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BLOW UP OF FRACTIONAL REACTION-DIFFUSION SYSTEMS WITH AND WITHOUT CONVECTION TERMS

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ABSTRACT. Based on the study of blow up of a particular system of ordinary differential equations, we give a sufficient condition for blow up of positive mild solutions to the Cauchy problem of a fractional reaction-diffusion system, and, by a comparison between the transition densities of the semigroups generated by $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ and $\Delta_\alpha + b(x)\cdot\nabla$ for $1 < \alpha < 2$, $d \geq 1$ and b in the Kato class on $\mathbb{R}^d$, we prove that this condition is also sufficient for the blow up of a fractional diffusion-convection-reaction system.

1. Introduction. Let d be a positive integer, $\beta_i > 1$, $\alpha_i \in (1,2)$ and

$$b_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

a function in the Kato class $\mathcal{K}_d^{\alpha_i - 1}$ on $\mathbb{R}^d$ (see Bogdan and Jakubowski [**3,** page 185]), $i = 1, 2$. In this paper, we study blow up in finite time of positive mild solutions to the Cauchy problem for the next fractional reaction-diffusion system with convection terms

(1.1)
$$\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= \left(\Delta_{\alpha_1} + b_1(x)\cdot\nabla\right) u_1(t,x) + u_2^{\beta_1}(t,x), \quad t>0,\ x\in\mathbb{R}^d,\\
\frac{\partial u_2(t,x)}{\partial t} &= \left(\Delta_{\alpha_2} + b_2(x)\cdot\nabla\right) u_2(t,x) + u_1^{\beta_2}(t,x), \quad t>0,\ x\in\mathbb{R}^d,\\
u_i(0,x) &= f_i(x), \qquad x\in\mathbb{R}^d,\ i=1,2,
\end{aligned}$$

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where

$$\Delta_{\alpha_i} := -(-\Delta)^{\alpha_i/2}, \quad 1 < \alpha_i < 2,$$

denotes the fractional power of the Laplacian, f_i is a nonnegative, not identically zero, bounded continuous function, $i = 1, 2$ and ∇ is the gradient operator, i.e.,

$$b_i(x) \cdot \nabla g(x) = \sum_{j=1}^{d} b_j^i(x) \frac{\partial g}{\partial x_j}(x),$$
$$x = (x_1, \ldots, x_d), \quad b_i \equiv \left(b_1^i, \ldots, b_d^i\right),$$

$i = 1, 2$, for any differentiable function g on $\mathbb{R}^d$.

Let $p_i(t, x, y)$ be the transition density of the semigroup generated by Δ_{α_i}, $i = 1, 2$. It is known (see Bogdan and Jakubowski [**3,** Theorems 1, 2]) that the semigroup generated by $\Delta_{\alpha_i} + b_i(x) \cdot \nabla$ has a continuous transition density $p_i^{b_i}(t, x, y)$ such that, for every $0 < T < \infty$, there is a $C_i = C_i(d, \alpha_i, b_i, T) > 1$ that satisfies

$$C_i^{-1} p_i(t, x, y) \le p_i^{b_i}(t, x, y) \le C_i p_i(t, x, y), \quad 0 < t \le T, \quad x, y \in \mathbb{R}^d, \tag{1.2}$$

and $C_i \to 1$ as $T \to 0$.

The associated integral system to (1.1) is given by

$$u_i(t, x) = \int_{\mathbb{R}^d} p_i^{b_i}(t, x, y) f_i(y)\, dy + \int_0^t \int_{\mathbb{R}^d} p_i^{b_i}(t - s, x, y) u_{i'}^{\beta_i}(s, y)\, dy\, ds, \tag{1.3}$$

$t \ge 0$, $x \in \mathbb{R}^d$, where $i \in \{1, 2\}$ and $i' = 3 - i$. A solution of integral system (1.3) is called a *mild solution* of (1.1). In this paper, solutions of (1.1) should be understood in this mild sense. If there exists a solution (u_1, u_2) of (1.3) defined in $[0, \infty) \times \mathbb{R}^d$, we say that (u_1, u_2) is a global solution, and, when there exists a number $T_b < \infty$ such that (1.3) has an unbounded solution in $[0, t] \times \mathbb{R}^d$ for every $t > T_b$, we say that (u_1, u_2) blows up in finite time.

The study of systems like (1.1) arise in several fields, such as heat conduction, chemical reaction processes, combustion theory, physics and engineering, see Beberbes and Eberly [**2**] and Samarskii, et al., [**23**]. Generators of the form Δ_{α_i} perturbed by gradient operators are used in models of anomalous growth of certain fractal interfaces and in hydrodynamic models with modified diffusivity, see, for example, Bardos, et al., [**1**] and Mann, Jr., and Woyczynski [**16**].

