-1}(U_*^{(r)})$ for $1 \leq r \leq n$. The most important special cases include ordinary order statistics (with $\gamma_i = n - i + 1$, $1 \leq i \leq n$) and record values $R_1 \leq \cdots \leq R_n$ (with $\gamma_i = 1$). Other special cases are kth record values, Pfeifer's record values, progressively censored type II order statistics and sequential order statistics. For details, see e.g., [**6**].

In the literature, many papers may be found devoted to the problem of characterization of probability distributions by properties of order statistics. Among the most interesting results are those related to characterizations of probability distributions by regression conditions involving order statistics. We are especially interested in regression conditions of the type

$$E\big(h(X_{s:n}) \mid X_{r:n} = x\big) = \xi(x), \tag{1.1}$$

for fixed $1 \leq r < s \leq n$, where h and ξ are known functions defined on the support of the underlying F. Beginning with Ferguson [**11**] who, in the case $s = r + 1$, characterized exponential, power and Pareto distributions as the only absolutely continuous distributions with $h(x) = x$ and ξ the linear function $\xi(x) = ax + b$, there is extensive literature on this subject. For a review of recent literature, the interested reader is referred to **[1]**. Among those results, we mention the paper by Dembińska and Wesołowski **[9]**, who extended Ferguson's result for continuous distributions to the non-adjacent case $s > r + 1$. Franco and Ruiz **[12, 14]** considered the adjacent case $s = r + 1$, but with general functions h and ξ, even in the case of arbitrary distributions (not necessarily continuous).

It appears that many results for order statistics have their analogues for the record values; thus, parallel research concerns characterizations of probability distributions by an analogous regression condition

$$E\big(h(R_s) \mid R_r = x\big) = \xi(x). \tag{1.2}$$

For instance, see **[10]** for linear regression ξ in the non-adjacent case and **[13]** for general regression in the adjacent case. However, in the non-adjacent case, the problem of characterization by either of the regressions (1.1) or (1.2) remains open.

The study of these regression conditions can be unified as follows. Fix $r, \ell \geq 1$ with $r + \ell \leq n$, and consider $X_*^{(r)}$ and $X_*^{(r+\ell)}$ based upon the continuous distribution function F concentrated on a fixed interval (α, β), where $-\infty \leq \alpha < \beta \leq \infty$ satisfy

$$\alpha = \inf\{x \in \mathbb{R} : F(x) > 0\}, \qquad \beta = \sup\{x \in \mathbb{R} : F(x) < 1\}.$$

Let $h : (\alpha, \beta) \to \mathbb{R}$ be a fixed, strictly increasing and continuous function such that $E|h(X_*^{(r+\ell)})| < \infty$. Define the regression function of $h(X_*^{(r+\ell)})$ on $X_*^{(r)}$ as the function $\xi : (\alpha, \beta) \to \mathbb{R}$, given by

$$\xi(x) = E\big(h(X_*^{(r+\ell)}) \mid X_*^{(r)} = x\big), \quad x \in (\alpha, \beta). \tag{1.3}$$

Obviously, this is a direct extension of (1.1) and (1.2). Moreover, the left-hand side of (1.3) depends upon r, ℓ and the parameters $\gamma_1, \ldots, \gamma_{r+\ell}$ as well, but, in order to avoid complicated symbols, we omit this dependence in the notation.

Cramer, et al., **[7]** proved that each continuous distribution F uniquely determines the continuous and strictly increasing version of the regression ξ. More precisely, ξ can be calculated as the expectation of the regular version of the corresponding conditional probability of $X_*^{(r+\ell)}$ given $X_*^{(r)} = x$. In this paper, we consider the inverse problem of the unique identification of F by the knowledge of single regression ξ given by (1.3).

This problem is completely solved in the adjacent case, i.e., for $\ell = 1$. Then, simple reasoning shows that F can be recovered from ξ and h as

$$F(x) = 1 - \exp\left(-\frac{1}{\gamma_{r+1}} \int_\alpha^x \frac{\mathrm{d}\xi(t)}{\xi(t) - h(t)} \right), \quad x \in (\alpha, \beta),$$

see [**7**]. Bieniek [**2**] proved that F is also uniquely determined by the reverse regression of $h(X_*^{(r)})$ given $X_*^{(r+1)}$. However, the arguments for the adjacent case cannot be extended to the non-adjacent case, even in simple special cases of order statistics and record values.

For $\ell \geq 2$, if $h(x) = x$, and ξ is a linear function of the form $\xi(x) = ax + b$, then either $a \in (0,1)$ and F is the unique power distribution, or $a = 1$ and F is exponential, or $a > 1$ and F is a Pareto distribution, see [**5, 7**]. Unfortunately, the method of proof introduced by Dembińska and Wesołowski [**9, 10**], utilizing a solution to the so-called integrated Cauchy functional equation, cannot be applied to the non-linear regression ξ. The reason is that, for general ξ, this approach leads to a functional equation whose solution is unknown. However, Bieniek [**3**] proved that F is uniquely determined by the knowledge of two regression functions:

$$\xi_1(x) = E\big(h(X_*^{(r+\ell)}) \mid X_*^{(r)} = x\big),$$

and

$$\xi_2(x) = E\big(h(X_*^{(r+\ell)}) \mid X_*^{(r+1)} = x\big),$$

namely, in this case,

$$F(x) = 1 - \exp\left(-\frac{1}{\gamma_{r+1}} \int_\alpha^x \frac{\mathrm{d}\xi_1(t)}{\xi_1(t) - \xi_2(t)} \right), \quad x \in (\alpha, \beta).$$

This poses the question of whether it is possible to determine ξ_2 when ξ_1 and h are given, which is the main idea of this paper.

Here, we continue the approach proposed by the authors in [**4**] for absolutely continuous distributions with continuous density function to study the uniqueness of the characterization by the single regression (1.3) in the case of arbitrary continuous distributions. We show that, for $\ell \geq 2$, the uniqueness of the characterization of the underlying F by the knowledge of the continuous regression ξ is equivalent to the uniqueness of solution to the corresponding system of $\ell - 1$ integral equations (see Theorem 4.6 below). In particular, for $\ell = 2$, the continuous regression ξ determines F uniquely if and only if the integral equation

$$y(x) = y(\alpha) + \frac{\gamma_{r+2}}{\gamma_{r+1}} \int_\alpha^x \frac{y(t) - h(t)}{\xi(t) - y(t)} \, \mathrm{d}\xi(t)$$

has the unique solution φ such that $h(x) < \varphi(x) < \xi(x)$ for all $x \in (\alpha, \beta)$. Moreover, then,

$$F(x) = 1 - \exp\left(- \frac{1}{\gamma_{r+1}} \int_\alpha^x \frac{\mathrm{d}\xi(t)}{\xi(t) - \varphi(t)} \right). \tag{1.4}$$

Note that, if ξ has continuous derivative, then the integral equation turns into the ordinary differential equation

$$y' = \frac{\gamma_{r+2}}{\gamma_{r+1}} \frac{y - h(t)}{\xi(t) - y} \xi'(t), \tag{1.5}$$

which was obtained in [**4**].

