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CENTRAL PART INTERPOLATION SCHEMES FOR INTEGRAL EQUATIONS WITH SINGULARITIES

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ABSTRACT. Two high order methods are constructed and analyzed for a class of Fredholm integral equations of the second kind with kernels that may have weak boundary and diagonal singularities. The proposed methods are based on improving the boundary behavior of the exact solution with the help of a change of variables, and on central part interpolation by polynomial splines on the uniform grid. A detailed error analysis for the proposed numerical schemes is given. This includes, in particular, error bounds under various types of assumptions on the equation, and shows that the proposed central part collocation approach has accuracy and numerical stability advantages compared with standard piecewise polynomial collocation methods, including the collocation at Chebyshev knots.

1. Introduction. We consider an integral equation of the form

$$u(x) = \int_0^1 [a(x,y)|x-y|^{-\nu} + b(x,y)]u(y)\,dy + f(x), \quad 0 \le x \le 1, \tag{1.1}$$

where $\nu \in (0,1)$, $f \in C[0,1] \cap C^m(0,1)$, $a, b \in C^m([0,1] \times [0,1])$, $m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{N} := \{1, 2, \ldots\}$. By $C^m(\Omega)$, we denote the set of m times continuously differentiable functions on Ω. The Banach space of continuous functions $u : [0,1] \to \mathbb{R} := (-\infty, \infty)$ with the norm $\|u\|_\infty = \{\max |u(x)| : 0 \le x \le 1\}$ is denoted by $C[0,1]$. Equations of the form (1.1) and related equations arise, for example, in potential

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problems **[13]**, nuclear physics **[4]**, atmospheric physics **[12]** and in radiative heat exchange **[40]**.

The exact solution to (1.1) (if it exists) is typically non-smooth at the endpoints of the interval of integration $[0,1]$, where its derivatives become unbounded, see, for example, **[11, 27, 29]**. In order to construct high-order numerical methods for these types of integral or integro-differential equations, the singular behavior of the exact solution **[3, 5, 8, 38, 39]** must be taken into account, see also **[1, 6, 15, 18, 20, 24, 40]**; in particular, the use of polynomial splines on special graded grids will work, see e.g., **[3, 5, 25, 34, 38]**. However, strongly non-uniform grids may cause serious rounding error problems and the unstable behavior of numerical results. With the aid of a suitable change of variables it is possible to convert the solution of (1.1) into $C^m[0,1]$-smooth function, after which the converted problem may be solved by standard piecewise polynomial collocation methods on uniform grids, see e.g., **[19, 28]**.

In the present paper, we follow the idea of smoothing transformation, but we apply so-called central part interpolation/collocation on uniform grids. This method was introduced in **[23]** for more specialized problems. Now, we actually consider a more complicated situation for (1.1), assuming that the coefficient functions $a(x,y)$ and $b(x,y)$ and their derivatives are continuous on $[0,1]\times(0,1)$. They may have some boundary singularities with respect to y, see Lemmas 2.1 and 2.2 below.

In central part interpolation method, for a given $n\in\mathbb{N}$, the uniform grid $\{jh : j=0,\dots,n\}$, $h=1/n$, is used on the interval $[0,1]$. For a given $m\in\mathbb{N}$, an interpolation operator $P_{h,m}$ is constructed for approximating a function $g\in C[0,1]$ by a piecewise polynomial $P_{h,m}g$ of degree $m-1$ such that, on each subinterval $[jh,(j+1)h]$, $j=0,\dots,n-1$, function $P_{h,m}g$ coincides with the Lagrange interpolation polynomial of degree $m-1$ that interpolates g at m grid points kh for integers k satisfying $-m/2<k-j\le m/2$. An extension of g is introduced for the evaluation of $P_{h,m}g$ for nodal points close to the end points of the interval $[0,1]$.

The collocation method based upon central part interpolation has accuracy and numerical stability advantages compared with standard piecewise polynomial collocation methods, including the collocation at Chebyshev knots, see the end of Sections 4 and 5 and Remark 6.2. The

Lagrange polynomial is exploited only for one subinterval of length h in the center of the full interpolation interval of length $(m-1)h$. In this central part, the polynomial interpolation loses its bad convergence property near the end points of the interval. A classical error formula for Lagrange interpolation of $g \in C^m[0,1]$ gives considerably more precise estimates in the central region than on the whole interval (see Lemma 4.1 below) and we have for $m \to \infty$ only logarithmic growth of norms of $P_{h,m}$ due to Runck **[9, 31, 32]** (see formula (4.8) below).

As usual, the collocation method for integral equations is only semi-discrete since the matrix entries of the resulting linear system require the evaluation of integrals of products of the kernel function of the integral operator against the Lagrange polynomials on small intervals. In order to derive a discrete collocation method **[3]**, we employ product integration techniques based on central part interpolation in a similar manner as in **[23]** with $a \in C^m([0,1] \times [0,1])$ and $b = 0$. Our approach here is close to that in **[35]**, where smooth splines are used as trial functions, see also **[21]**.

The rest of the paper is organized as follows. In Section 2, some results concerning the compactness of the integral operator of equation (1.1) are recalled, and a result regarding the smoothness of the exact solution to (1.1) is presented, see Theorem 2.3. Later on, these results will play a key role in the convergence analysis of the proposed algorithms. In Section 3, a smoothing transformation is introduced, and its properties are discussed. In Sections 4 and 5, central part interpolation by polynomials and piecewise polynomials is studied. In Sections 6 and 7, for (1.1), two numerical methods are constructed and justified. The main results of the paper are given by Theorems 6.1 and 7.2. Finally, in Section 8, the theoretical results are tested by numerical experimentation.

2. Smoothness of the solution. Denote by T the integral operator of equation (1.1):

$$(Tu)(x) = \int_0^1 [a(x,y)|x-y|^{-\nu} + b(x,y)]u(y)\,dy \qquad (2.1)$$

$$0 \le x \le 1, \quad 0 < \nu < 1.$$

We refer to **[27]** for the proofs of the following two lemmas.

Lemma 2.1. *Let T be defined by formula (2.1) with a fixed $\nu \in (0,1)$. Let $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1-\nu$, $\lambda_1 < 1-\nu$. Assume that $a, b \in C([0,1] \times (0,1))$ and*

$$|a(x,y)| + |b(x,y)| \le c y^{-\lambda_0}(1-y)^{-\lambda_1}, \quad (x,y) \in [0,1] \times (0,1), \tag{2.2}$$

where $c = c(a,b)$ is a positive constant. Then, T maps $C[0,1]$ into itself, and $T : C[0,1] \to C[0,1]$ is compact.

For $m \in \mathbb{N}$, $\theta_0, \theta_1 \in \mathbb{R}$, $\theta_0 < 1$, $\theta_1 < 1$, denote by $C^{m,\theta_0,\theta_1}(0,1)$ the weighted space of functions $u \in C[0,1] \cap C^m(0,1)$ satisfying for $x \in (0,1)$, $k = 1, \ldots, m$, the inequalities

$$\left|u^{(k)}(x)\right| \le c \begin{cases} 1 & \text{for } \theta_0 < 1-k, \\ 1 + |\log x| & \text{for } \theta_0 = 1-k, \\ x^{1-k-\theta_0} & \text{for } \theta_0 > 1-k, \end{cases}$$

$$\left|u^{(k)}(1-x)\right| \le c \begin{cases} 1 & \text{for } \theta_1 < 1-k, \\ 1 + |\log(1-x)| & \text{for } \theta_1 = 1-k, \\ (1-x)^{1-k-\theta_1} & \text{for } \theta_1 > 1-k, \end{cases}$$

where $c = c(u) > 0$ is a constant, in other words, $u \in C^{m,\theta_0,\theta_1}(0,1)$ if

$$|u|_{m,\theta_0,\theta_1} := \sum_{k=1}^{m} \sup_{0<x<1} \omega_{k-1+\theta_0}(x)\omega_{k-1+\theta_1}(1-x)|u^{(k)}(x)| < \infty,$$

where

$$\omega_\rho(r) = \begin{cases} 1 & \text{for } \rho < 0, \\ 1/1 + |\log r| & \text{for } \rho = 0, \\ r^\rho & \text{for } \rho > 0, \end{cases} \quad r,\ \rho \in \mathbb{R},\ r > 0.$$

Equipped with the norm

$$\|u\|_{C^{m,\theta_0,\theta_1}(0,1)} := \max_{0 \le x \le 1} |u(x)| + |u|_{m,\theta_0,\theta_1},$$

$C^{m,\theta_0,\theta_1}(0,1)$ becomes a Banach space. Note that

$$C^m[0,1] \subset C^{m,\theta_0,\theta_1}(0,1) \quad \text{for any } m \in \mathbb{N},\ \theta_0 < 1,\ \theta_1 < 1.$$

We introduce the notation

$$\partial_x^k \partial_y^l = \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^l, \quad k, l \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}.$$

Lemma 2.2. *Let T be defined by* (2.1) *with $\nu \in (0,1)$. Let $m \in \mathbb{N}$ and $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1-\nu$, $\lambda_1 < 1-\nu$. Assume that a, $b \in C^m([0,1]\times(0,1))$ and*

$$\left|\partial_x^k \partial_y^l a\,(x,y)\right| + \left|\partial_x^k \partial_y^l b\,(x,y)\right| \le c y^{-\lambda_0 - l}(1-y)^{-\lambda_1 - l}, \tag{2.3}$$

with $(x,y) \in [0,1] \times (0,1)$ and a positive constant $c = c(a,b)$ for all $k, l \in \mathbb{N}_0$ such that $k + l \le m$. Then operator T maps $C^{m,\theta_0,\theta_1}(0,1)$ with $\theta_0 = \nu + \lambda_0$ and $\theta_1 = \nu + \lambda_1$ into itself, and $T : C^{m,\theta_0,\theta_1}(0,1) \to C^{m,\theta_0,\theta_1}(0,1)$ is compact.