For a single equation, with $\alpha = 2$ and without a convection term, in his pioneering work, Fujita [**6**] showed the influence of spatial dimension on the finite time blow up versus global existence of solutions. Also see [**12, 13, 24, 21**] for cases with $0 < \alpha \leq 2$. Within the framework of a fractional diffusion equation, this spatial dimensional influence for the thermal blow up in a superdiffusive medium with a localized energy source was also shown in Olmstead and Roberts [**18**]. Introducing convection through a linear transport term that is proportional to the convection speed, under a one-dimensional domain of infinite extent and a nonlinear source term $g(u)$ satisfying $g(u) > 0$, $g'(u) > 0$, $g''(u) > 0$, $u \geq 0$ and

$$\int_0^\infty \frac{du}{g(u)} < \infty,$$

Kirk and Olmstead [**11**] have shown that there exists a critical convection speed above which blow up is avoided and below which blow up is guaranteed. For the case $\alpha = 2$, Tersenov [**25**] showed that, for the problem with Dirichlet boundary condition on a domain $\Omega \subset \mathbb{R}^d$, that lies in a strip, e.g., $|x_1| \leq l_1$, a large enough coefficient b can bring a sufficient cold substance from the boundary so as not to allow the term u^β to blow up the temperature. However, due to (1.2), in this paper (see Theorem 2.1), we will prove that when volume energy release is given by powers greater than one, and the convection terms are of the form $b_i(x) \cdot \nabla$ for b_i in the Kato class, $i = 1, 2$, the blow up in finite time of system (1.1) without convection terms ($b_i \equiv 0$, $i = 1, 2$) implies blow up in finite time of system (1.1) with convection terms (some b_i non-zero, $i = 1, 2$).

The finite time blow up of systems like (1.1) was initially considered by Escobedo and Herrero [**5**] for the case $\alpha_1 = \alpha_2 = 2$ without convection terms. In this paper, in Theorem 2.2 we have proved that the positive mild solution of the reaction-diffusion system

$$\begin{aligned} \frac{\partial v_i(t,x)}{\partial t} &= \Delta_{\alpha_i} v_i(t,x) + v_{i'}^{\beta_i}(t,x), \quad t > 0,\ x \in \mathbb{R}^d, \\ v_i(0,x) &= f_i(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.4}$$

$i \in \{1,2\}$ and $i' = 3-i$, where $0 < \alpha_i \le 2$ and d, β_i, f_i are as in system (1.1), blows up in finite time if

$$d < \frac{\alpha_1 \vee \alpha_2}{\beta_1 \vee \beta_2 - 1}. \tag{1.5}$$

Theorem 2.1 given below allows us to conclude (Corollary 2.3) that the positive mild solution of system (1.1) also blows up if (1.5) holds. Related cases involving perturbed and unperturbed Laplacian, fractional Laplacians and fractional derivatives may be found, for instance, in [**4, 7, 8, 10, 9, 17, 19, 20, 14, 22, 26**]. For instance, Villa [**26**], and Guedda and Kirane [**7**], have considered more general systems than (1.4), which, reduced to our case, imply, respectively, that the solution blows up in finite time if

$$d \le \frac{\beta_1 + \beta_2 + 2}{\beta_1(\beta_2+1)/(\alpha_1 \vee \alpha_2) + \beta_2(\beta_1+1)/(\alpha_1 \wedge \alpha_2) - 1/(\alpha_1 \vee \alpha_2)}$$

and

$$d \le \frac{(\beta_1 \vee \beta_2)\,(\alpha_1 \wedge \alpha_2)}{\beta_1\beta_2 - 1}.$$

Note that, when $\beta_1 = \beta_2$ and $\alpha_1 \ne \alpha_2$, the condition (1.5) is better. And, when $\beta_1 \ne \beta_2$ and $\alpha_1 = \alpha_2$, the blow up condition (1.5) is better if the difference between β_1 and β_2 is not very large; namely, $(\beta_1 \vee \beta_2)^2 < (\beta_1 \wedge \beta_2)(\beta_1 \vee \beta_2 + 2) + 1$ for the Villa condition, and $(\beta_1 \vee \beta_2)^2 < (\beta_1 \vee \beta_2)(\beta_1 \wedge \beta_2 + 1) - 1$ for the Guedda and Kirane condition.

On the other hand, Kakehi and Oshita [**8**] showed for the case $\alpha_1 = \alpha_2 = \alpha$ that the solution of system (1.4) blows up in finite time if

$$d \le \frac{\alpha\,(\beta_1 \vee \beta_2 + 1)}{\beta_1\beta_2 - 1}. \tag{1.6}$$

Note that, in this particular case, the Kakehi-Oshita condition (1.6) is better.

A good reference for global nonexistence of positive solutions for systems like (1.1) for the case $\alpha_1 = \alpha_2 = 2$, not considered in our paper, is Kirane and Qafsaoui [**10**]. In that paper, they additionally considered, as a particular case, the system (1.4), and they showed that, under the condition $d \le 2(\beta_1 \vee \beta_2 + 1)/(\beta_1\beta_2 - 1)$ there are no nontrivial global solutions. Note that this Kirane and Qafsaoui condition coincides (for $\alpha = 2$) with the blow up in finite time condition (1.6) given by Kakehi and Oshita.

2. Main theorems. The existence of nonnegative local mild solutions for the reaction-diffusion system (1.4) and for the reaction-convection-diffusion system (1.1) easily follows from the Banach fixed-point theorem (see, for example, [**20,** Theorem 1] or [**26,** Theorem 2.1]); thus, we omit this standard calculation.