Unfortunately, although our criterion for uniqueness of the characterization is surprisingly simple, it appears to be very difficult to implement, even in the special cases of h and ξ. This is due to the fact that our integral equations do not have classical initial conditions like $y(x_0) = y_0$, but non-classical like $h(x) < y < \xi(x)$ for $x \in (\alpha, \beta)$. At this time, we are able to prove the uniqueness of the solution only in the case when $\ell = 2$. If

$$h(\beta) = \lim_{x \to \beta} h(x) < \infty,$$

then $\xi(\beta) = h(\beta)$, which easily implies the desired uniqueness. On the other hand, if $h(\beta) = \infty$, then also, $\xi(\beta) = \infty$, and it is possible that

$$\xi(x) - h(x) \not\to 0 \quad \text{as } x \to \beta.$$

However, our approach will be applied to prove uniqueness in the case when $h(\beta) = \infty$ and ξ is a continuous increasing and asymptotically linear function of h, i.e., there exist $a \geq 1$ and $b > 0$ such that $\lim_{x \to \infty} [\xi(x) - (ah(x) + b)] = 0$. For instance, this yields a new characterization of gamma distributions.

Remark 1.1. Here and in the remainder of the paper we adopt the convention that all the limits as $x \to \beta$ should be understood as left-hand side limits.

The paper is organized as follows. In Section 2, we recall basic results on the distribution theory of GOSs, and we study basic properties of the regression function ξ, which are used in our considerations. Then, in Section 3, the Markov property of GOSs based on continuous distribution functions is applied to study the specific recurrence structure of the regressions of GOSs. In Section 4, our main results on the uniqueness of the characterization are stated and proven. Section 5 deals with the case $\ell = 2$, i.e., we consider the uniqueness of characterization by $E(h(X_*^{(r+2)}) \mid X_*^{(r)} = x)$. New characterizations of particular continuous distributions by the regression of record values or GOSs, obtained using our approach, are presented in Section 6. Finally, in Section 7, we summarize the results of the paper and discuss their relevance for researchers in both statistics and integral equations.

2. Auxiliary results. In this section, we recall some known results on the distribution of GOSs and essential properties of the regression function ξ defined by (1.3). Cramer and Kamps [**6**] proved that the marginal density function of the rth uniform generalized order statistic $U_*^{(r)}$ can be written as

$$f^{U_*^{(r)}}(x) = c_{r-1}\mathsf{G}_{r,r}^{r,0}\left(1-x \,\middle|\, \begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right), \quad x \in (0,1),$$

where $\mathsf{G}_{r,r}^{r,0}$ is a particular Meijer's G-function, defined by

$$\mathsf{G}_{r,r}^{r,0}\left(s \,\middle|\, \begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{s^z}{\prod_{i=1}^r(\gamma_j-1-z)}\,\mathrm{d}z, \tag{2.1}$$

and L is an appropriately chosen contour of integration (see, e.g., [**17,** Chapter 3] for the definition of a general G-function). In the remainder of the paper, for brevity, we write $G_r(x \mid \gamma_1,\ldots,\gamma_r)$ instead of $\mathsf{G}_{r,r}^{r,0}(1-x \mid {}^{\gamma_1,\ldots,\gamma_r}_{\gamma_1-1,\ldots,\gamma_r-1})$.

Remark 2.1. Due to (2.1), whenever this is necessary, without loss of generality, we may assume that $\gamma_{r+1} \le \cdots \le \gamma_{r+\ell}$.

Let $\overline{F} = 1 - F$ be the survival function, and let P_F denote the probability measure on $\mathbb{R}$, determined by F. If F is a continuous distribution function, then $X_*^{(r)}$ has density with respect to measure P_F, given by

$$f^{X_*^{(r)}}(x) = c_{r-1} G_r(F(x) \mid \gamma_1, \dots, \gamma_r) I_{(\alpha,\beta)}(x),$$

where I_A denotes the indicator function of a set A. In particular, the conditional P_F-density function of $X_*^{(r+\ell)}$, given $X_*^{(r)} = x$, $\ell \geq 1$, can be written as (see [**7**])

$$f^{X_*^{(r+\ell)} | X_*^{(r)}}(t \mid x) = \frac{c_{r+\ell-1}}{c_{r-1}} G_\ell \left(F_x(t) \mid \gamma_{r+1}, \dots, \gamma_{r+\ell}\right) \frac{1}{\overline{F}(x)} I_{(x,\beta)}(t),$$

for $x, t \in \mathbb{R}$, where

$$F_x(t) = \begin{cases} \frac{F(t)-F(x)}{\overline{F}(x)} & \text{for } t \geq x, \\ 0 & \text{for } t < x, \end{cases} \qquad \alpha < x, \ t < \beta.$$

Therefore, (1.3) is equivalent to

$$\xi(x) = \frac{c_{r+\ell-1}}{c_{r-1}} \frac{1}{\overline{F}(x)} \int_x^\beta h(t) G_\ell \left(F_x(t) \mid \gamma_{r+1}, \dots, \gamma_{r+\ell}\right) \mathrm{d}F(t). \tag{2.2}$$

Using representation (2.2), it is easy to prove the following properties of regressions of GOSs, defined by (1.3).

Lemma 2.2. *Let F be a continuous distribution function supported on (α, β), and let $\xi : (\alpha, \beta) \to \mathbb{R}$ be defined by* (2.2). *Then, ξ has the following properties*:

- (i) *ξ is continuous on (α, β)*;
- (ii) *ξ is increasing and it is constant on an interval $I \subset (\alpha, \beta)$ if and only if F is constant on I*;
- (iii) *$\xi(x) > h(x)$ for all $x \in (\alpha, \beta)$*;
- (iv) *if $h(\beta) < \infty$, then $\xi(\beta) = h(\beta)$.*

We conclude this section with the remark that, whenever it is necessary, without loss of generality, we may assume that $h(x) = x$, following the representation of ξ given in [**7,** page 2891].

3. The Markov property of GOSs. In what follows, we use the Markov property of GOSs based on continuous distribution functions. Note that, for distribution functions with possible jumps, the Markov property of GOSs does not hold, see [**8**].

Theorem 3.1. *If F is a continuous distribution function, then, for any $r \geq 2$, the conditional distribution of $X_*^{(r+1)}$ given $X_*^{(1)}, \ldots, X_*^{(r)}$ is the same as the conditional distribution of $X_*^{(r+1)}$ given $X_*^{(r)}$.*

For instance, for any $1 \leq p < r < s$ and for any function $h : (\alpha, \beta) \to \mathbb{R}$ for which the conditional expectations exist,

$$E\big[h\big(X_*^{(s)}\big) \mid X_*^{(p)}, X_*^{(r)}\big] = E\big[h\big(X_*^{(s)}\big) \mid X_*^{(r)}\big].$$

The next lemma is the crucial observation in the derivation of the results of this paper. It is proven in [**4**]; thus, here, we only give a short sketch of the proof. The lemma states that regressions of GOSs have a specific recurrence structure so that the regression of non-adjacent GOSs can be expressed as an appropriate regression of adjacent GOSs.