Denote $\mathcal{N}(I-T) = \{u \in C[0,1] : u = Tu\}$. The following theorem is a consequence of Lemmas 2.1 and 2.2.

Theorem 2.3. *Assume the conditions of Lemma* 2.2 *and $\mathcal{N}(I-T) = \{0\}$. Let $f \in C^{m,\theta_0,\theta_1}(0,1)$, $\theta_0 = \nu+\lambda_0$ and $\theta_1 = \nu+\lambda_1$. Then, equation* (1.1) *has a solution $u \in C^{m,\theta_0,\theta_1}(0,1)$ which is unique in $C[0,1]$, and*

$$\|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \le c\,\|f\|_{C^{m,\theta_0,\theta_1}(0,1)}\,, \tag{2.4}$$

with a constant c which is independent of f.

3. Smoothing transformation.

3. Smoothing transformation. Possible boundary singularities of the solution $u \in C^{m,\nu+\lambda_0,\nu+\lambda_1}(0,1)$ of equation (1.1) are generic; they occur for most of the free terms f even when f has no boundary singularities. In order to suppress the singularities of the solution we perform the following change of variables in equation (1.1):

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \le t \le 1,\ 0 \le s \le 1. \tag{3.1}$$

Here, $\varphi : [0,1] \to [0,1]$ is defined by the formula

$$\varphi(t) = \frac{1}{c_*} \int_0^t \sigma^{p_0 - 1}(1-\sigma)^{p_1 - 1}\, d\sigma, \quad 0 \le t \le 1,\ p_0, p_1 \in \mathbb{N}, \tag{3.2}$$

$$c_* = \int_0^1 \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma = B(p_0,p_1) = \frac{(p_0-1)!(p_1-1)!}{(p_0+p_1-1)!},$$

where B is the Euler beta function. Representing

$$(1-\sigma)^{p_1-1} = \sum_{k=0}^{p_1-1}(-1)^k \binom{p_1-1}{k}\sigma^k$$

and integrating, we see that φ is a polynomial,

$$\varphi(t) = \frac{1}{c_*}t^{p_0}\sum_{k=0}^{p_1-1}(-1)^k\frac{1}{k+p_0}\binom{p_1-1}{k}t^k.$$

Observe that

$$\begin{aligned} &0 \le \varphi(t) \le c_0 t^{p_0}, \\ &0 \le 1-\varphi(t) \le c'(1-t)^{p_1}, \quad 0 \le t \le 1, \\ \left|\varphi^{(k)}(t)\right| &\le c_k t^{p_0-k}(1-t)^{p_1-k}, \\ &0 < t < 1,\ k = 1,\dots,m,\ m \in \mathbb{N}. \end{aligned} \tag{3.3}$$

If $p_0 = p_1 = 1$, then $\varphi(t) = t$ for $0 \le t \le 1$. We are interested in transformations (3.2) with $p_0 > 1$ or/and $p_1 > 1$ since then the transformation (3.2) possesses a smoothing property for functions $u(x)$ with singularities of derivatives of $u(x)$ at $x = 0$ or/and $x = 1$, see Lemma 3.1; the proof may be found in [**35**].

Lemma 3.1. *Let $m \in \mathbb{N}$, $\theta_0, \theta_1 \in \mathbb{R}$, $\theta_0 < 1$, $\theta_1 < 1$. Let $u \in C^{m,\theta_0,\theta_1}(0,1)$ and $v(t) = u(\varphi(t))$, with φ defined by* (3.2). *Then, for $j = 1,\dots,m$, $0 < t < 1$,*

$$\left|v^{(j)}(t)\right| \le c\,\|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \begin{cases} t^{p_0-j} & \theta_0 < 0 \\ t^{p_0-j}(1+|\log t|) & \theta_0 = 0 \\ t^{(1-\theta_0)p_0-j} & \theta_0 > 0 \end{cases}$$

$$\times \begin{cases} (1-t)^{p_1-j} & \theta_1 < 0 \\ (1-t)^{p_1-j}(1+|\log(1-t)|) & \theta_1 = 0 \\ (1-t)^{(1-\theta_1)p_1-j} & \theta_1 > 0. \end{cases}$$

Theorem 3.2. *Let* $m \in \mathbb{N}$, $0 < \nu < 1$, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1 - \nu$, $\lambda_1 < 1 - \nu$. *Let* $u \in C^{m,\nu+\lambda_0,\nu+\lambda_1}(0,1)$ *and* $v(t) = u(\varphi(t))$, $t \in [0,1]$, *where* φ *is defined by* (3.2) *with the parameters* $p_0, p_1 \in \mathbb{N}$ *satisfying*

$$(3.4) \qquad p_i > \begin{cases} m & \text{for } \nu + \lambda_i \le 0, \\ m/(1-\nu-\lambda_i) & \text{for } 0 < \nu + \lambda_i < 1, \end{cases} \qquad i = 0,\ i = 1.$$

Then $v \in C^m[0,1]$ *and*

$$(3.5) \qquad v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1, \dots, m.$$

Proof. Let $u \in C^{m,\nu+\lambda_0,\nu+\lambda_1}(0,1)$ and $v(t) = u(\varphi(t))$, $t \in [0,1]$. Clearly, $v \in C[0,1] \cap C^m(0,1)$. Due to (3.4) and based upon Lemma 3.1, we obtain

$$v^{(j)}(0) := \lim_{t \to 0} v^{(j)}(t) = 0,$$
$$v^{(j)}(1) := \lim_{t \to 1} v^{(j)}(t) = 0, \quad j = 1, \dots, m,$$

in other words, the derivatives of v up to order m can be extended to the interval $[0,1]$ so that $v \in C^m[0,1]$, and (3.5) holds. $\square$

From (3.2), we see that $\varphi(0) = 0$, $\varphi(1) = 1$ and φ is strictly increasing. Hence, for $s \ne t$, we have

$$\frac{\varphi(t) - \varphi(s)}{t - s} > 0,$$
$$|\varphi(t) - \varphi(s)|^{-\nu} = \left(\frac{\varphi(t) - \varphi(s)}{t - s} \right)^{-\nu} |t - s|^{-\nu}.$$

After the change of variables (3.1), equation (1.1) takes the form

$$(3.6) \qquad v(t) = \int_0^1 K_\varphi(t,s) v(s)\, ds + f_\varphi(t), \qquad 0 \le t \le 1, \quad 0 < \nu < 1,$$

where $f_\varphi(t) = f(\varphi(t))$,

$$(3.7) \qquad K_\varphi(t,s) = \mathcal{A}(t,s)|t - s|^{-\nu} + \mathcal{B}(t,s),$$

$$(3.8) \qquad \mathcal{A}(t,s) = a(\varphi(t), \varphi(s))\Phi(t,s)^{-\nu}\varphi'(s),$$

$$(3.9) \qquad \mathcal{B}(t,s) = b(\varphi(t), \varphi(s))\varphi'(s)$$

and

$$\Phi(t,s) = \begin{cases} \varphi(t) - \varphi(s)/(t-s) & \text{for } t \neq s, \\ \varphi'(s) & \text{for } t = s, \end{cases} \qquad 0 \le t,\ s \le 1. \tag{3.10}$$

The solutions of (1.1) and (3.6) are related by

$$v(t) = u(\varphi(t)), \qquad u(x) = v(\varphi^{-1}(x)).$$

It follows from (3.2) and (3.10) that $\Phi(t,s) > 0$ everywhere in the square $0 \le t,\ s \le 1$ except two points $(0,0)$ and $(1,1)$ in which Φ vanishes causing singularities of $\Phi(t,s)^{-\nu}$. Note that Φ is a polynomial in s,t, since φ is a polynomial and $p_0, p_1 \in \mathbb{N}$. According to (3.2), (3.3) and (3.10),

$$\Phi(t,s) = \Phi(s,t), \quad 0 \le t, s \le 1.$$

Further, we have

$$\Phi\,(t,s) \asymp (t+s)^{p_0-1}\,((1-t)+(1-s))^{p_1-1} \quad \text{as } t,s \to 0 \text{ or as } t,s \to 1, \tag{3.11}$$

where $\Phi(t,s) \asymp \Psi(t,s)$ as $t,s \to 0$ or $t,s \to 1$ means that $\Phi(t,s)/\Psi(t,s)$ and $\Psi(t,s)/\Phi(t,s)$ are bounded as $t,s \to 0$ or $t,s \to 1$.