Theorem 2.1. *The positive mild solution of the reaction-convection-diffusion system* (1.1) *blows up in finite time if and only if the positive mild solution of the reaction-diffusion system* (1.4) *blows up in finite time.*

Proof. Let $0 < T < \infty$ be fixed. Let $p_i(t,x,y)$ be the transition density of the semigroup generated by Δ_{α_i}, $i = 1,2$, and $p_i^{b_i}(t,x,y)$ the transition density of the semigroup generated by $\Delta_{\alpha_i} + b_i(x) \cdot \nabla$, $i = 1,2$. From (1.2), we obtain

$$\begin{aligned} C_i^{-1}&\bigg[\int_{\mathbb{R}^d} p_i(t,x,y) f_i(y)\,dy + \int_0^t\int_{\mathbb{R}^d} p_i(t-s,x,y) u_{i'}^{\beta_i}(s,y)\,dy\,ds\bigg] \\ &\leq \int_{\mathbb{R}^d} p_i^{b_i}(t,x,y) f_i(y)\,dy + \int_0^t\int_{\mathbb{R}^d} p_i^{b_i}(t-s,x,y) u_{i'}^{\beta_i}(s,y)\,dy\,ds \\ &\leq C_i\bigg[\int_{\mathbb{R}^d} p_i(t,x,y) f_i(y)\,dy + \int_0^t \int_{\mathbb{R}^d} p_i(t-s,x,y) u_{i'}^{\beta_i}(s,y)\,dy\,ds\bigg], \end{aligned}$$

$0 < t \leq T$, $x \in \mathbb{R}^d$, $i = 1,2$.

Letting the integral system

$$\begin{aligned} v_i(t,x) = &\int_{\mathbb{R}^d} p_i(t,x,y) f_i(y)\,dy \\ &+ \int_0^t \int_{\mathbb{R}^d} p_i(t-s,x,y) v_{i'}^{\beta_i}(s,y)\,dy\,ds, \end{aligned} \tag{2.1}$$

$t \geq 0$, $x \in \mathbb{R}^d$, $i = 1,2$, we have that $v_1, v_2 \geq 0$ is a mild solution of the reaction-diffusion system (1.4). Thus, by comparison,

$$C_i^{-1} v_i(t,x) \leq u_i(t,x) \leq C_i v_i(t,x), \quad 0 \leq t \leq T,\ x \in \mathbb{R}^d.$$

Hence, (u_1, u_2) blows up in finite time if and only if (v_1, v_2) blows up in finite time. □

Theorem 2.2. *Suppose that* $0 < \alpha_i \leq 2$, $i = 1, 2$ *and let* $\alpha_1 \leq \alpha_2$. *Then, if* (v_1, v_2) *is a positive mild solution of system* (1.4) *and*

$$\frac{d}{\alpha_2} - \frac{d(\beta_1 \vee \beta_2)}{\alpha_2} + 1 > 0, \tag{2.2}$$

the mild solution (v_1, v_2) *blows up in finite time.*

As a direct consequence of Theorems 2.1 and 2.2, we obtain the next result.

Corollary 2.3. *Suppose that* $1 < \alpha_i < 2$, $i = 1, 2$, *and let* $\alpha_1 \leq \alpha_2$. *Then the positive mild solution of the reaction-convection-diffusion system* (1.1) *blows up in finite time if* (2.2) *holds.*

For the proof of Theorem 2.2, we need some preliminary results.

3. Preliminary results. In the sequel, we denote the transition density $p_i(t, x, y)$ of the semigroup generated by Δ_{α_i} as $p_i(t, x, y) \equiv p_i(t, x - y)$, $i = 1, 2$.

Lemma 3.1. *Let* $s, t > 0$ *and* $x, y \in \mathbb{R}^d$. *Then,*

(i) $p_i(ts, x) = t^{-d/\alpha_i} p_i(s, t^{-1/\alpha_i} x)$,
(ii) $p_i(t, x) \geq (s/t)^{d/\alpha_i} p_i(s, x)$ *for* $t \geq s$,
(iii) $p_i(t, (1/\tau)(x - y)) \geq p_i(t, x) p_i(t, y)$ *if* $p_i(t, 0) \leq 1$ *and* $\tau \geq 2$,
(iv) $p_1(t, x) \geq c p_2(t^{\alpha_2/\alpha_1}, x)$ *for some* $0 < c \leq 1$, *if* $\alpha_1 \leq \alpha_2$.