Lemma 3.2. *Fix $r \geq 1$, $\ell \geq 2$, and let $\varphi_\ell : (\alpha, \beta) \to \mathbb{R}$ be any Borel measurable function such that $E|\varphi_\ell(X_*^{(r+\ell)})| < \infty$. Moreover, for $i = 0, 1, \ldots, \ell - 1$ define*

$$\varphi_i(x) = E\big(\varphi_{i+1}(X_*^{(r+i+1)}) \mid X_*^{(r+i)} = x\big), \quad x \in (\alpha, \beta). \tag{3.1}$$

Then,

$$E\big(\varphi_\ell(X_*^{(r+\ell)}) \mid X_*^{(r)} = x\big) = \varphi_0(x), \quad x \in (\alpha, \beta).$$

Sketch of the proof. It suffices to note that

$$\begin{aligned} &E\big(\varphi_{\ell+1}(X_*^{(r+\ell+1)}) \bigm| X_*^{(r)}\big) \\ &\quad = E\big[E\big(\varphi_{\ell+1}(X_*^{(r+\ell+1)}) \bigm| X_*^{(r)}, X_*^{(r+\ell)}\big) \mid X_*^{(r)}\big] \\ &\quad = E\big[E\big(\varphi_{\ell+1}(X_*^{(r+\ell+1)}) \mid X_*^{(r+\ell)}\big) \mid X_*^{(r)}\big] \\ &\quad = E\big(\varphi_\ell(X_*^{(r+\ell)}) \mid X_*^{(r)}\big), \end{aligned} \tag{3.2}$$

where the first equality follows from the classical property of conditional expectations, the second from the Markov property of GOSs, and the third from definition (3.1). Now easy induction proof follows. □

4. Uniqueness of the characterization of continuous distributions. In this section, we state necessary and sufficient conditions for the uniqueness of the characterization of probability distributions by the knowledge of the single regression ξ given by (1.3). We assume that F is an arbitrary, continuous distribution function concentrated on (α, β) (possibly not absolutely continuous). Referring to (2.2), the regression ξ can be expressed in terms of the Riemann-Stieltjes integral, so first we recall some auxiliary results on integration with respect to functions which are monotone or of bounded variation.

Lemma 4.1. *Suppose that f and g are continuous on $[a, b]$ and A is a function of bounded variation on $[a, b]$. Define*

$$B(x) = \int_a^x f(t)\,\mathrm{d}A(t), \quad a \le x \le b.$$

Then, B is also of bounded variation and

$$\int_a^b g(t)\,\mathrm{d}B(t) = \int_a^b g(t)f(t)\,\mathrm{d}A(t).$$

Proof. This is a special case of a classical result in Lebesgue integration theory, see e.g., [**19,** Theorem 1.29]. See [**15,** Problems 1.2.26, 1.3.3] for an elementary proof based on Riemann-Stieltjes integral sums. □

The next lemma is an easy consequence of integration by parts for Riemann-Stieltjes integrals.

Lemma 4.2. *If f and g are continuous functions of bounded variation on $[a, b]$ and, additionally, $g(t) \ge c > 0$ for $a \le t \le b$, then*

$$\frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} = \int_a^b \frac{1}{g(t)}\,\mathrm{d}f(t) - \int_a^b \frac{f(t)}{(g(t))^2}\,\mathrm{d}g(t).$$

The next result is the key observation in further considerations. It gives a nontrivial expression of any regression of adjacent GOSs.

Lemma 4.3. *If $F:(\alpha,\beta)\to(0,1)$ is continuous distribution function, and*

$$\xi(x)=E\big(h(X_*^{(r+1)})\mid X_*^{(r)}=x\big),\quad x\in(\alpha,\beta),$$

for some continuous and strictly increasing functions $h,\xi:(\alpha,\beta)\to\mathbb{R}$ such that $h<\xi$, then

$$\xi(x)=\xi(\alpha)+\gamma_{r+1}\int_\alpha^x\big[\xi(t)-h(t)\big]\frac{\mathrm{d}F(t)}{\overline{F}(t)},\quad x\in(\alpha,\beta).$$

Proof. For simplicity, put $\gamma=\gamma_{r+1}$. Since $G_1(x\mid\gamma)=(1-x)^{\gamma-1}$, then, by (2.2) with $\ell=1$, we obtain

$$\xi(x)=\frac{\gamma}{\overline{F}(x)^\gamma}\int_x^\beta h(t)\overline{F}(t)^{\gamma-1}\,\mathrm{d}F(t). \tag{4.1}$$

Setting

$$f(x)=\gamma\int_x^\beta h(t)\overline{F}(t)^{\gamma-1}\,\mathrm{d}F(t),\quad g(x)=\overline{F}(x)^\gamma$$

by Lemma 4.2, for any $x\in(\alpha,\beta)$, we have

$$\xi(x)-\xi(\alpha)=\int_\alpha^x\frac{1}{g(t)}\,\mathrm{d}f(t)-\int_\alpha^x\frac{f(t)}{(g(t))^2}\,\mathrm{d}g(t). \tag{4.2}$$

However, by Lemma 4.1,

$$\begin{aligned}\int_\alpha^x\frac{1}{g(t)}\,\mathrm{d}f(t)&=-\gamma\int_\alpha^x\frac{1}{\overline{F}(t)^\gamma}h(t)\overline{F}(t)^{\gamma-1}\mathrm{d}F(t)\\&=-\gamma\int_\alpha^x h(t)\frac{\mathrm{d}F(t)}{\overline{F}(t)}.\end{aligned} \tag{4.3}$$

Moreover,

$$\int_\alpha^x\frac{f(t)}{(g(t))^2}\,\mathrm{d}g(t)=\int_\alpha^x\xi(t)\frac{\mathrm{d}g(t)}{g(t)}=-\gamma\int_\alpha^x\xi(t)\frac{\mathrm{d}F(t)}{\overline{F}(t)}. \tag{4.4}$$

Putting (4.3) and (4.4) into (4.2) easily completes the proof. ∎

Now, we state our main result in the special case $\ell = 2$. Denote by $H = -\log \overline{F}$ the so-called cumulative hazard function of F.

Theorem 4.4. *Assume that $h, \xi : (\alpha, \beta) \to \mathbb{R}$ are continuous and strictly increasing functions such that $h < \xi$. Then, the regression relation*

$$\xi(x) = E\big(h(X_*^{(r+2)}) \mid X_*^{(r)} = x\big), \tag{4.5}$$

characterizes continuous distribution function F uniquely if and only if the integral equation

$$y(x) = y(\alpha) + \frac{\gamma_{r+2}}{\gamma_{r+1}} \int_\alpha^x \frac{y(t) - h(t)}{\xi(t) - y(t)} \,\mathrm{d}\xi(t) \tag{4.6}$$

has exactly one solution $y = \varphi(x)$ such that

$$h(x) < \varphi(x) < \xi(x), \quad x \in (\alpha, \beta). \tag{4.7}$$

Then, F is given by the inversion formula

$$F(x) = 1 - \exp\left(- \int_\alpha^x \frac{\mathrm{d}\eta(t)}{\xi(t) - h(t)} \right), \quad x \in (\alpha, \beta), \tag{4.8}$$

where $\eta : (\alpha, \beta) \to \mathbb{R}$ is defined by

$$\eta(t) = \frac{\xi(t)}{\gamma_{r+1}} + \frac{\varphi(t)}{\gamma_{r+2}}. \tag{4.9}$$