Indeed, let $0 \le s < t \le 1/2$. According to (3.2) and (3.10), the following holds:

$$\Phi(t,s) = \frac{1}{t-s}\int_s^t \varphi'(\sigma)\,d\sigma = \frac{1}{c_*}\frac{1}{t-s}\int_s^t \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma.$$

Thus,

$$\Phi(t,s) \le \frac{1}{c_*}\frac{1}{t-s}\int_s^t \sigma^{p_0-1}d\sigma = \frac{1}{c_*}\frac{1}{p_0}\frac{t^{p_0}-s^{p_0}}{t-s}.$$

By Lagrange's mean value theorem we obtain

$$\frac{t^{p_0}-s^{p_0}}{t-s} \le p_0 t^{p_0-1} \le p_0(t+s)^{p_0-1},$$

and therefore, $\Phi(t,s) \le (t+s)^{p_0-1}/c_*$. Below, we also see that $\Phi(t,s) \ge c'(t+s)^{p_0-1}$ for a positive constant $c' > 0$ and $0 \le s < t \le 1/2$.

Indeed, for $0 \le s < t \le \delta < 1$, the following holds:

$$\Phi(t,s) = \frac{1}{t-s}\int_s^t \varphi'(\sigma)\,d\sigma = \frac{1}{c_*(t-s)}\int_s^t \sigma^{p_0-1}(1-\sigma)^{p_1-1}d\sigma$$
$$\ge \frac{(1-\delta)^{p_1-1}}{c_* p_0}\left(\frac{t^{p_0}-s^{p_0}}{t-s}\right).$$

Since $t^{p_0} - s^{p_0}/(t-s) \ge t^{p_0-1}$ for $0 \le s < t$, $p_0 \ge 1$, we obtain

$$\Phi(t,s) \ge c_\delta t^{p_0-1} \ge \frac{c_\delta}{2^{p_0-1}}(t+s)^{p_0-1} \text{ with } c_\delta = \frac{(1-\delta)^{p_1-1}}{c_* p_0}.$$

Therefore,

$$\Phi\,(t,s) \asymp (t+s)^{p_0-1} \quad \text{as } t,s \to 0.$$

In a similar manner, we obtain that $\Phi(t,s)\asymp[(1-t)+(1-s)]^{p_1-1}$ as $t,s \to 1$, and (3.11) follows.

Using (3.3), we obtain for $0 \le s,\, t \le 1$, $k = 1,\dots,m$, $m \in \mathbb{N}$, that

$$\left|\partial_s^k \Phi(t,s)\right| \le c(t+s)^{p_0-k-1}\left[(1-t)+(1-s)\right]^{p_1-k-1}. \tag{3.12}$$

Let g be an m times continuously differentiable function on an interval which contains the values of a function $\psi \in C^m[0,1]$. Then the composite function $g(\psi(s))$ is m times continuously differentiable on $[0,1]$, and it can be expressed by the Faà di Bruno differentiation formula, see e.g., **[16]**,

$$\left(\frac{d}{ds}\right)^j g(\psi(s)) = \sum \frac{j!}{k_1!\cdots k_j!} g^{(k_1+\ldots+k_j)}(\psi(s))\left(\frac{\psi'(s)}{1!}\right)^{k_1} \cdots \left(\frac{\psi^{(j)}(s)}{j!}\right)^{k_j}, \tag{3.13}$$

where $s \in [0,1]$, $j = 1,\dots,m$, and the sum is taken over all non-negative integers $k_1,\dots,k_j$ such that $k_1 + 2k_2 + \cdots + jk_j = j$.

Due to (3.11), (3.12) and (3.13), the following lemma holds.

Lemma 3.3. *For* $j = 0, \dots, m$, $m \in \mathbb{N}_0$, $0 \le t \le 1$, $0 < s < 1$, *the following holds*:
(3.14)
$$\left|\partial_s^j \left(\Phi(t,s)^{-\nu}\right)\right| \le c(t+s)^{-\nu(p_0-1)-j}\left[(1-t)+(1-s)\right]^{-\nu(p_1-1)-j}.$$

Since the factor $\varphi'(s) = (1/c_*)s^{p_0-1}(1-s)^{p_1-1}$ damps the singularities, it holds for $j = 0, \dots, m$ that

$$\left|\partial_s^j \left(\Phi(t,s)^{-\nu}\right)\varphi'(s)\right| \le cs^{(p_0-1)(1-\nu)-j}(1-s)^{(p_1-1)(1-\nu)-j}.$$

Lemma 3.4. *Let* a *and* b *satisfy the conditions of Lemma* 2.2, *and let* φ *be defined by* (3.2). *Then, for* $j = 0, \dots, m$, $0 \le t \le 1$, $0 < s < 1$, *the following holds*:

$$\left|\partial_s^j a(\varphi(t),\varphi(s))\right| + \left|\partial_s^j b(\varphi(t),\varphi(s))\right| \le cs^{-p_0\lambda_0-j}(1-s)^{-p_1\lambda_1-j}. \tag{3.15}$$

Proof. Estimate (3.15) is a consequence of (2.3), (3.2) and the formula of Faà di Bruno (3.13). □

Lemma 3.5. *Let* a *and* b *satisfy the conditions of Lemma 2.1. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be defined by the formula* (3.8). *Let* φ *be defined by* (3.2). *Then the following holds true.*

(i) *If*

$$p_0 > (1-\nu)/(1-\nu-\lambda_0), \qquad p_1 > (1-\nu)/(1-\nu-\lambda_1), \tag{3.16}$$

then with $\delta_0 := (1-\nu-\lambda_0)p_0-(1-\nu) > 0$, $\delta_1 := (1-\nu-\lambda_1)p_1-(1-\nu) > 0$, *the following holds*:

$$|\mathcal{A}(t,s)| \le cs^{\delta_0}(1-s)^{\delta_1}, \quad (t,s) \in [0,1]\times(0,1). \tag{3.17}$$

(ii) *If*

$$p_0 > 1/(1-\lambda_0), \qquad p_1 > 1/(1-\lambda_1), \tag{3.18}$$

then, with $\delta_0 := (1-\lambda_0)p_0 - 1 > 0$, $\delta_1 := (1-\lambda_1)p_1 - 1 > 0$, *the following holds*:

$$|\mathcal{B}(t,s)| \le cs^{\delta_0}(1-s)^{\delta_1}, \quad (t,s) \in [0,1]\times(0,1). \tag{3.19}$$

Proof. By inequalities (3.3), (3.15) and (3.14), we have

$$|\mathcal{A}(t,s)| \leq cs^{-p_0\lambda_0+p_0-1}(t+s)^{-\nu p_0+\nu}(1-s)^{-p_1\lambda_1+p_1-1}(2-t-s)^{-\nu p_1+\nu}$$

that, for p_0, p_1 satisfying (3.16), which yields (3.17). Similarly, by (3.3) and (3.15),

$$|\mathcal{B}(t,s)| \leq cs^{-p_0\lambda_0+p_0-1}(1-s)^{-p_1\lambda_1+p_1-1}$$

for p_0, p_1 satisfying (3.18), we obtain (3.19). □

From (3.17) and (3.19), we can define $\mathcal{A}(t,s) = 0$ and $\mathcal{B}(t,s) = 0$ for $t \in [0,1]$, $s = 0$ and, for $t \in [0,1]$, $s = 1$. Moreover, we extend $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$ with respect to s outside $[0,1]$ by the value of zero. The corresponding extensions of $\mathcal{A}$ and $\mathcal{B}$ will again be denoted by $\mathcal{A}$ and $\mathcal{B}$. Thus, under conditions (3.16) and (3.18), we obtain that

$$\mathcal{A}, \mathcal{B} \in C([0,1] \times [-\delta, 1+\delta]) \quad \text{for any } \delta \geq 0. \tag{3.20}$$

Under the conditions of Theorems 2.3 and 3.2, the solution $v(t) = u(\varphi(t))$, $t \in [0,1]$, of (3.6) belongs to $C^m[0,1]$ and satisfies (3.5). Continuing v for $t < 0$ by the constant value $v(0)$ and for $t > 1$ by the constant value $v(1)$, the extended function belongs to $C^m(\mathbb{R})$. This circumstance is helpful for the central part interpolation on the uniform grid by polynomials and piecewise polynomials treated in the next two sections.

Remark 3.6. While the change of variables (3.1) eliminates the boundary singularities of K_φ, f_φ and exact solution v of equation (3.6), the diagonal singularity of the kernel K_φ will still be present, see (3.7).

Remark 3.7. Instead of (3.2), other transformations may also be used. We refer the reader to **[26, 28]** for a general discussion in this connection, see also, **[7, 10, 14, 19, 22, 33]**.