Proof. For (i)–(iii), see [**24,** pages 46, 47] and, for (iv), see [**15,** page 1699]. □

Using Lemma 3.1 (iv), it follows from (2.1) that

$$
\begin{aligned}
v_1(t,x) &\geq \int_{\mathbb{R}^d} c p_2\left(t^{\alpha_2/\alpha_1}, x-y\right) f_1(y)\,dy \\
&\quad + \int_0^t \int_{\mathbb{R}^d} c p_2\left((t-s)^{\alpha_2/\alpha_1}, x-y\right) v_2^{\beta_1}(s,y)\,dy\,ds, \\
v_2(t,x) &\geq \int_{\mathbb{R}^d} p_2(t,x-y) f_2(y)\,dy \\
&\quad + \int_0^t \int_{\mathbb{R}^d} p_2(t-s,x-y) v_1^{\beta_2}(s,y)\,dy\,ds.
\end{aligned}
\tag{3.1}
$$

Lemma 3.2. *If $v_1, v_2 \geq 0$ is a solution of the integral system* (2.1)*, then there exist some positive constants c_0, γ_0 and t_0, with $t_0 > 1$, such that*

$$\min\left\{v_1(t_0,x), v_2(t_0,x)\right\} \geq c_0 p_2(\gamma_0, x), \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. By (i) of Lemma 3.1, we can fix $t_0 > 1$ such that

$$p_2\left(t_0^{\alpha_2/\alpha_1}, 0\right) \leq 1 \quad \text{and} \quad p_2(t_0, 0) \leq 1,$$

and thus, using (i) and (iii) of Lemma 3.1, we obtain

$$
\begin{aligned}
p_2\left(t_0^{\alpha_2/\alpha_1}, x-y\right) &\geq 2^{-d} p_2\left(\frac{t_0^{\alpha_2/\alpha_1}}{2^{\alpha_2}}, x\right) p_2\left(t_0^{\alpha_2/\alpha_1}, 2y\right), \\
p_2(t_0, x-y) &\geq 2^{-d} p_2\left(\frac{t_0}{2^{\alpha_2}}, x\right) p_2(t_0, 2y).
\end{aligned}
$$

Using these inequalities, it follows from (3.1) that

$$
\begin{aligned}
v_1(t_0,x) &\geq p_2\left(\frac{t_0^{\alpha_2/\alpha_1}}{2^{\alpha_2}}, x\right) 2^{-d} c \int_{\mathbb{R}^d} p_2\left(t_0^{\alpha_2/\alpha_1}, 2y\right) f_1(y)\,dy, \\
v_2(t_0,x) &\geq p_2\left(\frac{t_0}{2^{\alpha_2}}, x\right) 2^{-d} \int_{\mathbb{R}^d} p_2(t_0, 2y) f_2(y)\,dy.
\end{aligned}
$$

We consider

$$a = \min\left\{2^{-d} c \int_{\mathbb{R}^d} p_2\left(t_0^{\alpha_2/\alpha_1}, 2y\right) f_1(y)\,dy,\ 2^{-d} \int_{\mathbb{R}^d} p_2(t_0, 2y) f_2(y)\,dy\right\}$$

and $\gamma_0 = t_0/2^{\alpha_2}$. Then, from (ii) of Lemma 3.1, we obtain

$$\begin{aligned} v_1(t_0, x) &\geq a t_0^{[d(\alpha_1-\alpha_2)]/(\alpha_1\alpha_2)} p_2(\gamma_0, x), \\ v_2(t_0, x) &\geq a p_2(\gamma_0, x). \end{aligned}$$

Finally, the desired result is obtained by taking $c_0 = a t_0^{[d(\alpha_1-\alpha_2)]/(\alpha_1\alpha_2)}$.

□

Let $t_0 > 1$ be as in Lemma 3.2. The semigroup property implies

$$\begin{aligned} v_i(t+t_0, x) = &\int_{\mathbb{R}^d} p_i(t, x-y) v_i(t_0, y)\, dy \\ &+ \int_0^t \int_{\mathbb{R}^d} p_i(t-s, x-y) v_{i'}^{\beta_i}(s, y)\, dy\, ds, \end{aligned}$$

$t > 0$, $x \in \mathbb{R}^d$, $i = 1, 2$. From Lemma 3.1 (iv), we have that $p_1(t^{\alpha_1/\alpha_2}, x) \geq c p_2(t, x)$. Thus,

$$\begin{aligned} v_1\left(t^{\alpha_1/\alpha_2} + t_0, x\right) \geq &\int_{\mathbb{R}^d} c p_2(t, x-y) v_1(t_0, y)\, dy \\ &+ \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} c p_2\left((t^{\alpha_1/\alpha_2} - s)^{\alpha_2/\alpha_1}, x-y\right) v_2^{\beta_1}(s+t_0, y)\, dy\, ds, \\ v_2\left(t^{\alpha_1/\alpha_2} + t_0, x\right) \geq &\int_{\mathbb{R}^d} p_2(t^{\alpha_1/\alpha_2}, x-y) v_2(t_0, y)\, dy \\ &+ \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} p_2(t^{\alpha_1/\alpha_2} - s, x-y) v_1^{\beta_2}(s+t_0, y)\, dy\, ds. \end{aligned}$$