Proof. Assume that the regression condition (4.5) holds, and define

$$\varphi(x) = E\big(h(X_*^{(r+2)}) \mid X_*^{(r+1)} = x\big). \tag{4.10}$$

Then, by Lemma 3.2 with $\ell = 2$ (see (3.2) with $\varphi_2 = h$) we get

$$\xi(x) = E\big(\varphi(X_*^{(r+1)}) \mid X_*^{(r)} = x\big). \tag{4.11}$$

Moreover, by twice applying Lemma 2.2 (iii) with $\ell = 1$, we obtain $\xi > \varphi > h$ on (α, β). From Lemma 4.3, equations (4.10) and (4.11) can be written as

$$\xi(x) = \xi(\alpha) + \gamma_{r+1} \int_\alpha^x \big[\xi(t) - \varphi(t)\big] \frac{\mathrm{d}F(t)}{\overline{F}(t)},$$

and

$$\varphi(x) = \varphi(\alpha) + \gamma_{r+2} \int_\alpha^x \left[\varphi(t) - h(t)\right] \frac{\mathrm{d}F(t)}{\bar{F}(t)},$$

respectively. Therefore, subtracting and then again applying Lemma 4.1, we get

(4.12)

$$\begin{aligned} \xi(x) - \varphi(x) &= \xi(\alpha) - \varphi(\alpha) + \int_\alpha^x \left(1 - \frac{\gamma_{r+2}}{\gamma_{r+1}} \frac{\varphi(t) - h(t)}{\xi(t) - \varphi(t)}\right) \mathrm{d}\xi(t) \\ &= \xi(x) - \varphi(\alpha) - \frac{\gamma_{r+2}}{\gamma_{r+1}} \int_\alpha^x \frac{\varphi(t) - h(t)}{\xi(t) - \varphi(t)} \,\mathrm{d}\xi(t); \end{aligned}$$

thus, φ satisfies the integral equation (4.6). Moreover, with η defined by (4.9), we have

$$\begin{aligned} \int_\alpha^x \frac{\mathrm{d}\eta(t)}{\xi(t) - h(t)} &= \frac{1}{\gamma_{r+1}} \int_\alpha^x \frac{\mathrm{d}\xi(t)}{\xi(t) - h(t)} + \frac{1}{\gamma_{r+2}} \int_\alpha^x \frac{\mathrm{d}\varphi(t)}{\xi(t) - h(t)} \\ &= \int_\alpha^x \frac{\xi(t) - \varphi(t)}{\xi(t) - h(t)} \frac{\mathrm{d}F(t)}{\overline{F}(t)} + \int_\alpha^x \frac{\varphi(t) - h(t)}{\xi(t) - h(t)} \frac{\mathrm{d}F(t)}{\overline{F}(t)} \\ &= \int_\alpha^x \frac{\mathrm{d}F(t)}{\overline{F}(t)} = H(x), \end{aligned}$$

since $H(\alpha) = 0$, where the second equality follows from Lemma 4.1. This easily implies the inversion formula (4.8). □

Remark 4.5. The underlying F can also be recovered from (1.4), or from

$$F(x) = 1 - \exp\left(- \frac{1}{\gamma_{r+2}} \int_\alpha^x \frac{\mathrm{d}\varphi(t)}{\varphi(t) - h(t)}\right).$$

Now, we prove the extension of the last theorem for arbitrary $\ell \geq 3$. This is the main result of the paper.

Theorem 4.6. *The regression relation* (1.3), *i.e.,* $\xi(x) = E(h(X_*^{(r+\ell)}) \mid X_*^{(r)} = x)$, *for* $x \in (\alpha, \beta)$, *for the continuous and strictly increasing function* $\xi : (\alpha, \beta) \to \mathbb{R}$ *characterizes the continuous distribution function* F, *uniquely supported on* (α, β), *if and only if the system of* $\ell - 1$

integral equations

$$y_i(x) = y_i(\alpha) + \frac{\gamma_{r+i+1}}{\gamma_{r+1}} \int_\alpha^x \frac{y_i(t) - y_{i+1}(t)}{\xi(t) - y_1(t)} \, \mathrm{d}\xi(t), \tag{4.13}$$

$1 \le i \le \ell - 1$, *with* $y_\ell = h$ *has exactly one solution* $(\varphi_1, \ldots, \varphi_{\ell-1})$ *satisfying the condition*

$$h(x) < \varphi_{\ell-1}(x) < \cdots < \varphi_1(x) < \xi(x), \quad x \in (\alpha, \beta). \tag{4.14}$$

Then, F is given by the inversion formula

$$F(x) = 1 - \exp\left(- \int_\alpha^x \frac{\mathrm{d}\eta(t)}{\xi(t) - h(t)} \right), \quad x \in (\alpha, \beta), \tag{4.15}$$

where $\eta : (\alpha, \beta) \to \mathbb{R}$ *is given by*

$$\eta(t) = \frac{\xi(t)}{\gamma_{r+1}} + \sum_{i=1}^{\ell-1} \frac{\varphi_i(t)}{\gamma_{r+i+1}}. \tag{4.16}$$

Proof. Denote again $\varphi_\ell = h$, and refer to Lemma 3.2 to conclude that $\varphi_0 = \xi$. By Lemma 4.3 applied to (3.1), we obtain

$$\varphi_i(x) = \varphi_i(\alpha) + \gamma_{r+i+1} \int_\alpha^x [\varphi_i(t) - \varphi_{i+1}(t)] \frac{\mathrm{d}\overline{F}(t)}{\overline{F}(t)},$$

for $i = 0, 1, \ldots, \ell - 1$. Now, consider the differences $\xi(x) - \varphi_i(x) = \varphi_0(x) - \varphi_i(x)$ for $i = 1, \ldots, \ell - 1$. Performing analogous calculations as in the proof of equality (4.12), we easily prove that $(\varphi_1, \ldots, \varphi_{\ell-1})$ solves system (4.13). Moreover, for η defined by (4.16), we have

$$\begin{aligned}
\int_\alpha^x \frac{\mathrm{d}\eta(t)}{\xi(t) - h(t)} &= \sum_{i=0}^{\ell-1} \frac{1}{\gamma_{r+i+1}} \int_\alpha^x \frac{1}{\xi(t) - h(t)} \, \mathrm{d}\varphi_i(t) \\
&= \sum_{i=0}^{\ell-1} \int_\alpha^x \frac{\varphi_i(t) - \varphi_{i+1}(t)}{\xi(t) - h(t)} \frac{\mathrm{d}F(t)}{\overline{F}(t)} \\
&= \int_\alpha^x \frac{1}{\xi(t) - h(t)} \left(\sum_{i=0}^{\ell-1} [\varphi_i(t) - \varphi_{i+1}(t)] \right) \frac{\mathrm{d}F(t)}{\overline{F}(t)} \\
&= \int_\alpha^x \frac{\mathrm{d}F(t)}{\overline{F}(t)} = H(x).
\end{aligned}$$

This suffices to easily complete the proof. □

Remark 4.7. If we, additionally, assume that the regression ξ is differentiable, then the system of integral equations turns into the system of $\ell - 1$ differential equations

$$y_i' = \frac{\gamma_{r+i+1}}{\gamma_{r+1}} \frac{y_i - y_{i+1}}{\xi(x) - y_1} \xi'(x), \quad 1 \leq i \leq \ell - 1, \tag{4.17}$$

satisfying the condition (4.14). The uniqueness of the solution to system (4.17) with (4.14) ensures the uniqueness of the characterization of the absolutely continuous distribution F, given by (4.15) with $d\eta(t)$ replaced by $\eta'(t)\,dt$. Moreover, if $\ell = 2$, then system (4.17) reduces to the single ordinary differential equation (1.5) with condition (4.7). Therefore, the results of this section are generalizations of the results of [**4**].

5. Uniqueness of the characterization for $\ell = 2$. In the previous section, the equivalent condition for the uniqueness of the characterization of continuous distributions by the regression (1.3) has been proved. Unfortunately, although our criterion is surprisingly simple, it appears to be very difficult to implement, even in the special cases of h and ξ. This criterion is formulated in terms of integral equations which are non-linear with respect to the unknown functions. Moreover, the solution to (4.6) is required to satisfy the non-classical condition given by inequality (4.7).

In the remainder of this paper we consider the desired uniqueness of the characterization of continuous distributions by regression of GOSs in the case when $\ell = 2$ and, additionally, one of the following conditions holds:

(i) h is bounded so that $h(\beta) = \lim_{x\to\beta} h(x) < \infty$;
(ii) h is unbounded so that $h(\beta) = \infty$, and ξ is asymptotically linear transformation of h, i.e.,

$$\lim_{x\to\beta} [\xi(x) - (ah(x) + b)] = 0$$

for some $a \geq 1$.