4. Central part interpolation by polynomials.

4. Central part interpolation by polynomials. Given an interval $[a,b]$, $a < b$, and $m \in \mathbb{N}$, we introduce the uniform grid consisting of m points

$$x_i = a + \left(i - \frac{1}{2}\right)h, \quad i = 1, \ldots, m, \; h = \frac{b-a}{m}. \tag{4.1}$$

Denote by $\mathcal{P}_{m-1}$ the set of polynomials of degree not exceeding $m-1$ and by Π_m the Lagrange interpolation projection operator assigning to any $g \in C[a,b]$ the polynomial $\Pi_m g \in \mathcal{P}_{m-1}$ which interpolates g at points (4.1):

$$(\Pi_m g)(x) = \sum_{j=1}^{m} g(x_j) \prod_{\substack{k=1 \\ k\neq j}}^{m} \frac{x - x_k}{x_j - x_k}, \quad a \le x \le b,\ m \ge 2,$$

$$(\Pi_1 g)(x) = g(x_1), \quad a \le x \le b.$$

Lemma 4.1. *In the case of interpolation knots* (4.1) *with* $m \in \mathbb{N}$, *for* $g \in C^m[a,b]$, *the following holds*:

$$\max_{a\le x\le b} |g(x) - (\Pi_m g)(x)| \le \theta_m h^m \max_{a\le x\le b} |g^{(m)}(x)|, \tag{4.2}$$

with

$$\theta_m = \frac{(2m)!}{2^{2m}(m!)^2} \cong (\pi m)^{-1/2},$$

where $\theta_m \cong \epsilon_m$ *means that* $\theta_m/\epsilon_m \to 1$ *as* $m \to \infty$.

Further, for $m = 2k$, $k \ge 1$, *the non-improvable estimate*

$$\max_{x_k\le x\le x_{k+1}} |g(x) - (\Pi_m g)(x)| \le \vartheta_m h^m \max_{a\le x\le b} |g^{(m)}(x)|, \tag{4.3}$$

holds with

$$\vartheta_m = 2^{-2m} \frac{m!}{((m/2)!)^2} \cong \sqrt{2/\pi}\, m^{-1/2} 2^{-m}, \tag{4.4}$$

whereas, for $m = 2k+1$, $k \ge 1$, *the non-improvable estimate*

$$\max_{x_k\le x\le x_{k+2}} |g(x) - (\Pi_m g)(x)| \le \vartheta_m h^m \max_{a\le x\le b} |g^{(m)}(x)|, \tag{4.5}$$

holds with

$$\vartheta_m = \frac{2\sqrt{3}}{9} \frac{(k!)^2}{(2k+1)!} \cong \frac{2\sqrt{6\pi}}{9} m^{-1/2} 2^{-m}. \tag{4.6}$$

Remark 4.2. Without a detailed proof, similar results were formulated in **[23]**.

Proof. These estimates are consequences of the following well-known error formula, see, e.g., **[30]**:

$$g(x) - (\Pi_m g)(x) = \frac{g^{(m)}(\xi)}{m!}(x - x_1)\dots(x - x_m),$$
$$x \in [a, b], \quad \xi \in (a, b).$$

Indeed, for points (4.1), the maximum of $|(x - x_1)\cdots(x - x_m)|$ on $[a, b]$ is attained at the end points of the interval; thus,

$$\max_{a\le t\le b} |(x - x_1)\dots(x - x_m)| = \frac{h}{2}\cdot\frac{3}{2}h\cdot\dots\cdot\frac{2m-1}{2}h$$
$$= \frac{1\cdot 3\cdot 5\cdot\dots\cdot(2m-1)}{2^m}h^m,$$

and (4.2) holds with

$$\theta_m = \frac{1\cdot 3\cdot\dots\cdot(2m-1)}{2^m\, m!} = \frac{(2m)!}{2^{2m}\,(m!)^2}.$$

The Stirling formula

$$m! \cong \sqrt{2\pi m}\, m^m e^{-m} \tag{4.7}$$

yields

$$\theta_m \cong \frac{2\sqrt{\pi m}\,(2m)^{2m}e^{-2m}}{2^m\sqrt{2\pi m}\, m^m e^{-m} 2^m\sqrt{2\pi m}\, m^m e^{-m}} = \frac{2\sqrt{\pi m}}{2\pi m} = (\pi m)^{-1/2}.$$

We now prove (4.3) and (4.4) for $m = 2k$, $k \in \mathbb{N}$. Note that the maximum of $|(x - x_1)\cdots(x - x_{2k})|$ on $[x_k, x_{k+1}]$ is attained at the center of $[x_k, x_{k+1}]$ and equals

$$\left(\frac{1}{2}h\right)^2\left(\frac{3}{2}h\right)^2\dots\left(\frac{2k-1}{2}h\right)^2 = \frac{[1\cdot 3\cdot 5\cdot\dots\cdot(2k-1)]^2}{2^{2k}}h^m.$$

Thus, (4.3) holds with

$$\vartheta_m = \frac{[1\cdot 3\cdot\dots\cdot(2k-1)]^2}{2^m\, m!} = \frac{[(2k)!]^2}{2^m\, m!(2\cdot 4\cdot\dots\cdot 2k)^2} = \frac{m!}{2^{2m}\,[(m/2)!]^2}.$$

This, together with (4.7), yields (4.4):

$$\vartheta_m \cong \frac{\sqrt{2\pi m}\, m^m e^{-m}}{2^{2m}\left[\sqrt{2\pi(m/2)}\ (m/2)^{m/2}\, e^{-m/2}\right]^2} = \frac{\sqrt{2}\sqrt{\pi m}\, m^m\, e^{-m}}{2^{2m}\, \pi m\, m^m\, 2^{-m}\, e^{-m}}$$

$$= 2^{-m} m^{-1/2}\left(\frac{2}{\pi}\right)^{1/2}.$$

Finally, we will prove (4.5). Let $m = 2k+1, k \geq 1$. We estimate separately $|(x-x_k)(x-x_{k+1})(x-x_{k+2})|$ and $|x-x_1|\cdots|x-x_{k-1}||x-x_{k+3}|\cdots|x-x_m|$ on the interval $[x_k, x_{k+2}]$. Taking $x - x_{k+1} = y$, we have

$$\max_{x_k \leq x \leq x_{k+2}} |(x-x_k)(x-x_{k+1})(x-x_{k+2})| = \max_{-h\leq y\leq h} |(y-h)y(y+h)|$$

$$= \max_{-h\leq y\leq h} |y^3 - h^2 y|.$$

The cubic function $\phi(y) = y^3 - h^2 y$ vanishes at $-h$, 0 and h, has a local maximum at $y = -(\sqrt{3}/3)h$ in $[-h, h]$ with

$$\phi\left(-\frac{\sqrt{3}}{3}h\right) = \frac{2\sqrt{3}}{9}h^3,$$

and a local minimum at

$$y = \frac{\sqrt{3}}{3}h$$

with

$$\phi\left(\frac{\sqrt{3}}{3}h\right) = -\frac{2\sqrt{3}}{9}h^3.$$

Thus,

$$\max_{x_k \leq x \leq x_{k+1}} |(x-x_k)(x-x_{k+1})(x-x_{k+2})| = \frac{2\sqrt{3}}{9}h^3.$$

The maximum of $|x-x_1|\cdots|x-x_{k-1}||(x-x_{k+3})|\cdots|(x-x_m)|$ on $[x_k, x_{k+2}]$ is attained at the center x_{k+1} of $[x_k, x_{k+2}]$ and equals

$$(2h)^2(3h)^2\cdots(kh)^2 = (k!)^2 h^{2(k-1)} = (k!)^2 h^{m-3}.$$

This results in the estimate

$$\max_{x_k \le x \le x_{k+2}} |(x-x_1)\cdots(x-x_m)| \le \frac{2\sqrt{3}}{9}(k!)^2 h^m$$

and (4.5) with

$$\vartheta_m = \frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!}.$$

Due to the Stirling formula (4.7),

$$\begin{aligned}\vartheta_m &= \frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!}\\ &\cong \frac{2\sqrt{3}}{9}\frac{2\pi\, k\, k^{2k}e^{-2k}}{\sqrt{2\pi(2k+1)}\,(2k+1)^{2k+1}e^{-(2k+1)}}\\ &= \frac{2\sqrt{3}}{9}\frac{\sqrt{2\pi}}{\sqrt{2k+1}\,e^{-1}}\left(\frac{k}{2k+1}\right)^{2k+1}.\end{aligned}$$

Since

$$\frac{k}{2k+1} = \frac{k}{2k}\left(\frac{2k}{2k+1}\right) = \frac{1}{2}\left(1-\frac{1}{2k+1}\right),$$

$$\left(\frac{k}{2k+1}\right)^{2k+1} = \left(\frac{1}{2}\right)^{2k+1}\left(1-\frac{1}{2k+1}\right)^{2k+1} \cong \left(\frac{1}{2}\right)^{2k+1} e^{-1},$$

we obtain (4.6) due to:

$$\begin{aligned}\frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!} &\cong \frac{2\sqrt{3}}{9\,e^{-1}}\frac{\sqrt{2\pi}}{\sqrt{2k+1}}\left(\frac{k}{2k+1}\right)^{2k+1}\\ &\cong \frac{2\sqrt{6\pi}}{9}\, m^{-1/2}2^{-m}.\end{aligned}$$

□

In what follows, we will denote by $\mathcal{L}(X,Y)$ the Banach space of linear bounded operators A from a Banach space X into a Banach space Y with the norm $\|A\|_{\mathcal{L}(X,Y)} = \sup_{x\in X, \|x\|_X \le 1} \|Ax\|_Y$.