Hence, Lemma 3.2 implies

(3.2)

$$\begin{aligned} v_1\left(t^{\alpha_1/\alpha_2} + t_0, x\right) \geq &\, c c_0 p_2(t+\gamma_0, x) \\ &+ \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} c p_2\left((t^{\alpha_1/\alpha_2} - s)^{\frac{\alpha_2}{\alpha_1}}, x-y\right) v_2^{\beta_1}(s+t_0, y)\, dy\, ds, \\ v_2\left(t^{\alpha_1/\alpha_2} + t_0, x\right) \geq &\, c_0 p_2\left(t^{\alpha_1/\alpha_2} + \gamma_0, x\right) \\ &+ \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} p_2(t^{\alpha_1/\alpha_2} - s, x-y) v_1^{\beta_2}(s+t_0, y)\, dy\, ds. \end{aligned}$$

We define

$$\overline{v}_1(t) = \int_{\mathbb{R}^d} p_2(t,x)v_1(t,x)\,dx, \quad t \geq 0,$$
$$\overline{v}_2(t) = \int_{\mathbb{R}^d} p_2(t,x)v_2(t,x)\,dx, \quad t \geq 0.$$

Lemma 3.3. *If there exists a* $T_0 > 0$ *such that* $\overline{v}_1(t) = \infty$ *or* $\overline{v}_2(t) = \infty$ *for all* $t \geq T_0$*, then the mild solution* (v_1, v_2) *of system* (1.4) *blows up in finite time.*

Proof. Suppose that $\overline{v}_1(t) = \infty$ for all $t \geq T_0$. Let

$$T_0 \leq t \leq s \leq \frac{6t}{2^{\alpha_2}+1} \quad \text{and} \quad \tau = \left(\frac{6t-s}{s}\right)^{1/\alpha_2}.$$

From Lemma 3.1 (i), we have

$$p_2(6t-s, x-y) = \left(\frac{s}{6t-s}\right)^{d/\alpha_2} p_2\left(s, \frac{1}{\tau}(x-y)\right).$$

Since $\tau \geq 2$, it follows from Lemma 3.1 (iii) that

$$p_2(6t-s, x-y) \geq \left(\frac{s}{6t-s}\right)^{d/\alpha_2} p_2(s,x)p_2(s,y).$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^d} p_2(6t-s, x-y)v_1(s,y)\,dy \\
&\quad \geq \left(\frac{s}{6t-s}\right)^{d/\alpha_2} p_2(s,x) \int_{\mathbb{R}^d} p_2(s,y)v_1(t,y)\,dy \\
&\quad = \left(\frac{s}{6t-s}\right)^{d/\alpha_1} p_2(s,x)\overline{v}_1(s).
\end{aligned}$$

Since $T_0 \leq t \leq s$, we have that $\overline{v}_1(s) = \infty$, and thus,

$$\int_{\mathbb{R}^d} p_2(6t-s, x-y)v_1(s,y)\,dy = \infty. \tag{3.3}$$

On the other hand, from (2.1) and the fact that $f_2 \geq 0$, we get

$$v_2(6t,x) \geq \int_0^{6t} \int_{\mathbb{R}^d} p_2(6t-s,x-y) v_1^{\beta_2}(s,y)\,dy\,ds.$$

Finally, from Jensen's inequality and (3.3), we obtain

$$v_2(6t,x) \geq \int_0^{6t/(2^{\alpha_2}+1)} \left(\int_{\mathbb{R}^d} p_2(6t-s,x-y) v_1(s,y)\,dy \right)^{\beta_2} ds = \infty,$$

so that $v_2(t,x) = \infty$ for all $t \geq 6T_0$ and $x \in \mathbb{R}^d$. Clearly, blow up of v_2 implies blow up of v_1. Similarly, it can be shown that, if $\overline{v}_2(t) = \infty$ for all $t \geq 6T_0$, then $v_1(t,x) = \infty$ for all $t \geq 6T_0$ and $x \in \mathbb{R}^d$. □

4. Proof of Theorem 2.2.

Proof. Multiplying equations (3.2) by $p_2(t^{\alpha_1/\alpha_2} + t_0, x)$ gives

$$\begin{aligned} &p_2(t^{\alpha_1/\alpha_2} + t_0, x) v_1(t^{\alpha_1/\alpha_2} + t_0, x) \\ &\quad \geq c c_0 p_2(t^{\alpha_1/\alpha_2} + t_0, x) p_2\left(t + \gamma_0, x\right) \\ &\qquad + \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} c p_2(t^{\alpha_1/\alpha_2} + t_0, x) p_2((t^{\alpha_1/\alpha_2} - s)^{\alpha_2/\alpha_1}, x-y) \\ &\quad \cdot v_2^{\beta_1}\left(s + t_0, y\right)\,dy\,ds \end{aligned}$$

and

$$\begin{aligned} &p_2(t^{\alpha_1/\alpha_2} + t_0, x) v_2(t^{\alpha_1/\alpha_2} + t_0, x) \\ &\quad \geq c_0 p_2(t^{\alpha_1/\alpha_2} + t_0, x) p_2(t^{\alpha_1/\alpha_2} + \gamma_0, x) \\ &\qquad + \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} p_2(t^{\alpha_1/\alpha_2} + t_0, x) p_2(t^{\alpha_1/\alpha_2} - s, x-y) \\ &\quad \cdot v_1^{\beta_2}\left(s + t_0, y\right)\,dy\,ds. \end{aligned}$$

