JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 30, Number 1, Spring 2018

ON THE CONTRIBUTIONS OF W. EDWARD OLMSTEAD

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ABSTRACT. With this article, we wish to honor the many contributions of our mentor, colleague and dear friend, Professor W. Edward Olmstead, on the occasion of his retirement from Northwestern University. Ed has spent over five decades at Northwestern University, first as a graduate student and then as a member of the faculty. During this time he completed his PhD, played a key role in the formation of the Department of Engineering Science and Applied Mathematics (ESAM), developed several courses in applied mathematics, participated in the education of numerous students, and made vast and important contributions in the field of applied mathematics. Here, we give a (mostly chronological) account of some of Ed's major research interests and contributions, primarily in the field of Integral Equations.

1. Introduction. W. Edward Olmstead's interests in applied mathematics have primarily centered on problems at the intersection of physics and engineering, as well as the mathematics needed to solve these problems. His work has included extensive formulation and analysis of integral and integrodifferential equations with applications to a wide range of topics including various areas of fluid and solid mechanics and numerous thermal problems associated with radiation, combustion and anomalous diffusion phenomena. In the following sections, we will expound on some of Olmstead's major contributions. This is by no means an exhaustive account of his work in applied mathematics. Our focus will be on his work in integral equations where he has developed both novel formulations that have elucidated phenomena such as blow-up behavior in a variety of physical applications, as well as new analytical techniques that provide for a deeper understanding of the properties of solutions to integral equations.

2. A history of research in integral equations and their applications. Olmstead first became interested in integral equations in the early 1960s during his doctoral research at Northwestern University.

DOI:10.1216/JIE-2018-30-1-3

This early work focused on a gas-liquid interaction problem. Specifically, he derived a mathematical model that describes the observed shape of a liquid surface as it is being depressed by an impinging gas jet. Through the use of conformal mapping and finite Hilbert transforms, he formulated a model that gives rise to a nonlinear Fredholm integral equation with a singular kernel. By applying asymptotic analyses and obtaining numerical solutions of the integral equation, the profile of the free surface of the liquid as well as some qualitative characteristics were recovered [**36**].

2.1. Early career. After completing his PhD in 1963, Olmstead took a postdoctoral fellowship at Johns Hopkins University where he studied the Oseen linearization of the Navier-Stokes equations. In the study of the steady flow of a viscous incompressible fluid past a half-plane, he found that various configurations could be formulated as linear Fredholm equations amenable to solution by the Weiner-Hopf method [**24, 29**]. In 1964, he returned to Northwestern University, taking a faculty position and continuing to study the integral equations arising from various Oseen flow problems. Olmstead and his first PhD student, David Hector, found that the Fredholm equation that governs flow past a half-plane does not have a unique solution [**34**]. This nonuniqueness of the solution was very much an unexpected and noteworthy result. In [**34**], it was shown that the non-trivial solution of the homogeneous problem described a pattern of circulation around the half-plane. Later, Olmstead was able to find an exact solution to the flow field in [**25**].

Olmstead continued to study linearized viscous flows into the late 1960s and obtained additional significant results with his PhD student, Arthur Gautesen. In 1968, they revealed a new paradox in viscous hydrodynamics [**30**] by converting the problem of Oseen flow past an obstacle of arbitrary shape to the solution of a Fredholm integral equation in vector form. From this formulation, they were able to prove that the drag force on an obstacle was invariant under flow direction reversal in the far field independent of the shape or symmetry of the obstacle. Given that the flow field itself is not reversible, the paradoxical invariance of the drag force was a remarkable finding. In another notable collaboration with Gautesen in 1971, their research resulted in the solution of the Fredholm equation governing the potential of two in-line strips [**5**]. Their result corrected an earlier, erroneous solution by F.G. Tricomi, a luminary in the field of integral equations.

Olmstead was a Visiting Member of the Courant Institute at NYU in 1967–68 where he collaborated with Professor Joseph Keller. Together, they investigated a nonlinear Volterra equation governing the temperature at the leading edge of a semi-infinite rod subject to heat loss by means of thermal radiation. While the radiation problem had been under consideration for several decades, Olmstead and Keller obtained more detailed properties of the solution [**9**]. They proved existence and uniqueness and also obtained asymptotic behavior of the temperature in various limits. This early work is extensively cited in the literature (with in excess of seventy citations continuing well into the 2000s) and was a catalyst for some of Olmstead's later work in the development of asymptotic techniques for analyzing integral equations.