Comparing estimates (4.2)–(4.5), we observe that, in the central parts of $[a,b]$, estimates for the error $g - \Pi_m g$ are approximately 2^m times more precise than on the entire interval. In the central parts of $[a,b]$, the interpolation process on the uniform grid also has good stability properties as m increases; in contrast to the exponential growth, see **[9]**, of $\|\Pi_m\|_{\mathcal{L}(C[a,b],C[a,b])}$ as $m \to \infty$, the following holds for $rh^{1/2} \le (b-a)/2$ by Runck's theorem (see **[9, 31, 32]**):

$$\|\Pi_m\|_{\mathcal{L}(C[a,b],C[(a+b)/2-rh^{1/2},(a+b)/2+rh^{1/2}])} \le c_r(1+\log m), \tag{4.8}$$

where the constant c_r depends only upon $r > 0$.

As is well known, logarithmic growth is the slowest which holds for projectors $P_m : C[a,b] \to \mathcal{P}_{m-1}$ and, for example, Chebyshev interpolation projectors $\overline{\Pi}_m$ have this slowest growth.

Recall, see e.g, **[9, 30]**, that the Chebyshev interpolant $\overline{\Pi}_m g$ is a polynomial of degree $m-1$ that interpolates $g \in C[a,b]$ at m distinct knots $\{x_i\} \subset (a,b)$:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2i-1}{2m}\pi\right), \quad i = 1,\dots,m. \tag{4.9}$$

Actually, these knots are zeros of the Chebyshev polynomial of the first kind $T_m((2x-a-b)/(b-a))$, where:

$$T_m(t) = \cos(m \arccos t), \quad t \in [-1,1], \ m = 0,1,\dots\,. \tag{4.10}$$

In the case of Chebyshev knots (4.9) the non-improvable estimate

$$\max_{a\le x\le b} \left|g(x) - (\overline{\Pi}_m g)(x)\right| \le \frac{2(b-a)^m}{m!\,4^m} \max_{a\le x\le b} |g^{(m)}(x)|, \quad g \in C^m[a,b] \tag{4.11}$$

holds, and

$$\|\overline{\Pi}_m\|_{\mathcal{L}(C[a,b],C[a,b])} \le 8 + \frac{4}{\pi}\log m, \quad m = 1,2,\dots\,.$$

On the other hand, it is known that, for any projection operator $P_m : C[a,b] \to \mathcal{P}_{m-1}$, that is, for any operator $P_m : C[a,b] \to C[a,b]$ such that $P_m^2 = P_m$ and the range $\mathcal{R}(P_m) = \mathcal{P}_{m-1}$, the following holds:

$$\|P_m\|_{\mathcal{L}(C[a,b],C[a,b])} \ge c_0(1+\log m), \quad m = 1,2,\dots,$$

where $c_0 > 0$ is independent of m, see e.g, **[9, 30]**. Thus, in the case of Chebyshev knots (4.9), the norm $\|\overline{\Pi}_m\|_{\mathcal{L}(C[a,b],C[a,b])}$ is of minimal possible growth order as $m \to \infty$.

Further, in the central part $[x_k, x_{k+1}]$ or $[x_k, x_{k+2}]$, the interpolation error $g - \Pi_m g$ is even smaller than the error on $[a,b]$ of the Chebyshev interpolant $\overline{\Pi}_m g$ of degree $m-1$. Indeed, with respect to $b-a=mh$ the estimate (4.11) for $g \in C^m[a,b]$ reads:

$$\max_{a\le x\le b}\left|g(x)-(\overline{\Pi}_m g)(x)\right| \le \overline{\theta}_m h^m \max_{a\le x\le b}\left|g^{(m)}(x)\right|, \tag{4.12}$$

$$\overline{\theta}_m = \frac{2\,m^m}{m!\,4^m}. \tag{4.13}$$

Therefore, due to (4.4), for even m we have

$$\frac{\overline{\theta}_m}{\vartheta_m} = \frac{2^{2m+1}m^m\,((m/2)!)^2}{4^m\,(m!)^2} = \frac{2\,m^m\,((m/2)!)^2}{(m!)^2} \cong \left(\frac{e}{2}\right)^m. \tag{4.14}$$

For instance, $\overline{\theta}_4/\vartheta_4 = 32/9$ for $m=4$, and the Chebyshev interpolant $\overline{\Pi}_m g$ is approximately 3.56 times coarser than the central part of the interpolant $\Pi_m g$.

Due to (4.6) for odd m, we have

$$\frac{\overline{\theta}_m}{\vartheta_m} = \frac{2\,m^m\,9\,m!}{m!\,4^m 2\sqrt{3}(((m-1)/2)!)^2} \cong \frac{3\sqrt{3}}{2\pi}\left(\frac{e}{2}\right)^m. \tag{4.15}$$

For instance, $\overline{\theta}_5/\vartheta_5 = (9375\sqrt{3})/4096$ for $m=5$, and Chebyshev interpolant $\overline{\Pi}_m g$ is approximately 3.96 times coarser than the central part of the interpolant $\Pi_m g$.

5. Central part interpolation by piecewise polynomials. We introduce in $\mathbb{R}$ the uniform grid:

$$\{jh : j\in\mathbb{Z}\}, \quad h=\frac{1}{n},\ n\in\mathbb{N}. \tag{5.1}$$

Let $m\in\mathbb{N}$, $m\ge 2$, be fixed. Given a function $g\in C[-\delta,1+\delta]$, $\delta>0$, we define a piecewise polynomial interpolant $\Pi_{h,m}g\in C[0,1]$ for $h=1/n<(2\delta)/m$ as follows. On every subinterval, $[jh,(j+1)h]$, $0\le j\le n-1$, the function $\Pi_{h,m}g$ is defined independently from other subintervals as a polynomial $\Pi_{h,m}^{[j]}g\in\mathcal{P}_{m-1}$ of degree $\le m-1$ by the conditions

$$\Pi_{h,m}^{[j]}g(lh)=g(lh), \quad l=j-\frac{m}{2}+1,\dots,j+\frac{m}{2} \quad \text{if } m \text{ is even,}$$

$$\Pi_{h,m}^{[j]}g(lh)=g(lh), \quad l=j-\frac{m-1}{2},\dots,j+\frac{m-1}{2} \quad \text{if } m \text{ is odd.}$$

A unified writing form of these interpolation conditions is

$$\Pi^{[j]}_{h,m} g(lh) = g(lh) \quad \text{for } l \in \mathbb{Z} \text{ such that } l - j \in \mathbb{Z}_m, \tag{5.2}$$

where

$$\mathbb{Z}_m = \left\{ k \in \mathbb{Z} : -\frac{m}{2} < k \leq \frac{m}{2} \right\},$$
$$\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}.$$

For an "interior" knot jh, $1 \leq j \leq n-1$, interpolation conditions (5.2) contain the condition $(\Pi^{[j-1]}_{h,m} g)(jh) = g(jh)$ as well as the condition $(\Pi^{[j]}_{h,m} g)(jh) = g(jh)$; thus, $\Pi_{h,m} g$ is uniquely defined at interior knots, and $\Pi_{h,m} g$ is continuous on $[0,1]$, namely, for the "interior" knots jh, $1 \leq j \leq n-1$, interpolation conditions (5.2) yield

$$(\Pi_{h,m} g)(jh) = g(jh)$$

for $\Pi_{h,m} g$ as a function on $[(j-1)h, jh]$ as well as a function on $[jh, (j+1)h]$. The one side derivatives of the interpolant $\Pi_{h,m} g$ at the interior knots may be different.

We introduce the Lagrange fundamental polynomials $L_{k,m} \in \mathcal{P}_{m-1}$, $k \in \mathbb{Z}_m$, satisfying $L_{k,m}(l) = \delta_{k,l}$ for $l \in \mathbb{Z}_m$, where $\delta_{k,l}$ is the Kronecker symbol, $\delta_{k,l} = 0$ for $k \neq l$ and $\delta_{k,k} = 1$. An explicit formula for $L_{k,m}$ is given by

$$L_{k,m}(t) = \prod_{l \in \mathbb{Z}_m \setminus \{k\}} \frac{t-l}{k-l}, \quad k \in \mathbb{Z}_m. \tag{5.3}$$

For $0 \leq j \leq n-1$, we claim that

$$(\Pi^{[j]}_{h,m} g)(t) = \sum_{k \in \mathbb{Z}_m} g((j+k)h) L_{k,m}(nt - j), \quad t \in [jh, (j+1)h]. \tag{5.4}$$

Indeed, $\Pi^{[j]}_{h,m} g$ defined by (5.4) is really a polynomial of degree $\leq m-1$, and it satisfies interpolation conditions (5.2): for l with $l - j \in \mathbb{Z}_m$, the following holds:

$$(\Pi^{[j]}_{h,m} g)(lh) = \sum_{k \in \mathbb{Z}_m} g((j+k)h) L_{k,m}(l-j)$$

$$= \sum_{k \in \mathbb{Z}_m} g\left((j+k)\,h\right)\delta_{k,l-j}$$
$$= g\left((j+(l-j))\,h\right) = g\left(lh\right).$$

For $m = 2$, the interpolant $\Pi_{h,2}g$ is the usual piecewise linear function joining the pair of points

$$(jh, g\,(jh)) \in \mathbb{R}^2 \quad \text{and} \quad ((j+1)\,h, g\,((j+1)\,h)) \in \mathbb{R}^2$$

for $0 \le j \le n-1$ by a straight line; $\Pi_{h,2}g$ does not use the values of g outside $[0,1]$, and $\Pi_{h,2}g$ is a projection operator in $C[0,1]$, i.e., $\Pi_{h,2}^2 = \Pi_{h,2}$.