Integrating with respect to x, we have

$$\begin{aligned} &\overline{v}_1(t^{\alpha_1/\alpha_2} + t_0) \geq c c_0 p_2(t + t^{\alpha_1/\alpha_2} + t_0 + \gamma_0, 0) + \int_0^{t^{\alpha_1/\alpha_2}} \\ &\quad \cdot \int_{\mathbb{R}^d} c p_2(t^{\alpha_1/\alpha_2} + t_0 + (t^{\alpha_1/\alpha_2} - s)^{\alpha_2/\alpha_1}, y) v_2^{\beta_1}\left(s + t_0, y\right)\,dy\,ds \end{aligned}$$

and

$$\overline{v}_2(t^{\alpha_1/\alpha_2}+t_0) \geq c_0 p_2(2t^{\alpha_1/\alpha_2}+t_0+\gamma_0, 0) + \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} p_2(2t^{\alpha_1/\alpha_2}+t_0-s, y) v_1^{\beta_2}(s+t_0, y)\, dy\, ds.$$

Using (i) and (ii) of Lemma 3.1, we obtain

$$\overline{v}_1(t^{\alpha_1/\alpha_2}+t_0) \geq cc_0(t+t^{\alpha_1/\alpha_2}+t_0+\gamma_0)^{-d/\alpha_2} p_2(1,0) + \int_0^{t^{\alpha_1/\alpha_2}} \int_{\mathbb{R}^d} c\left(\frac{s+t_0}{t^{\alpha_1/\alpha_2}+t_0+(t^{\alpha_1/\alpha_2}-s)^{\alpha_2/\alpha_1}}\right)^{d/\alpha_2} p_2(s+t_0, y) \cdot v_2^{\beta_1}(s+t_0, y)\, dy\, ds$$

and

$$\overline{v}_2(t^{\alpha_1/\alpha_2}+t_0) \geq c_0(2t^{\alpha_1/\alpha_2}+t_0+\gamma_0)^{-d/\alpha_2} p_2(1,0) + \int_0^{t^{\alpha_1\alpha_2}} \int_{\mathbb{R}^d} \left(\frac{s+t_0}{2t^{\alpha_1/\alpha_2}+t_0-s}\right)^{d/\alpha_2} p_2(s+t_0, y) \cdot v_1^{\beta_2}(s+t_0, y)\, dy\, ds.$$

Now, applying Jensen's inequality gives

$$\overline{v}_1(t^{\alpha_1/\alpha_2}+t_0) \geq cc_0(t+t^{\alpha_1/\alpha_2}+t_0+\gamma_0)^{-d/\alpha_2} p_2(1,0) + \int_0^{t^{\alpha_1/\alpha_2}} c\left(\frac{s+t_0}{t^{\alpha_1/\alpha_2}+t_0+(t^{\alpha_1/\alpha_2}-s)^{\alpha_2/\alpha_1}}\right)^{d/\alpha_2} \overline{v}_2(s+t_0)^{\beta_1}\, ds$$

and

$$\overline{v}_2(t^{\alpha_1/\alpha_2}+t_0) \geq c_0(2t^{\alpha_1/\alpha_2}+t_0+\gamma_0)^{-d/\alpha_2} p_2(1,0) + \int_0^{t^{\alpha_1/\alpha_2}} \left(\frac{s+t_0}{2t^{\alpha_1/\alpha_2}+t_0-s}\right)^{d/\alpha_2} \overline{v}_1(s+t_0)^{\beta_2}\, ds.$$

Using the facts that $t^{\alpha_1/\alpha_2}-s \leq t^{\alpha_1/\alpha_2}$ and $\alpha_1/\alpha_2 \leq 1$, we obtain

$$\overline{v}_1(t^{\alpha_1/\alpha_2}+t_0) \geq cc_0(2t+t_0+\gamma_0)^{-d/\alpha_2} p_2(1,0)$$

$$+\int_0^{t^{\alpha_1/\alpha_2}} c\left(\frac{s+t_0}{2\,(t+t_0)}\right)^{d/\alpha_2} \overline{v}_2\,(s+t_0)^{\beta_1}\,ds$$

and

$$\begin{aligned}\overline{v}_2(t^{\alpha_1/\alpha_2}+t_0) &\geq c_0(2t^{\alpha_1/\alpha_2}+t_0+\gamma_0)^{-d/\alpha_2}p_2(1,0)\\ &+\int_0^{t^{\alpha_1/\alpha_2}} \left(\frac{s+t_0}{2(t^{\alpha_1/\alpha_2}+t_0)}\right)^{d/\alpha_2} \overline{v}_1\,(s+t_0)^{\beta_2}\,ds.\end{aligned}$$