In order to prove it, we need the following lemma, which discusses auxiliary properties of solutions to the problem

$$\begin{cases} y(x) = y(\alpha) + \gamma \displaystyle\int_\alpha^x \frac{y(t) - h(t)}{\xi(t) - y(t)} \, \mathrm{d}\xi(t), \\ h(x) < y(x) < \xi(x), \end{cases} \qquad x \in (\alpha, \beta). \tag{5.1}$$

Lemma 5.1. *Assume that $h, \xi : (\alpha, \beta) \to \mathbb{R}$ are two arbitrary, strictly increasing and continuous functions such that $h(x) < \xi(x)$ for all $x \in (\alpha, \beta)$. Suppose that $\gamma > 0$, and $\varphi : (\alpha, \beta) \to \mathbb{R}$ is a fixed solution to problem* (5.1)*. If y is any other solution, then*

(i) *$y(x) \neq \varphi(x)$ for all $x \in (\alpha, \beta)$;*
(ii) *either $y < \varphi$ or $y > \varphi$ on (α, β);*
(iii) *if $z = |y - \varphi|$, then z is strictly increasing on (α, β);*
(iv) *if $h < y < \varphi$ and $w = (\varphi - y)/(\varphi - h)$, then $w(\alpha) < w < 1$, and w is increasing on (α, β);*
(v) *if $\varphi < y < \xi$, and additionally $\gamma \geq 1$ and $w = (y - \varphi)/(\xi - \varphi)$, then $w(\alpha) < w < 1$, and w is increasing on (α, β).*

Remark 5.2. In what follows, for brevity, we write $\int_\alpha^x f \, \mathrm{d}g$ instead of $\int_\alpha^x f(t) \, \mathrm{d}g(t)$, if it does not lead to confusion.

Proof. Fix arbitrary $x_0 \in (\alpha, \beta)$, and denote $\varphi_0 = \varphi(x_0)$. Let

$$D = \{(x, y) : x \in (\alpha, \beta),\ h(x) < y < \xi(x)\},$$

and choose $\varepsilon > 0$ small enough such that the ball

$$B = B((x_0, \varphi_0), \varepsilon) \subset D.$$

Consider the function $K : D \to (0, \infty)$, given by

$$K(x, y) = \frac{y - h(x)}{\xi(x) - y},$$

so that the integral equation in (5.1) becomes

$$y(x) = y(\alpha) + \gamma \int_\alpha^x K(t, y(t)) \, \mathrm{d}\xi(t). \tag{5.2}$$

Then, for y_1, y_2 such that (x, y_1) and (x, y_2) are in D, we have

$$|K(x, y_1) - K(x, y_2)| = \frac{\xi(x) - h(x)}{(\xi(x) - y_1)(\xi(x) - y_2)} |y_1 - y_2|,$$

so that K satisfies the Lipschitz condition with respect to y on B. Standard methods show that the integral equation (5.2) with the initial condition $y(x_0) = \varphi_0$ has the unique solution on the interval $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$ sufficiently small. Thus, if y is any other solution to (5.1), then $y(x_0) \neq \varphi(x_0)$. This proves part (i) of the lemma.

In order to prove part (ii), it suffices to note that φ and y are continuous. Since their difference is never zero, it must be either positive or negative on the entire interval of interest.

To prove part (iii), we first note that, if y and φ are any two solutions to (5.1)), then

$$y(x) - \varphi(x) = y(\alpha) - \varphi(\alpha) + \gamma \int_\alpha^x \frac{(\xi - h)(y - \varphi)}{(\xi - y)(\xi - \varphi)} \, \mathrm{d}\xi. \tag{5.3}$$

Thus, assume that $y > \varphi$, and consider $z = y - \varphi$. Since ξ is strictly increasing and the integrand is positive on (α, β), it follows that z is also strictly increasing. If $y < \varphi$, then it suffices to consider $z = \varphi - y$.

(iv) Let $h < y < \varphi < \xi$ and $w = (\varphi - y)/(\varphi - h)$. Setting $f = \varphi - y$ and $g = \varphi - h$ in Lemma 4.2 for any $x \in (\alpha, \beta)$, we obtain

$$w(x) - w(\alpha) = \int_\alpha^x \frac{1}{\varphi - h} \, \mathrm{d}(\varphi - y) - \int_\alpha^x \frac{\varphi - y}{(\varphi - h)^2} \, \mathrm{d}(\varphi - h). \tag{5.4}$$

The first integral in (5.4) is equal to

$$\begin{aligned} \int_\alpha^x \frac{1}{\varphi - h} \, \mathrm{d}(\varphi - y) &= \gamma \int_\alpha^x \frac{(\xi - h)(\varphi - y)}{(\varphi - h)(\xi - y)(\xi - \varphi)} \, \mathrm{d}\xi \\ &= \int_\alpha^x \frac{(\xi - h)(\varphi - y)}{(\varphi - h)^2(\xi - y)} \, \mathrm{d}\varphi, \end{aligned} \tag{5.5}$$

where the first equality follows from (5.3) applied to $\varphi - y$, and the second equality follows from Lemma 4.1. Substituting (5.5) for (5.4) yields

$$w(x) - w(\alpha) = \int_\alpha^x \frac{(\varphi - y)(y - h)}{(\varphi - h)^2(\xi - y)} \, \mathrm{d}\varphi + \int_\alpha^x \frac{\varphi - y}{(\varphi - h)^2} \, \mathrm{d}h.$$

Due to the assumption $h < y < \varphi$, both of the integrands in the last formula are positive. Therefore, w is increasing since φ and h are increasing on (α, β).

(v) If $\varphi < y < \xi$, then a similar approach applied to $w = (y-\varphi)/(\xi-\varphi)$ yields

$$w(x) - w(\alpha) = \int_\alpha^x \frac{y-\varphi}{\xi-\varphi}\, \mathrm{d}\lambda,$$

where $\lambda(t) = y(t) + (\gamma-1)\xi(t) + \varphi(t)$. Since $\gamma \geq 1$, the function λ is increasing. Due to the assumption $\varphi < y < \xi$, the integrand of the last integral is positive; thus, w is strictly increasing. □

This lemma allows us to prove the uniqueness of the characterization of distribution by regression of GOSs in the case when $\ell = 2$. First, we consider the case when h is bounded, i.e., $h(\beta) < \infty$.

Theorem 5.3. *Assume that $h, \xi : (\alpha, \beta) \to \mathbb{R}$ are continuous and strictly increasing functions such that $h < \xi$. If $h(\beta) < \infty$, then the regression condition* (4.5)*, i.e.,*

$$E\big(h(X_*^{(r+2)}) \mid X_*^{(r)} = x\big) = \xi(x),$$

uniquely determines the continuous F by the inversion formula (4.8)*.*

Proof. Recall that, by Lemma 2.2 (iv), we also have $\xi(\beta) = h(\beta)$. Let φ be any solution to integral equation (4.6) satisfying (4.7). Assume that there exists another solution y. Then, for $z = (y-\varphi)^2$, we have $0 \leq z(x) \leq (\xi(x) - h(x))^2$; thus, $\lim_{x\to\beta} z(x) = 0$. On the other hand, $z(\alpha) > 0$, and, according to Lemma 5.1 (iii), z is strictly increasing, a contradiction. The conclusion follows from Theorem 4.4. □

Unfortunately, we were not able to find the proof of a corresponding result in the case when h is not bounded, so that $h(\beta) = +\infty$, which is valid for any possible regression ξ. However, below, we prove the uniqueness of the characterization in the case when the regression ξ is asymptotically equivalent to a linear transformation of h, more precisely,

$$\lim_{x\to\beta} \big[\xi(x) - (ah(x) + b)\big] = 0. \tag{5.6}$$

Obviously, if this is the case, then necessarily, we have $a \geq 1$ and $b > 0$ since $\xi > h$ on (α, β). First, we need the following lemma.