For $m \ge 3$, $\Pi_{h,m}g$ uses values of g outside of $[0,1]$. For $g \in C[0,1]$, $\Pi_{h,m}g$ obtains a sense after an extension of g onto $[-\delta, 1+\delta]$ with $\delta \ge (m/2)h$. In our work, we will consider the functions $g \in C^m[0,1]$ that satisfy the boundary conditions (recall Theorem 3.2):

$$g^{(j)}\,(0) = g^{(j)}\,(1) = 0, \quad j = 1, \ldots, m.$$

This fortuitously yields that the simplest extension operator

$$E_\delta : C\,[0,1] \to C\,[-\delta, 1+\delta]\,, \tag{5.5}$$

$$(E_\delta g)\,(t) = \begin{cases} g\,(0) & \text{for } -\delta \le t \le 0, \\ g\,(t) & \text{for } 0 \le t \le 1, \\ g\,(1) & \text{for } 1 \le t \le 1+\delta, \end{cases} \tag{5.6}$$

maintains the smoothness of g. The operator

$$P_{h,m} := \Pi_{h,m} E_\delta : C\,[0,1] \to C\,[0,1] \tag{5.7}$$

is well defined, and $P_{h,m}^2 = P_{h,m}$, i.e., $P_{h,m}$ is a projector in $C[0,1]$.

For $w_h \in \mathcal{R}(P_{h,m})$ (the range of $P_{h,m}$), we have $w_h = P_{h,m}w_h = \Pi_{h,m}E_\delta w_h$ and, due to (5.4), we obtain for $t \in [jh, (j+1)h]$, $j = 0, \ldots, n-1$, that

$$w_h(t) = \sum_{k \in \mathbb{Z}_m} (E_\delta w_h)((j+k)h) L_{k,m}(nt-j), \tag{5.8}$$

where

$$(E_\delta w_h)(ih) = \begin{cases} w_h(ih) & \text{for } i = 0, \dots, n, \\ w_h(0) & \text{for } i < 0, \\ w_h(1) & \text{for } i > n. \end{cases}$$

Thus, $w_h \in \mathcal{R}(P_{h,m})$ is uniquely determined on $[0, 1]$ by its knot values $w_h(ih)$, $i = 0, \dots, n$. We conclude that $\dim \mathcal{R}(P_{h,m}) = n+1$. It is also clear that, for $w_h \in \mathcal{R}(P_{h,m})$, we have $w_h = 0$ if and only if $w_h(ih) = 0$, $i = 0, \dots, n$.

For $g \in C[-\delta, 1+\delta]$, the interpolant $\Pi_{h,m} g$ is closely related to the central part interpolation of g on the uniform grid treated in Section 4. On $[jh, (j+1)h]$, the interpolant $\Pi_{h,m} g = \Pi_{h,m}^{[j]} g$ coincides with the polynomial interpolant $\Pi_m g$ constructed for g on the interval $[a_j, b_j]$, where

$$\begin{aligned} a_j &= \left(j - \frac{m-1}{2}\right) h, \\ b_j &= \left(j + \frac{m+1}{2}\right) h \qquad \text{in the case of even } m, \\ a_j &= \left(j - \frac{m}{2}\right) h, \\ b_j &= \left(j + \frac{m}{2}\right) h \qquad \text{in the case of odd } m. \end{aligned}$$

Moreover, $[jh, (j+1)h]$ is contained in the central part of $[a_j, b_j]$ on which the interpolation error can be estimated by (4.3) and (4.5). In this way, we obtain the following result.

Lemma 5.1.

(i) *For* $g \in C^m[-\delta, 1+\delta]$*,* $m \geq 2$*,* $\delta > 0$*,* $h = 1/n < (2\delta)/m$*,*

$$\max_{0 \leq t \leq 1} |g(t) - (\Pi_{h,m} g)(t)| \leq \vartheta_m h^m \max_{-\delta \leq t \leq 1+\delta} |g^{(m)}(t)|, \tag{5.9}$$

with ϑ_m *defined by* (4.4) *and* (4.6)*, respectively, for even and odd* m*.*

(ii) *Let*

$$V^{(m)} := \left\{ v \in C^m[0,1] : v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1, \dots, m \right\}.$$

Then, for $g \in V^{(m)}$*, the following holds:*

$$\max_{0\le t\le 1} |g(t) - (P_{h,m}g)(t)| \le \vartheta_m h^m \max_{0\le t\le 1} |g^{(m)}(t)|. \tag{5.10}$$

Proof. Claim (i) is a direct consequence of Lemma 4.1. Further, to prove estimate (5.10), we have $E_\delta g \in C^m[-\delta, 1+\delta]$ for $g \in V^{(m)}$ and

$$\max_{-\delta\le t\le \delta} |(E_\delta g)^{(m)}(t)| = \max_{0\le t\le 1} |g^{(m)}(t)|,$$
$$(E_\delta g)(t) = g(t) \quad \text{for } 0 \le t \le 1.$$

Applying (5.9) to $E_\delta g$, yields:

$$\max_{0\le t\le 1} |(E_\delta g)(t) - (\Pi_{h,m} E_\delta g)(t)| \le \vartheta_m h^m \max_{-\delta\le t\le 1+\delta} |(E_\delta g)^{(m)}(t)|.$$

We can rewrite it as

$$\max_{0\le t\le 1} |g(t) - (P_{h,m}g)(t)| \le \vartheta_m h^m \max_{0\le t\le 1} |g^{(m)}(t)|,$$

which completes the proof. □

From (4.8), (5.5), (5.6) and (5.7), we obtain that, with respect to n (with respect to $h = 1/n$) the norms $\|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])}$ are uniformly bounded:

$$\|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])} \le c\,(1 + \log m)\,, \tag{5.11}$$

with a constant c which is independent of h (of n) and of m.

Together with (5.10), noting that $V^{(m)}$ is dense in $C[0,1]$, the Banach-Steinhaus theorem yields the next result.

Lemma 5.2. *For any* $g \in C[0,1]$,

$$\max_{0\le t\le 1} |g(t) - (P_{h,m}g)(t)| \longrightarrow 0 \quad \textit{as } n \to \infty.$$

It is natural to compare the accuracy of $\Pi_{h,m}g$, not with the accuracy of $\overline{\Pi}_{h,m}g$ (the Chebyshev interpolation), but with the accuracy of $\overline{\Pi}_{\bar{h},m}g$, $\overline{h} = mh$, since $\Pi_{h,m}g$ and $\overline{\Pi}_{\bar{h},m}g$ need, respectively, $n+1$ and n values of g. Due to (4.14) and (4.15), we have similar relations between the accuracies of $g - \Pi_{h,m} g$ and of $g - \overline{\Pi}_{\bar{h},m} g$:

$$\frac{\|g - \overline{\Pi}_{\bar{h},m} g\|_\infty}{\|g - \Pi_{h,m} g\|_\infty} \cong \left(\frac{e}{2}\right)^m, \qquad m = 2k,\ k \geq 1;$$
$$\frac{\|g - \overline{\Pi}_{\bar{h},m} g\|_\infty}{\|g - \Pi_{h,m} g\|_\infty} \cong \frac{3\sqrt{3}}{2\pi}\left(\frac{e}{2}\right)^m, \quad m = 2k+1,\ k \geq 1.$$

6. Collocation method based on central part interpolation.

6.1. Operator form of the method and convergence analysis. We rewrite equation (3.6) in the operator form

$$v = T_\varphi v + f_\varphi, \tag{6.1}$$

where operator T_φ is defined by

$$(T_\varphi v)(t) = \int_0^1 K_\varphi(t,s) v(s)\, ds, \quad 0 \leq t \leq 1, \tag{6.2}$$

with $K_\varphi(t,s)$, given by (3.7). Using $P_{h,m}$, see (5.7), we approximate (6.1) by equation

$$v_h = P_{h,m} T_\varphi v_h + P_{h,m} f_\varphi. \tag{6.3}$$

This is the operator form of our first method based on collocation techniques and central part interpolation on the uniform grid.