Thus,

$$\begin{aligned}&\overline{v}_1(t^{\alpha_1/\alpha_2}+t_0)\,(t+t_0)^{d/\alpha_2}\\ &\quad\geq cc_0\left(\frac{t+t_0}{2t+t_0+\gamma_0}\right)^{d/\alpha_2} p_2(1,0)\\ &\qquad+2^{-d/\alpha_2}c\int_0^{t^{\alpha_1/\alpha_2}} (s+t_0)^{d/\alpha_2}\,\overline{v}_2\,(s+t_0)^{\beta_1}\,ds\end{aligned}$$

and

$$\begin{aligned}&\overline{v}_2(t^{\alpha_1/\alpha_2}+t_0)\,(t+t_0)^{d/\alpha_2}\\ &\quad\geq c_0\left(\frac{t+t_0}{2t^{\alpha_1/\alpha_2}+t_0+\gamma_0}\right)^{d/\alpha_2} p_2(1,0)\\ &\qquad+2^{-d/\alpha_2}\int_0^{t^{\alpha_1/\alpha_2}} (s+t_0)^{d/\alpha_2}\,\overline{v}_1\,(s+t_0)^{\beta_2}\,ds.\end{aligned}$$

Assume that $t\geq 1$. Since $\gamma_0=t_0/2^{\alpha_2}$, we have that $t_0>\gamma_0$. Thus,

$$\begin{aligned}\overline{v}_1(t^{\alpha_1/\alpha_2}+t_0)\,(t+t_0)^{d/\alpha_2} &\geq 2^{-d/\alpha_2}cc_0p_2(1,0)\\ &+2^{-d/\alpha_2}c\int_1^{t^{\alpha_1/\alpha_2}} (s+t_0)^{d/\alpha_2}\\ &\cdot(s+t_0)^{-d\beta_1/\alpha_2}\left[(s+t_0)^{d/\alpha_2}\,\overline{v}_2\,(s+t_0)\right]^{\beta_1}ds\end{aligned}$$

and

$$\begin{aligned}\overline{v}_2(t^{\alpha_1/\alpha_2}+t_0)\,(t+t_0)^{d/\alpha_2} &\geq 2^{-d/\alpha_2}c_0p_2(1,0)\\ &+2^{-d/\alpha_2}\int_1^{t^{\alpha_1/\alpha_2}} (s+t_0)^{d/\alpha_2}\\ &\cdot(s+t_0)^{-d\beta_2/\alpha_2}\left[(s+t_0)^{d/\alpha_2}\,\overline{v}_1\,(s+t_0)\right]^{\beta_2}ds.\end{aligned}$$

Let $\eta = \min\{2^{-d/\alpha_2} c c_0 p_2(1,0),\ 2^{-d/\alpha_2} c\}$. Since $0 < c \le 1$, we have

$$\begin{aligned}
&\overline{v}_1(t^{\alpha_1/\alpha_2} + t_0)\,(t+t_0)^{d/\alpha_2} \\
&\quad \ge \eta + \eta \int_1^{t^{\alpha_1/\alpha_2}} (s+t_0)^{(d/\alpha_2)-(d\beta_1/\alpha_2)}\,[(s+t_0)^{d/\alpha_2}\,\overline{v}_2\,(s+t_0)]^{\beta_1}\,ds, \\
&\qquad\qquad t \ge 1, \\
&\overline{v}_2\left(t^{\alpha_1/\alpha_2} + t_0\right)(t+t_0)^{d/\alpha_2} \\
&\quad \ge \eta + \eta \int_1^{t^{\alpha_1/\alpha_2}} (s+t_0)^{(d/\alpha_2)-(d\beta_2/\alpha_2)}\,[(s+t_0)^{d/\alpha_2}\,\overline{v}_1\,(s+t_0)]^{\beta_2}\,ds,
\end{aligned}$$

$t \ge 1$, or, equivalently,

$$\begin{aligned}
&\overline{v}_1(t^{\alpha_1/\alpha_2})t^{d/\alpha_2} \\
&\quad \ge \eta + \eta \int_{t_0+1}^{t^{\alpha_1/\alpha_2}} s^{(d/\alpha_2)-(d\beta_1/\alpha_2)}[s^{d/\alpha_2}\overline{v}_2(s)]^{\beta_1}\,ds, \quad t^{\alpha_1/\alpha_2} \ge t_0+1, \\
&\overline{v}_2(t^{\alpha_1/\alpha_2})t^{d/\alpha_2} \\
&\quad \ge \eta + \eta \int_{t_0+1}^{t^{\alpha_1/\alpha_2}} s^{(d/\alpha_2)-(d\beta_2/\alpha_2)}[s^{d/\alpha_2}\overline{v}_1(s)]^{\beta_2}\,ds, \quad t^{\alpha_1/\alpha_2} \ge t_0+1.
\end{aligned}$$