Lemma 5.4. *Assume that $\xi, \Phi : (\alpha, \beta) \to \mathbb{R}$ are any continuous functions and ξ is strictly increasing with $\lim_{x\to\beta} \xi(x) = +\infty$. Define $g : (\alpha, \beta) \to \mathbb{R}$ by*

$$g(x) = g(\alpha) + \int_{\alpha}^{x} \Phi(t)\, \mathrm{d}\xi(t).$$

If $\lim_{x\to\beta} \Phi(x) = A \in (0, \infty)$, then $\lim_{x\to\beta} g(x)/\xi(x) = A$.

Proof. This is an easy application of Stolz's theorem. Let $\{x_n\}_{n\geq 1}$ be any strictly increasing sequence in (α, β) such that $\lim_{n\to\infty} x_n = \beta$. Then, for $n \geq 1$, we have

$$g(x_{n+1}) - g(x_n) = \int_{x_n}^{x_{n+1}} \Phi(t)\, \mathrm{d}\xi(t).$$

By the mean value theorem for Stieltjes integrals, for every $n \geq 1$, there exists a $\theta_n \in (x_n, x_{n+1})$ such that

$$\frac{g(x_{n+1}) - g(x_n)}{\xi(x_{n+1}) - \xi(x_n)} = \Phi(\theta_n).$$

Obviously, $\theta_n \to \beta$ as $n \to \infty$; thus, due to the assumptions on Φ, we have $\Phi(\theta_n) \to A$ as $n \to \infty$. By the classical Stolz lemma we obtain

$$\lim_{n\to\infty} \frac{g(x_n)}{\xi(x_n)} = A,$$

which completes the proof. □

Theorem 5.5. *Assume that $h(\beta) = +\infty$, and $\xi : (\alpha, \beta) \to \mathbb{R}$ is a continuous and strictly increasing function satisfying* (5.6) *for some $a \geq 1$. Assume that $\gamma \geq 1$ and the problem* (5.1) *has a solution φ such that*

$$\lim_{x\to\beta} \big[\varphi(x) - (ch(x) + d)\big] = 0 \tag{5.7}$$

for some $c \geq 1$ and $d \in (0, b)$.

(a) *If $a = 1$, then $c = 1$ and $d = b/(1+\gamma)$, and φ is the unique solution to* (5.1).

(b) *If $a > 1$, then $c \in (1, a)$ is the unique solution to the quadratic equation*

$$c^2 + a(\gamma - 1)c - a\gamma = 0 \tag{5.8}$$

in $(1, a)$, and φ is the unique solution to (5.1).

Proof.

(a) Obviously,

$$1 < \frac{\varphi(x)}{h(x)} < \frac{\xi(x)}{h(x)};$$

thus, if $a = 1$, then, trivially, $c = 1$, and by (5.6) and (5.7), we obtain

$$\lim_{x \to \beta} \frac{\varphi(x)}{\xi(x)} = 1$$

and

$$\lim_{x \to \infty} \big[\xi(x) - \varphi(x)\big] = b - d > 0.$$

Setting

$$\Phi(x) = \frac{\varphi(x) - h(x)}{\xi(x) - \varphi(x)}, \tag{5.9}$$

we have $\Phi(x) \to d/(b-d)$ as $x \to \beta$ and

$$\varphi(x) - \varphi(\alpha) = \gamma \int_\alpha^x \Phi(t)\, \mathrm{d}\xi(t).$$

Applying Lemma 5.4, we get

$$\lim_{x \to \beta} \frac{\varphi(x)}{\xi(x)} = \gamma \frac{d}{b-d}.$$

On the other hand, we know that this limit must be equal to 1; hence,

$$1 = \gamma \frac{d}{b-d} \tag{5.10}$$

and $d = b/(1+\gamma)$.

Now, if y is any other solution to (5.1) satisfying $h < y < \xi$, then

$$1 < \frac{y(x)}{h(x)} < \frac{\xi(x)}{h(x)};$$

thus, also, $\lim_{x\to\infty} y(x)/h(x) = 1$ and $\lim_{x\to\beta} y(x)/\xi(x) = 1$. If $\varphi < y < \xi$, then we consider $z = y - \varphi$. From Lemma 5.1 (iii), z increases from $z(\alpha) > 0$ to a constant $C \in (z(\alpha), b-d]$. Defining

$$\Psi = \frac{z + \varphi - h}{\xi - \varphi - z}, \tag{5.11}$$

and letting $x \to \infty$, we obtain

$$\Psi(x) \longrightarrow \frac{C+d}{b-d-C}.$$

Again applying Lemma 5.4 to

$$y(x) - y(\alpha) = \gamma \int_\alpha^x \Psi(t)\,\mathrm{d}\xi(t),$$

and taking into account that $y(x)/\xi(x) \to 1$ as $x \to \beta$, we obtain

$$1 = \gamma \frac{C+d}{b-d-C}.$$

Comparing this equality with (5.10), we obtain $C = 0$, a contradiction. If $h < y < \varphi$, it suffices to consider $z = \varphi - y$ to obtain another contradiction. Therefore, φ is the unique solution to (4.6).

(b) The proof for the case $a > 1$ follows the ideas of the proof of part (a); thus, here, we only sketch it. Note that

$$\lim_{x\to\beta} \frac{\xi(x)}{h(x)} = a \quad \text{and} \quad \lim_{x\to\beta} \frac{\varphi(x)}{h(x)} = c.$$

This implies

$$\lim_{x\to\beta} \frac{\varphi(x)}{\xi(x)} = \frac{c}{a}. \tag{5.12}$$

Rewriting Φ, defined by (5.9) in the form

$$\Phi(x) = \frac{\varphi(x)/h(x) - 1}{(\xi(x)/h(x)) - (\varphi(x)/h(x))},$$

and passing to the limit as $x \to \beta$, we obtain

$$\Phi(x) \longrightarrow \frac{c-1}{a-c}.$$

Once again, by Lemma 5.4, we obtain

$$\frac{\varphi(x)}{\xi(x)} \longrightarrow \gamma\frac{c-1}{a-c}$$

which, together with (5.12), leads to the quadratic equation (5.8). Since $\Delta = a^2(\gamma-1)^2 + 4a\gamma > 0$, this equation has two solutions of opposite signs. Define $f(c)$ as the left-hand side of (5.8). Since $a > 1$, then $f(1) = 1 - a < 0$ and $f(a) = a\gamma(a-1) > 0$; therefore, (5.8) has exactly one solution in $(1, a)$.

Now, if y is any other solution to (4.6), and, if the limit $L(y) = \lim_{x\to\beta} y(x)/h(x)$ exists, then similar reasoning using Lemma 5.4 shows that $L(y) \in (1, a)$ and

$$\frac{L(y)}{a} = \gamma\frac{L(y)-1}{a-L(y)}.$$

However, this is the equation defining a unique c; thus, $L(y) = c$ for any solution y to (4.6).