It follows from (3.4) that (3.16) and (3.18) hold. From (3.20) with $\delta = 0$, we obtain that $\mathcal{A}, \mathcal{B} \in C([0,1] \times [0,1])$. Therefore, T_φ, given by (6.2), is compact as an operator from $C[0,1]$ into $C[0,1]$. Assuming that $\mathcal{N}(I - T) = \{0\}$, or equivalently, $\mathcal{N}(I - T_\varphi) = \{0\}$, the bounded inverse $(I - T_\varphi)^{-1} : C[0,1] \to C[0,1]$ exists due to the Fredholm alternative. Denote

$$\kappa = \|(I - T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1],C[0,1])}.$$

The compactness of $T_\varphi : C[0,1] \to C[0,1]$ and the pointwise convergence $P_{h,m}$ to I (the identity mapping) in $C[0,1]$, see Lemma 5.2, imply the norm convergence

$$\epsilon_h := \|T_\varphi - P_{h,m} T_\varphi\|_{\mathcal{L}(C[0,1],C[0,1])} \longrightarrow 0 \quad \text{as } n \to \infty$$
$$(\text{as } h = 1/n \to 0).$$

Hence, there is an n_0 such that $\kappa\epsilon_h < 1$ for $n > n_0$. We conclude that $I - P_{h,m}T_\varphi$ is invertible in $C[0,1]$ for $n \geq n_0$ and

$$\kappa_h := \left\|(I - P_{h,m}T_\varphi)^{-1}\right\|_{\mathcal{L}(C[0,1],C[0,1])} \longrightarrow \kappa \quad \text{as } n \to \infty \quad (\text{as } h = 1/n \to 0). \tag{6.4}$$

This proves the unique solvability of the collocation equation (6.3) for $n \geq n_0$.

Let v and v_h be the solutions of (6.1) and (6.3), respectively. Then,

$$(I - P_{h,m}T_\varphi)(v - v_h) = v - P_{h,m}v,$$
$$v - v_h = (I - P_{h,m}T_\varphi)^{-1}(v - P_{h,m}v)$$

and

$$\|v - v_h\|_\infty \leq \kappa_h \|v - P_{h,m}v\|_\infty, \quad h = 1/n,\ n \geq n_0. \tag{6.5}$$

Note also that

$$\|v - v_h\|_\infty \geq \frac{1}{\|I - P_{h,m}T_\varphi\|_{\mathcal{L}(C[0,1],C[0,1])}} \|v - P_{h,m}v\|_\infty, \tag{6.6}$$

$$\|I - P_{h,m}T_\varphi\|_{\mathcal{L}(C[0,1],C[0,1])} \longrightarrow \|I - T_\varphi\|_{\mathcal{L}(C[0,1],C[0,1])} \quad \text{as } n \to \infty;$$

thus (6.5) essentially cannot be improved.

Further, let the assumptions of Lemma 2.2 be fulfilled, and let $f \in C^{m,\theta_0,\theta_1}(0,1)$, $m \in \mathbb{N}$, $m \geq 2$, $\theta_0 = \nu + \lambda_0$, $\theta_1 = \nu + \lambda_1$. Then, it follows from Theorem 2.3 that the solution u of (1.1) belongs to $C^{m,\theta_0,\theta_1}(0,1)$. By Theorem 3.2, for $v(t) = u(\varphi(t))$, we have $v \in C^m[0,1]$ and $v^{(j)}(0) = v^{(j)}(1) = 0$, $j = 1,\ldots,m$; by Lemma 5.1 (ii), $\|v - P_{h,m}v\|_\infty \leq \vartheta_m h^m \|v^{(m)}\|_\infty$. Now, (6.5) yields

$$\|v - v_h\|_\infty \leq \kappa_h \vartheta_m h^m \|v^{(m)}\|_\infty, \quad h = 1/n,\ n \geq n_0. \tag{6.7}$$

We summarize the above-obtained results as follows.

Theorem 6.1. *Let the assumptions of Lemma* 2.2 *be fulfilled. Moreover, assume that* $f \in C^{m,\theta_0,\theta_1}(0,1)$, *with* $m \in \mathbb{N}$, $m \geq 2$, $\theta_0 = \nu + \lambda_0$, $\theta_1 = \nu + \lambda_1$. *Let* $\mathcal{N}(I - T) = \{0\}$, *or equivalently,* $\mathcal{N}(I - T_\varphi) = \{0\}$. *Finally, let* φ *be defined by the formula* (3.2) *with parameters* $p_0, p_1 \in \mathbb{N}$ *satisfying* (3.4).

Then, equation (6.1) (*equation* (3.6)) *has a unique solution* $v \in C[0,1]$, *and there exists an* n_0 *such that, for* $n \geq n_0$, *equation* (6.3) *has a unique solution* v_h. *The accuracy of* v_h *can be estimated by* (6.7) *where* κ_h *is defined in* (6.4) *and* ϑ_m *is given by the formulae* (4.4) *and* (4.6) *for even and odd* m, *respectively.*

Remark 6.2. Method (6.3) contains $n+1$ unknowns and is more precise than the standard piecewise polynomial collocation method

$$v_{\bar{h}} = \overline{\Pi}_{\bar{h},m} T_\varphi v_{\bar{h}} + \overline{\Pi}_{\bar{h},m} f_\varphi, \quad \bar{h} = mh, \tag{6.8}$$

with $n = 1/h$ unknowns and arbitrary choice of m collocation points in subintervals $[j\bar{h}, (j+1)\bar{h}]$, even in the case of Chebyshev collocation points. For example, as was just shown, the accuracy of method (6.3) is guided by $\|v - P_{h,m}v\|_\infty$ where v is the solution of the equation (6.1).

Similarly, the accuracy of method (6.8) is guided by $\|v - \overline{\Pi}_{\bar{h},m}v\|_\infty$. At the end of Sections 4 and 5 we compared $\|v - P_{h,m}v\|_\infty$ with $\|v - \overline{\Pi}_{\bar{h},m}v\|_\infty$ in the Chebyshev case: $\|v - \overline{\Pi}_{\bar{h},m}v\|_\infty$ is coarser than $\|v - P_{h,m}v\|_\infty$ approximately $(e/2)^m$ times for even m and approximately $(3\sqrt{3})/(2\pi)(e/2)^m$ times for odd m. For other choices of interpolation points the accuracy of the standard piecewise polynomial collocation method is, as a rule, more coarse. For instance, in the case of uniform location of collocation points, the error is 2^m times coarser than the error $\|v - P_{h,m}v\|_\infty$.

Remark 6.3. With respect to $u_h(x) := v_h(\varphi^{-1}(x))$, $0 \leq x \leq 1$, estimate (6.7) on conditions of Theorem 6.1 reads as

$$\max_{0\leq x\leq 1} |u(x) - u_h(x)| = \max_{0\leq t\leq 1} |v(t) - v_h(t)| \leq ch^m,$$
$$h = 1/n, \quad n \geq n_0,$$

where c is a positive constant which does not depend on n.

6.2. Matrix form of the method. The solution v_h of equation (6.3) belongs to $\mathcal{R}(P_{h,m})$; therefore, the knot values $v_h(ih)$ $(i = 0, \ldots, n)$ uniquely determine v_h. Equation (6.3) is equivalent to a system of linear algebraic equations with respect to $v_h(ih)$, $i = 0, \ldots, n$, and our task is to write this system.

For $w_h \in \mathcal{R}(P_{h,m})$, we have $w_h = 0$ if and only if $w_h(ih) = 0$, $i = 0, \dots, n$. Since $(P_{h,m}w)(ih) = w(ih)$, $i = 0, \dots, n$, equation (6.3) is equivalent to the condition:

$$v_h(ih) = (T_\varphi v_h)(ih) + f_\varphi(ih), \quad i = 0, \dots, n,$$

i.e., $v_h \in \mathcal{R}(P_{h,m})$ satisfies equation (6.1) (equation (3.6)) at the knots ih, $i = 0, \dots, n$. Using representation (5.8) for v_h, we obtain

$$\begin{aligned}(T_\varphi v_h)(ih) &= \int_0^1 K_\varphi(ih, s) v_h(s)\, ds \\ &= \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} K_\varphi(ih, s) v_h(s)\, ds \\ &= \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}_m} \int_{jh}^{(j+1)h} K_\varphi(ih, s) L_{k,m}(ns - j)\, ds (E_\delta v_h)((j+k)h) \\ &= \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}_m} \alpha_{i,j,k} \begin{cases} v_h(0) & \text{for } j + k \le 0, \\ v_h((j+k)h) & \text{for } 1 \le j + k \le n - 1, \\ v_h(1) & \text{for } j + k \ge n. \end{cases}\end{aligned}$$