In a pair of papers [**7, 33**] in the early 1970s, Richard Handelsman and Olmstead analyzed the asymptotic behavior of solutions to a large family of nonlinear Volterra equations of the form

$$\phi(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(s) - \phi^n(s)}{\sqrt{t-s}}\, ds. \tag{2.1}$$

They developed a general asymptotic method that utilizes properties of Mellin transforms. Using this approach, they provided a thorough cataloging of both the small and long time behavior of $\phi(t)$. It was also observed that, depending upon the nature of the data function $f(t)$, the useful asymptotic representation may break down at $t \to 0$, giving rise to the study of singularly perturbed equations of this structure [**32**].

2.2. The 1970s and beyond: Singular equations, bifurcation and blow-up. From the mid 1970s through the early 1980s, Olmstead worked on a variety of related problems, including bifurcation problems for nonlinear differential equations. He and his PhD student, David Mescheloff, converted the differential equations to equivalent Fredholm equations of the second kind. Using this approach, they analyzed the buckled states of a nonlinearly elastic rod [**35**]. Olmstead also considered some control problems with thermal applications that give rise to Volterra equations. These problems involved optimal control criteria [**44**] as well as boundary controllability [**6, 26**].

As noted, some of Olmstead's earlier work had involved the appearance of singularly perturbed equations. In the mid 1980s, Olmstead and his PhD student, Jeffrey Angell, provided important contributions in the area of singularly perturbed integral equations of both Volterra and Fredholm type. They developed analytical techniques for the study of such problems. While there had been an extensive history of progress on singularly perturbed differential equations, until the work of Olmstead and Angell, there was very little literature on singularly perturbed integral equations.

They considered problems in which an equation of the second kind degenerates into an equation of the first kind in the limit as the perturbation parameter vanishes. Typically, this singular limit is associated with a boundary layer effect near the origin in the case of a Volterra equation or near one or both endpoints for a Fredholm equation. The method of matched asymptotics (MMA), used to obtain inner and outer solutions for singularly perturbed differential equations, relies on the local nature of the differential equation description. By contrast, the integral equation formulation is necessarily nonlocal, thereby rendering the MMA approach inapplicable. To deal with this complexity, they derived appropriate *inner* and *outer* integral equations. They then developed a two-timing approach, alternately solving the integral equations in the outer and inner variables to ultimately produce a uniformly valid asymptotic solution [**1, 2, 31**]. The use of and reference to the techniques they developed have continued in the literature in relation to various applications, numerical methods for integral equations and extensions to broader classes of equations.

Also in the 1980s, Olmstead developed an interest in systems exhibiting thermal runaway, and many of Olmstead's contributions up to the present involve the analysis of blow-up solutions to nonlinear Volterra equations. Of particular note is the 1983 paper [**27**], in which he modeled the problem of surface temperature of a region subject to a combustible reaction as a nonlinear Volterra integral equation whose solution becomes unbounded in finite time. Several interesting questions arise in the study of blow-up solutions. One of these is whether the blow-up time can be predicted, at least within some bounding window. Olmstead developed a two step process to derive analytic bounds on the blow-up time. First, he demonstrated how to obtain a lower bound, say t^*, on the time at which blow-up may occur by establishing the existence of a unique, non-negative, continuous solution valid for all time t satisfying $0 < t < t^* < \infty$. Next, an upper bound $t^{**} < \infty$ on the blow-up time is determined by proving, via a contradiction argument, that the solution cannot exist for $t > t^{**}$.

In several studies spanning the 1980s, Olmstead alone and together with his PhD student, Glenn Lasseigne, studied the ignition of a combustible solid under various conditions, including the influence of convection heating **[18]**, material stimulated by an arbitrary energy flux at one end **[27]**, and reactant consumption **[19]** among others (see **[20, 21, 17]**). Using large activation energy asymptotics, nonlinear integral equations arise that describe the temperature of a combustible material near the ignition state. The integral equations from such problems generally take the form

$$u(t) = \int_{t_0}^{t} k(t-s)F[u(s), s]\, ds, \quad t \geq t_0, \tag{2.2}$$

where $u(t)$ is the temperature at a location within the material that experiences the maximum thermal impact. The kernel $k(t)$ reflects the diffusive ability of the material, typically $k(t) \geq 0$, $k'(t) \leq 0$. The nonlinear function $F[u, t]$ represents the high energy input arising from the reactive nature of the material, typically $F > 0$, $F_u > 0$, $F_{uu} \geq 0$, $F_t > 0$. The solution of (2.2) will experience a blow-up in finite time if the diffusive ability of the material is unable to keep pace with the high energy input. Mathematically, there is a competition between the kernel and the nonlinearity that determines whether or not the integral in (2.2) becomes unbounded in finite time.