If $\varphi < y < \xi$, then, we consider $w = (y-\varphi)/(\xi-\varphi)$. Then, by Lemma 5.1 (v), w increases from $w(\alpha) > 0$ to $C \in (w(\alpha), 1]$. The function Ψ defined by (5.11) can be rewritten as

$$\Psi = \frac{(y-\varphi)/(\xi-\varphi) + \Phi}{1-(y-\varphi)/(\xi-\varphi)} = \frac{w+\Phi}{1-w}.$$

Therefore, the limit of Ψ at β exists and is equal to $[C + (c/\gamma a)]/(1-C)$. On the other hand,

$$y(x) = y(\alpha) + \gamma\int_\alpha^x \Psi(t)\,\mathrm{d}\xi(t);$$

thus, from Lemma 5.4, we obtain the existence of the limit

$$\lim_{x\to\beta}\frac{y(x)}{\xi(x)} = \gamma\frac{C + c/\gamma a}{1-C}.$$

Writing $y/\xi = y/h \cdot h/\xi$, we obtain the existence of the limit

$$\lim_{x\to\beta}\frac{y(x)}{h(x)},$$

which implies

$$\frac{c}{a} = \gamma\frac{C + c/\gamma a}{1-C}.$$

However, this is only possible if $C = 0$, a contradiction. If $h < y < \varphi$, then it suffices to apply analogous considerations to $w = (\varphi - y)/(\varphi - h)$ to arrive at another contradiction. □

Remark 5.6. Note that the assumption $\gamma \geq 1$ is used only in part (b) of the above theorem. We want to apply it with $\gamma = \gamma_{r+2}/\gamma_{r+1}$; however, according to Remark 2.1, in general, we need not assume that $\gamma_{r+1} \leq \gamma_{r+2}$.

Corollary 5.7. *Assume that $h, \xi : (\alpha, \beta) \to \mathbb{R}$ are continuous and strictly increasing functions such that $h < \xi$, $h(\beta) = \infty$ and ξ satisfies* (5.6) *for some $a \geq 1$. If the problem* (5.1) *has a solution φ which satisfies* (5.7)*, then the regression condition* (4.5)*, i.e.,*

$$E\big(h(X_*^{(r+2)}) \mid X_*^{(r)} = x\big) = \xi(x),$$

uniquely determines continuous F by the inversion formula (4.8).

6. New characterizations of particular distributions. In this section, we give new characterizations of particular distributions by the single regression (1.3) of record values in the case when $\ell = 2$, and, for simplicity, $h(x) = x$, i.e., by the condition

$$\xi(x) = E\big(R_{n+2} \mid R_n = x\big).$$

We know that record values are GOSs with $\gamma_i = 1$; hence, using (2.2) (see also (4.1) with $\gamma = 1$) we have a corresponding regression of adjacent record values

$$\varphi(x) = E\big(R_{n+2} \mid R_{n+1} = x\big) = \frac{1}{\overline{F}(x)} \int_x^\infty t \, \mathrm{d}F(t). \tag{6.1}$$

Utilizing Lemma 3.2 with $\ell = 2$, we obtain

$$\xi(x) = E\big(\varphi(R_{n+1}) \mid R_n = x\big) = \frac{1}{\overline{F}(x)} \int_x^\infty \varphi(t) \, \mathrm{d}F(t). \tag{6.2}$$

In the first example, we show that, when F is a continuous distribution with density which is not continuous, then the regression ξ need not be differentiable at some points.

Example 6.1. Consider a mixture of uniform distributions with density and distribution function, respectively, given by

$$f(x) = \frac{1}{3} I_{[0,1)}(x) + \frac{2}{3} I_{[1,2]}(x),$$

and

$$F(x) = \begin{cases} x/3 & \text{for } x \in [0,1), \\ (2x-1)/3 & \text{for } x \in [1,2]. \end{cases} \tag{6.3}$$

Using (6.1), and after some elementary calculations, we obtain

$$\varphi(x) = \begin{cases} (x^2 - 7)/(2(x-3)) & \text{for } x \in [0,1), \\ (x/2) + 1 & \text{for } x \in [1,2]. \end{cases}$$

Then, by (6.2), a simple computation leads to

$$\xi(x) = \begin{cases} (x^2 + 6x - 21 + 4\ln((3-x)/2))/(4(x-3)) & \text{for } x \in [0,1), \\ (x+6)/4 & \text{for } x \in [1,2]. \end{cases}$$

It is easy to verify that ξ is continuous, but not differentiable, at $x = 1$. However, $\beta = 2$; thus, $h(\beta) < \infty$, and the uniqueness of the characterization of the continuous distribution (6.3) by this regression of R_{n+2} on R_n immediately follows from Theorem 5.3.

In the next example, we give a new characterization of a particular gamma distribution by the regression of record values. Recall that gamma distribution $\Gamma(k,\theta)$ on $(0,\infty)$, with parameters $k, \theta > 0$, is defined by the density function

$$f(x) = \frac{\theta^k}{\Gamma(k)} x^{k-1} \mathrm{e}^{-\theta x}, \quad x \geq 0.$$

Example 6.2. Consider gamma $\Gamma(2,1)$ distribution with $f(x) = x\mathrm{e}^{-x}$, and $\overline{F}(x) = (1+x)\mathrm{e}^{-x}$ for $x > 0$. We also use the following notation:

$$E(x) = \frac{\mathrm{e}^x}{x} \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\, \mathrm{d}t, \quad x > 0.$$

Elementary computations show that

$$E'(x) = \left(1 - \frac{1}{x}\right)E(x) - \frac{1}{x^2},$$

and $\lim_{x\to\infty} E(x) = \lim_{x\to\infty} E'(x) = 0$. A series of elementary calculations shows that $\varphi(x) = x + 1 + 1/(x+1)$, and therefore, if $F \sim \Gamma(2,1)$, then

$$\xi(x) = x + 2 + \frac{2}{x+1} - E(x+1), \quad x > 0.$$

Now, we prove that this regression determines the gamma distribution $\Gamma(2,1)$ uniquely. Note that we cannot use Theorem 5.3 since $h(\beta) = +\infty$. In addition, computations show that $\varphi(x) = x + 1 + 1/(x+1)$ is a solution to

$$\varphi'(x) = \frac{\varphi(x) - x}{\xi(x) - \varphi(x)}\, \xi'(x) \tag{6.4}$$

such that $x < \varphi(x) < \xi(x)$ for $x > 0$. On the other hand, we have

$$\lim_{x\to\infty} [\xi(x) - (x+2)] = 0, \qquad \lim_{x\to\infty} [\varphi(x) - (x+1)] = 0.$$

Therefore, by Theorem 5.5 (a), we see that φ is the unique solution to (6.4). From Corollary 5.7, we see that $\Gamma(2,1)$ is the only distribution for which

$$E\big(R_{n+2} \mid R_n = x\big) = x + 2 + \frac{2}{x+1} - E(x+1), \quad x > 0.$$

In the next example, we extend the conclusions of the previous one and show that each gamma distribution $\Gamma(k,\theta)$ is uniquely determined by the corresponding regression

$$\xi(x) = E\big(X_*^{(r+2)} \mid X_*^{(r)} = x\big)$$

of GOSs. However, in contrast to the previous example, due to the complicated computations involved, we do not determine an explicit form for ξ and $\varphi(x) = E(X_*^{(r+2)} \mid X_*^{(r+1)} = x)$.