Consider the integral system

(4.1)

$$\begin{aligned}
w_1(t^{\alpha_1/\alpha_2}) &= \eta + \eta \int_{t_0+1}^{t^{\alpha_1/\alpha_2}} \left(s^{\theta_1} \wedge s^{\theta_2}\right) w_2^{\beta_1}(s)\,ds, \quad t^{\alpha_1/\alpha_2} \ge t_0+1, \\
w_2(t^{\alpha_1/\alpha_2}) &= \eta + \eta \int_{t_0+1}^{t^{\alpha_1/\alpha_2}} \left(s^{\theta_1} \wedge s^{\theta_2}\right) w_1^{\beta_2}(s)\,ds, \quad t^{\alpha_1/\alpha_2} \ge t_0+1,
\end{aligned}$$

where $\theta_1 = (d/\alpha_2) - (d\beta_1/\alpha_2)$ and $\theta_2 = (d/\alpha_2) - (d\beta_2/\alpha_2)$. The differential expression of (4.1) is

$$\begin{aligned}
&\frac{\alpha_1}{\alpha_2} t^{\alpha_1/\alpha_2 - 1} w_1'(t^{\alpha_1/\alpha_2}) \\
&\quad = \frac{\alpha_1}{\alpha_2} t^{\alpha_1/\alpha_2 - 1} \eta[(t^{\alpha_1/\alpha_2})^{\theta_1} \wedge (t^{\alpha_1/\alpha_2})^{\theta_2}] w_2^{\beta_1}(t^{\alpha_1/\alpha_2}),\ t^{\alpha_1/\alpha_2} \ge t_0+1, \\
&\frac{\alpha_1}{\alpha_2} t^{\alpha_1/\alpha_2 - 1} w_2'(t^{\alpha_1/\alpha_2}) \\
&\quad = \frac{\alpha_1}{\alpha_2} t^{\alpha_1/\alpha_2 - 1} \eta[(t^{\alpha_1/\alpha_2})^{\theta_1} \wedge (t^{\alpha_1/\alpha_2})^{\theta_2}] w_1^{\beta_2}(t^{\alpha_1/\alpha_2}),\ t^{\alpha_1/\alpha_2} \ge t_0+1,
\end{aligned}$$

$w_1(t_0+1) = \eta = w_2(t_0+1)$, or, equivalently,

$$\begin{aligned} w_1'(t) &= \eta\left(t^{\theta_1} \wedge t^{\theta_2}\right) w_2^{\beta_1}(t), \quad t \geq t_0+1, \\ w_2'(t) &= \eta\left(t^{\theta_1} \wedge t^{\theta_2}\right) w_1^{\beta_2}(t), \quad t \geq t_0+1, \\ w_1(t_0+1) &= \eta = w_2(t_0+1), \end{aligned} \tag{4.2}$$

whose solution satisfies

$$\frac{w_1^{\beta_2+1}(t) - \eta^{\beta_2+1}}{\beta_2+1} = \frac{w_2^{\beta_1+1}(t) - \eta^{\beta_1+1}}{\beta_1+1}.$$

Assume, without loss of generality, that $\beta_2 \geq \beta_1$. Since $0 < \eta < 1$, we have

$$\frac{w_1^{\beta_2+1}(t)}{\beta_2+1} \leq \frac{w_2^{\beta_1+1}(t)}{\beta_1+1}$$

or, equivalently,

$$w_2(t) \geq \left(\frac{\beta_1+1}{\beta_2+1}\right)^{1/(\beta_1+1)} w_1^{(\beta_2+1)/(\beta_1+1)}(t), \quad t \geq t_0+1.$$

Substituting this into the first equation of (4.2), we obtain

$$w_1'(t) \geq \eta\left(t^{\theta_1} \wedge t^{\theta_2}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\beta_1/(\beta_1+1)} w_1^{\beta_1(\beta_2+1)/(\beta_1+1)}(t), \quad t \geq t_0+1.$$

Due to the assumption that $\beta_2 \geq \beta_1$, we have that $\theta_2 \leq \theta_1$, and since $t \geq t_0+1$, $t^{\theta_1} \wedge t^{\theta_2} = t^{\theta_2}$. Thus,

$$w_1^{-\beta_1(\beta_2+1)/(\beta_1+1)}(t) w_1'(t) \geq \eta\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\beta_1/(\beta_1+1)} t^{\theta_2}, \quad t \geq t_0+1.$$

Integrating from t_0+1 to t yields

$$\frac{\beta_1+1}{1-\beta_1\beta_2}\left[w_1^{(1-\beta_1\beta_2)/(\beta_1+1)}(t) - \eta^{(1-\beta_1\beta_2)/(\beta_1+1)}\right] \geq \eta\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\beta_1/(\beta_1+1)} \int_{t_0+1}^{t} s^{\theta_2} ds.$$

Thus (recalling that $\beta_1\beta_2 > 1$), we obtain

$$w_1(t) \geq \left[\eta^{(1-\beta_1\beta_2)/(\beta_1+1)} - \eta\left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\right.$$

$$\cdot \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\beta_1/(\beta_1+1)} \int_{t_0+1}^{t} s^{\theta_2}\,ds\Bigg]^{(\beta_1+1)/(1-\beta_1\beta_2)}.$$

Since $\theta_2+1=(d/\alpha_2)-(d\beta_2/\alpha_2)+1>0$, it follows that

$$\int_{t_0+1}^{t} s^{\theta_2}\,ds \longrightarrow \infty$$

when $t\to\infty$. Thus, there exists a $T_0>t_0+1$ such that $w_1(t)=\infty$ for $t=T_0$. By comparison, we have

$$t^{d/\alpha_1}\overline{v}_1(t)\ge w_1(t)=\infty \quad \text{for } t=T_0,$$

and Lemma 3.3 implies that (v_1,v_2) blows up in finite time. □

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