Blow-up solutions here are indicative of ignition, which may or may not occur. Thus, determining conditions under which the solution becomes unbounded in finite time, or alternatively remains bounded for all time, is of particular interest. Olmstead and Lasseigne took a novel analytical approach to determining whether ignition would occur and to characterizing the critical ignition time as it depends on various thermal parameters. Prior research in this area had generally been restricted to determining an ignition time value numerically.

This work was extended in the early 2000s into another type of application when Olmstead collaborated with colleague, Vladimir Volpert, and a joint PhD student, Lake Ritter, on a polymerization process in which a monomer is converted into a polymer by means of a self-propagating thermal reaction wave. Although the activation energies are smaller than those arising in the combustion problem, the kinetics of the polymerization problem are similar to the combustion problem but with two reactants. A family of integral equations of the type (2.2) are found to govern the temperature of the reacting mixture at the potential initiation site. This family, presented in [**46**] extended the family of equations derived in [**19**] (contained the latter as a subfamily). Blow-up in this context is interpreted as initiation of a propagating polymerization wave. As in the combustion problem, conditions for initiation to occur as well as characterization of the initiation time were made possible in large part due to the analytic techniques Olmstead and Handelsman had developed in the 1970s.

2.3. The 1990s into the 21st century: Novel formulations in combustion, anomalous diffusion, and materials science. In the early 1990s, Olmstead began a highly productive and long lasting collaboration with his then PhD student, Catherine Roberts. Together, they began to examine some fundamental aspects of blow-up problems formulated as nonlinear Volterra equations in the form of (2.2). In an early study of explosion, they revealed competition between two key characteristics of a system: the diffusive ability of the medium involved, and the strength of an applied nonlinear energy source. In [**47**], they showed that, if the diffusive ability is sufficiently high, there will be no blow-up, and the integral equation will have a global solution. On the other hand, if the material is unable to diffuse enough energy away from the nonlinear source, a blow-up will occur within a finite period of time.

Of note in this work is the introduction of the Dirac delta function into a related nonlinear parabolic partial differential equation (pde) model of the form

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \delta(x - x_0)G[u], \quad 0 \le x,\ x_0 \le l,\ t > 0, \tag{2.3}$$

subject to certain initial and boundary conditions. The Dirac delta term serves to intensify and highly localize the nonlinearity, providing a general model of a highly localized nonlinear heat source applied to a reactive-diffusive medium. This modeling approach was quite novel. Previously, results related to parabolic pdes with nonlinear source terms required the nonlinearity not to depend explicitly on space or to have very smooth spatial dependence. Analyzing the problem in this new way clearly reveals how the solution changes character depending on various parameters in the original problem, such as domain size, characteristics of the material, and placement of the source term.

Olmstead and Roberts continued to investigate problems involving nonlinear Volterra equations subject to blow-up. In their 1994 work [**38**], they considered a finite strip of diffusive material with an applied nonlinear energy source concentrated at a single point using the delta function formulation. They demonstrated that blow-up depends on proximity of the source to a cold edge of the strip. In further study, they examined integral equations with kernels of fractional type and with exponential or power law nonlinearities. For various general cases, they derived the asymptotic growth rate of the solution near blow-up [**48**]. In [**39**], they extended their consideration to include in the source term nonlocal properties of the temperature, and, in [**43**], they collaborated with Keng Deng on a problem involving blow-up in coupled Volterra equations.

Olmstead and Roberts also adapted their research methodology for determining blow-up solutions and applied it to quenching type problems [**37, 49**]. The phenomenon of quenching occurs when the solution of the problem remains bounded while progressing toward a finite quenching value as the first order time derivative of the solution becomes unbounded in finite time.

In the late 1990s and into the 2000s, Olmstead became interested in thermal models involving a moving energy source, i.e., those for which the position of the source is time dependent. This modification of the basic mathematical model permits the consideration of a much wider variety of applications. Moving the concentrated heat source within the medium, for example, models the behavior in certain laser and manufacturing applications [**22, 23**]. In [**28**], Olmstead demonstrated that, if the speed of the heat source exceeds a certain critical value, no blow-up can occur. This is due to the fact that the source continually moves into cooler surroundings and cannot generate sufficient heating to produce a blow-up. Olmstead and his PhD student, Colleen Kirk, analyzed variations of this problem that further considered the influence of moving heat sources in [**10**], [**11**], and [**12**]. Moving sources were incorporated into quenching problems with Roberts in [**40**].