Example 6.3. Recall that, for any distribution function F with density f, we define the failure rate of F as

$$\lambda_F(x) = \frac{f(x)}{\overline{F}(x)}, \quad x \in (\alpha, \beta). \tag{6.5}$$

For $\Gamma(k, \theta)$, it is elementary to prove that, for any $k > 0$, we have

$$\lim_{x\to\infty} \lambda_F(x) = \theta.$$

Applying the arguments from the proof of [**4,** Lemma 5], we see that

$$\lim_{x\to\infty} \big(\varphi(x) - x\big) = \frac{1}{\gamma_{r+2}} \lim_{x\to\infty} \frac{1}{\lambda_F(x)} = \frac{1}{b\gamma_{r+2}},$$

and similarly,

$$\lim_{x\to\infty} \big(\xi(x) - x\big) = \left(\frac{1}{\gamma_{r+1}} + \frac{1}{\gamma_{r+2}}\right) \lim_{x\to\infty} \frac{1}{\lambda_F(x)} = \frac{1}{b}\left(\frac{1}{\gamma_{r+1}} + \frac{1}{\gamma_{r+2}}\right).$$

Therefore, for any gamma distribution, we see that the corresponding regressions ξ and φ satisfy (5.6) and (5.7) with $a = c = 1$. Moreover, by the proof of Theorem 4.4, we know that φ is a solution to problem (5.1). From Corollary 5.7, we infer that this is the unique solution to (5.1), so the gamma distribution $\Gamma(a, b)$ is uniquely determined by the corresponding regression ξ.

In the next example, we identify the unique distribution on the positive half-axis determined by the specific regression of record values satisfying condition (5.6) of asymptotic linearity with $a = 4$ and $b = 3$.

Example 6.4. Set $D = x^2 + 2x + 2$, and let F be the distribution with density function $f(x) = 2\sqrt{2}(1 + 4x + 2x^2)D^{-5/2}$, for $x > 0$, and $\overline{F}(x) = 2\sqrt{2}(1 + x)D^{-3/2}$.

Then, putting $\gamma_{r+1} = \gamma_{r+2} = 1$, we obtain

$$\varphi(x) = E\big(R_{n+2} \mid R_{n+1} = x\big) = 2x + 1 + \frac{1}{x+1}$$

and

$$\begin{aligned} \xi(x) &= E\big(R_{n+2} \mid R_n = x\big) \\ &= \frac{1}{x+1}\left(8 + 9x + 5x^2 + D^{\frac{3}{2}} \ln\left(\frac{1+x}{1+\sqrt{D}}\right)\right). \end{aligned} \tag{6.6}$$

Then, $\lim_{x\to\infty}[\xi(x) - (4x+3)] = 0$ and $\lim_{x\to\infty}[\varphi(x) - (2x+1)] = 0$; thus, by Corollary 5.7, we see that F is the only distribution for which (6.6) holds.

In the last example, we show that, in general, the regression ξ need not be an asymptotically linear function of h, so, in general, the problem of uniqueness of the characterization by (4.5) is still open.

Example 6.5. Consider the distribution F such that

$$\overline{F}(x) = \frac{\mathrm{e}}{x(\log x)^{1+a}}, \quad x \geq \mathrm{e}.$$

In this case, from the results of Nagaraja [**18**], we have $E(R_n) < \infty$ if and only if $n < a + 1$. Therefore, $E(R_3) < \infty$ if and only if $a > 2$. Then, for $h(x) = x$, we have

$$\varphi(x) = E\big(R_3 \mid R_2 = x\big) = x\left(1 + \frac{1}{a}\log x\right)$$

and

$$\xi(x) = E\big(R_3 \mid R_1 = x\big) = x\left(1 + \frac{1+2a}{a^2}\log x + \frac{1}{a(a-1)}(\log x)^2\right).$$

Therefore, ξ does not satisfy (5.6); hence, we cannot apply Corollary 5.7 to claim that this regression characterizes F.

7. Summary and discussion of the results. We present an entirely new approach to the classic problem of characterization of continuous probability distributions by the regression functions of ordered statistical data. This approach allows for a reformulation of the problem of the uniqueness of the characterization as the problem of the uniqueness of the solution to an integral equation or a system of integral equations with a non-classical "initial" condition. Although our results are proven for a general model of generalized order statistics, they are new even in the most important special cases of order statistics and record values. As a result of the new approach, we obtain new characterizations of distributions, and we prove that gamma distributions are uniquely characterized by the corresponding regression of GOSs.

The main results of the paper are illustrated with examples which show that our considerations are necessary due to the possible lack of differentiability of the regression functions (Example 6.1), but the problem needs further study (Example 6.5). Examples 6.2, 6.3 and 6.4 show new results that were not possible to obtain without the results derived in this paper, especially Theorem 5.5 and Corollary 5.7.

Now, we discuss the significance of our results from the point of view of statistics. We prove new characterizations of distributions by the regression of non-adjacent GOSs with ξ, other than the linear one. We stress that, in this paper, no assumptions are imposed on either the underlying F (except for the continuity) or on the parameter vector of GOSs (except for the obvious integrability conditions), while in [**4**], a quite restrictive assumption was imposed that F has a continuous density (or, equivalently, that the regression ξ being considered has continuous first derivative). Also, in [**4**], it was shown that the condition $\lambda_F(x) \to +\infty$ as $x \to \beta$, where λ_F is the failure rate defined by (6.5), is sufficient for the uniqueness of the characterization, and Example 6.3 shows that the condition is unnecessary.

We also underline the fact that we have presented a new technique of the proof of the uniqueness of the characterization without knowledge of the explicit form of ξ. This technique requires Theorem 5.5, which cannot be proven in the present form without the use of integral equations. It is possible to prove it using only differential equations, but with the additional assumptions on the existence of the limits of the derivatives ξ' and φ' at β.

On the other hand, the practical applicability of our results is somewhat limited, as is explained in [**4,** Section 5]. Any potential application would require a large amount of observed data, and it would involve many numerical computations. Therefore, it would only give approximations of the characterized distributions.

Finally, we discuss the relevance of our paper to the theory of integral equations. We show an unexpected application of integral equations in the area of statistical distribution theory since we have translated the problem in probability and statistics to the language of integral equations. In our opinion, its complete solution demands additional insight from integral equations rather than from the point of view of statistics. Also, in Section 5, we provide an exemplary analysis of the properties of the solutions to a non-linear integral equation with non-classical "initial" conditions. This type of analysis may appear to be of independent interest to the integral equations community. Moreover, since our main results are quite difficult to implement, there are some open problems remaining to be solved. For instance:

(1) Is it possible to extend Theorem 5.3 for $\ell \geq 3$?
(2) Does this (1) also apply in the case when $h(\beta) = +\infty$ for $\ell \geq 2$?
(3) For given $r, \ell \geq 1$, the parameter models $\gamma_1, \ldots, \gamma_{r+\ell}$, and the function h determines necessary and sufficient conditions for any function ξ to be a possible regression function of GOSs.
(4) Assume that $\ell \geq 3$ and $\xi(x) = ax + b$ for some $a > 0$. From [**7,** Theorem 5.1] and our Theorem 4.6, we see that system (4.13) with the condition (4.14) has the unique solution $\varphi = (\varphi_1, \ldots, \varphi_{\ell-1})$. It is elementary to prove that $\varphi_i(x) = c_i x + d_i$ for some appropriately chosen c_i, d_i, $1 \leq i \leq \ell - 1$. The open problem is to prove the uniqueness of φ directly, as this would give an independent proof of [**7,** Theorem 5.1].

All of these problems most likely require a deeper knowledge of integral equations than the average statistician has. Ordinary differential equations are too weak of a tool to tackle such problems since they are insufficient, even for absolutely continuous distributions, and for continuous distributions, integral equations appear to be an appropriate tool.

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