Another area of Olmstead's research during this time involved shear band formation. High shear stress, as might be seen during ballistic impact, can result in localized bands of plastic deformation in a high-strength metal. Together with PhD student James DiLellio, he proposed a model in which the deformation is induced by an inhomogeneity in the thermal flux. They describe the temperature in the shear band as the solution of a nonlinear Volterra equation [**3**]. The steep increase in temperature, as observed during the formation of a shear band, is signaled by a blow-up solution. Using boundary layer analysis, they characterized the temporal evolution of the thickness of a shear band in [**4**].

Moving further into the 2000s, Olmstead continued his collaborations with former students, Roberts and Kirk. In 2006, he became interested in anomalous diffusion. Subdiffusion, for example, is a type of anomalous diffusion in which the diffusion of heat is significantly retarded. Subdiffusive materials exhibit thermal transport behavior in which the mean square displacement associated with Brownian diffusion evolves on a slower-than-normal time scale. In the mathematical model, the classical heat operator of (2.3) is replaced by one that includes a fractional derivative in time. In [**41**], Olmstead and Roberts considered thermal blow-up in a medium that exhibits subdiffusive properties. They introduced a localized energy source into these fractional diffusion equations and showed that the temperature at the site of the source could be formulated as a nonlinear Volterra equation in the form of (2.2). They showed that the approach used to explore blow-up behavior in classical diffusion could be adapted to anomalous diffusion. Olmstead, Roberts and Kirk investigated modified versions of the subdiffusion problem in a sequence of papers [**50, 45, 15**].

Further investigations also considered blow-up phenomena involving materials exhibiting superdiffusive properties [**13, 42**]. In superdiffusive materials, the capability of thermal energy transport is dramatically enhanced, which is mathematically modeled by replacing the classical heat operator of (2.3) with a fractional derivative in space. On a molecular scale, a mean square displacement law associated with Levy flights governs the superdiffusion. Physical processes exhibiting superdiffusion include turbulent diffusion, slip diffusion on a solid surface, and diffusion in porous glasses.

In other recent work, Olmstead has considered systems of Volterra equations that can exhibit blow-up behavior. Joint work with Kirk and Roberts in [**14**] extended the results for the system in [**43**] to include a more general class of kernels. In [**8**], Olmstead and Kirk collaborated with Mokhtar Kirane and Abdelouahab Kadem on a system of initial value problems for fractional differential equations with exponential nonlinearities that could be analyzed as Volterra equations. Olmstead continues to be active in research. To this day, Olmstead and his collaborators continue to study a wide variety of problems related to anomalous diffusion and blow-up phenomena.

3. A legacy of teaching and mentorship. One can hardly consider Ed Olmstead's legacy over the past half century without including ESAM (the department of Engineering Science and Applied Mathematics) itself as part of his lasting imprint. In the early 1970s, still early in his own career, Ed chaired the committee for the graduate program in Applied Mathematics that existed as part of Northwestern's Technological Institute. He served as the coordinator of the department's Applied Mathematics Program during the 1975–76 year just preceding the inception of the ESAM name. At the time, Applied Mathematics was one of three programs that constituted the ESAM department, and Ed was one of its founding applied mathematics faculty. Together with Professor Bernard Matkowsky, who joined the young department in 1977, Ed worked to expand the course offerings in applied mathematics. As each program grew, and the other programs shifted to other or independent departments, the Applied Mathematics Program became the sole focus of ESAM. Ed has served with distinction from its inception until his retirement. He chaired the department from 1991 through 1993. His prowess in the classroom was recognized with his appointment as the Charles Deering McCormick Professor of Teaching Excellence from 1994 to 1997. When the history of the department was chronicled in 1999, Professor Emeritus Raymond Kliphardt recounted [**16**] of Ed that "[h]is reputation for superior instruction is well known." As an instructor and mentor, Ed has participated in the development of numerous undergraduate and graduate students, including 16 doctoral descendants listed in Table 1.

As our friend and mentor begins this new chapter of his life, we wish him all the best. The opportunity we have had to learn from and to work with him has been invaluable. His contributions in the area of integral equations and the phenomena they model have been prodigious and important. Likewise, his impact on the careers and

TABLE 1. W.E. Olmstead's doctoral students with year.

Student	Year	Student	Year
D.L. Hector	1966	E. Ammicht	1978
R.G. Burman	1967	David Lasseigne	1985
J.P. Dugan	1968	Jeffrey Angell	1986
A.K. Gautesen	1969	Catherine Roberts	1992
J.E. Hartka	1971	James DiLellio	1997
David Mescheloff	1973	Robert Flemming	1998
D.W. Kucera	1977	Colleen Kirk	1999
V.D. Panico	1978	Lake Ritter	2003

lives of his students and colleagues has been significant. Both he and his work will be long remembered and appreciated.

Acknowledgments. We would like to thank Ed for his helpful insight and perspective as we prepared this article.

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