Thus,

$$(T_\varphi v_h)(ih) = \sum_{l=0}^{n} b_{i,l} v_h(lh), \quad i = 0, \dots, n,$$

where, for $k \in \mathbb{Z}_m$, we denote

$$\alpha_{i,j,k} = \int_{jh}^{(j+1)h} K_\varphi(ih, s) L_{k,m}(ns - j)\, ds, \tag{6.9}$$
$$i = 0, \dots, n, \quad j = 0, \dots, n - 1,$$

$$b_{i,l} = \begin{cases} \sum_{k \in \mathbb{Z}_m} \sum_{\{j: 0 \le j \le n-1, j+k \le 0\}} \alpha_{i,j,k} & \text{for } l = 0 \\ \sum_{k \in \mathbb{Z}_m} \sum_{\{j: 0 \le j \le n-1, j+k = l\}} \alpha_{i,j,k} & \text{for } 1 \le l \le n - 1 \\ \sum_{k \in \mathbb{Z}_m} \sum_{\{j: 0 \le j \le n-1, j+k \ge n\}} \alpha_{i,j,k} & \text{for } l = n \end{cases} \tag{6.10}$$

with $i, l = 0, \ldots, n$. We see that the matrix form of method (6.3) is given by

$$v_h(ih) = \sum_{l=0}^{n} b_{i,l} v_h(lh) + f_\varphi(ih), \quad i = 0, \ldots, n, \tag{6.11}$$

with $b_{i,l}$ defined by (6.9)–(6.10). Having determined $v_h(ih)$, $i = 0, \ldots, n$, through solving the system (6.11), the collocation solution $v_h(t)$ at any intermediate point $t \in [jh, (j+1)h]$, $j = 0, \ldots, n-1$, is given by

$$v_h(t) = \sum_{k \in \mathbb{Z}_m} L_{k,m}(nt - j) \begin{cases} v_h(0) & \text{for } j + k \leq 0, \\ v_h((j+k)h) & \text{for } 1 \leq j + k \leq n - 1, \\ v_h(1) & \text{for } j + k \geq n, \end{cases} \tag{6.12}$$

with $L_{k,m}$, $k \in \mathbb{Z}_m$, defined by (5.3).

7. Product integration based on the central part interpolation.

7.1. Operator form of the method and convergence analysis. We present in the following lemma some estimates for functions $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$, see (3.8), as well as $\partial_s^m[\mathcal{A}(t,s)v(s)]$, $\partial_s^m[\mathcal{B}(t,s)v(s)]$ in a somewhat specific form for the requirements of Theorem 7.2 below.

Lemma 7.1. *Let a and b satisfy the conditions of Lemma* 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be defined by the formulas* (3.8) *and* (3.9), *respectively. Let φ be defined by* (3.2). *Finally assume that $u \in C^{m,\theta_0,\theta_1}(0,1)$, $m \in \mathbb{N}$, $\theta_0 = \nu + \lambda_0$, $\theta_1 = \nu + \lambda_1$, and let $v(t) = u(\varphi(t))$.*

Then, the following estimates hold for $(t,s) \in [0,1] \times (0,1)$.

(i) *If $p_0, p_1 \in \mathbb{N}$*

$$p_0 > m/(1 - \nu - \lambda_0), \quad p_1 > m/(1 - \nu - \lambda_1), \tag{7.1}$$

then, with $\delta_0 := (1-\nu-\lambda_0)p_0 - m > 0$ and $\delta_1 := (1-\nu-\lambda_1)p_1 - m > 0$,

$$|\mathcal{A}(t,s)| \leq c s^{m-(1-\nu)+\delta_0}(1-s)^{m-(1-\nu)+\delta_1} \tag{7.2}$$

and

$$|\partial_s^m[\mathcal{A}(t,s)v(s)]| \leq c s^{-(1-\nu)+\delta_0}(1-s)^{-(1-\nu)+\delta_1} \|u\|_{C^{m,\theta_0,\theta_1}} . \tag{7.3}$$

(ii) *If* $p_0, p_1 \in \mathbb{N}$,

$$p_0 > m/(1-\lambda_0), \qquad p_1 > m/(1-\lambda_1), \tag{7.4}$$

then with $\delta_0 := (1-\lambda_0)p_0 - m > 0$ *and* $\delta_1 := (1-\lambda_1)p_1 - m > 0$, *the following holds*:

$$|\mathcal{B}(t,s)| \le c s^{m-1+\delta_0}(1-s)^{m-1+\delta_1} \tag{7.5}$$

and

$$|\partial_s^m[\mathcal{B}(t,s)v(s)]| \le c s^{-1+\delta_0}(1-s)^{-1+\delta_1} \|u\|_{C^{m,\theta_0,\theta_1}} . \tag{7.6}$$

Proof. These estimates are direct consequences of Lemmas 3.3–3.4. □

We determine the approximate solution v_h for equation (3.6) by solving the following problem:

$$\begin{aligned} v_h(t) = &\int_0^1 |t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v_h(s))\,ds \\ &+ \int_0^1 P_{h,m}(\mathcal{B}(t,s)v_h(s))\,ds + f_\varphi(t), \end{aligned} \tag{7.7}$$

where $0 \le t \le 1$ and $\mathcal{A}, \mathcal{B} \in C([0,1] \times [-\delta, 1+\delta])$, $\delta > 0$, see (3.20). Here, $P_{h,m}$, see (5.7), is applied to the products $\mathcal{A}(t,s)v_h(s)$ and $\mathcal{B}(t,s)v_h(s)$ as functions of s treating t as a parameter. This is the operator form of a product integration method corresponding to the piecewise polynomial central part interpolation on the uniform grid $\{jh : j = 0, \ldots, n\}$, $h = 1/n$, $n \in \mathbb{N}$. The convergence behavior of method (7.7) is characterized by the next theorem.

Theorem 7.2.

(i) *Let* $0 < \nu < 1$, λ_0, $\lambda_1 \in \mathbb{R}$, $\lambda_0 < 1-\nu$, $\lambda_1 < 1-\nu$. *Let* $f \in C[0,1]$. *Assume that* $a, b \in C([0,1] \times (0,1))$ *satisfy* (2.2). *Let* $\mathcal{N}(I-T) = 0$, *with* T *given by* (2.1). *Finally, let* φ *be defined by the formula* (3.2) *with parameters* $p_0, p_1 \in \mathbb{N}$ *such that*

$$p_0 > \max\left\{\frac{1}{1-\lambda_0}, \frac{1-\nu}{1-\nu-\lambda_0}\right\},$$

$$p_1 > \max\left\{\frac{1}{1-\lambda_1}, \frac{1-\nu}{1-\nu-\lambda_1}\right\}.$$

Then, for sufficiently large n, say for $n \geq n_0$, equation (7.7) *has a unique solution $v_h \in C[0,1]$, $h = 1/n$, and*

$$\|v - v_h\|_\infty = \max_{t\in[0,1]} |v(t) - v_h(t)| \longrightarrow 0 \text{ as } n \to \infty, \tag{7.8}$$

where $v \in C[0,1]$ is the solution of (3.6)*.*

(ii) *Let $m \in \mathbb{N}$, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 < 1-\nu$, $\lambda_1 < 1-\nu$, $0 < \nu < 1$. Assume that $a, b \in C^m([0,1]\times(0,1))$ satisfy* (2.3)*. Let $f \in C^{m,\theta_0,\theta_1}(0,1)$ with $\theta_0 = \nu + \lambda_0$, $\theta_1 = \nu + \lambda_1$. Let $\mathcal{N}(I-T) = \{0\}$, for T given by* (2.1)*. Finally, let φ be defined by formula* (3.2) *with parameters p_0 and p_1 satisfying* (7.1) *and* (7.4)*. Then,*

$$\|v - v_h\|_\infty \leq ch^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad n \geq n_0, \tag{7.9}$$

with a positive constant c which is independent of n and f.

Proof. We consider equations (3.6) and (7.7) as operator equations

$$v = \mathcal{T}v + f_\varphi \tag{7.10}$$

and

$$v_h = \mathcal{T}_h v_h + f_\varphi, \tag{7.11}$$

where $f_\varphi(t) = f(\varphi(t))$, $0 \leq t \leq 1$, and $\mathcal{T} = T_\varphi$ and $\mathcal{T}_h$ are defined by the formulae

$$(\mathcal{T}v)(t) = \int_0^1 [|t-s|^{-\nu}\mathcal{A}(t,s) + \mathcal{B}(t,s)]v(s)\,ds, \quad 0 \leq t \leq 1, \tag{7.12}$$

(7.13)

$$(\mathcal{T}_h v)(t) = \int_0^1 [|t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v(s)) + P_{h,m}(\mathcal{B}(t,s)v(s))]\,ds,$$

where $0 \leq t \leq 1$. Since $f \in C[0,1]$, $f_\varphi(t) = f(\varphi(t))$, $0 \leq t \leq 1$, it follows from (3.2) that $f_\varphi \in C[0,1]$. Since $\mathcal{A}, \mathcal{B} \in C([0,1]\times[0,1])$, we obtain that $\mathcal{T}$ and $\mathcal{T}_h$ are compact as operators from $C[0,1]$ into $C[0,1]$.

Next, we show that $\mathcal{T}_h \to \mathcal{T}$ compactly in $C[0,1]$, i.e.,

$$\|\mathcal{T}_h v - \mathcal{T} v\|_\infty \longrightarrow 0 \quad \text{for every } v \in C[0,1] \text{ as } h = 1/n \to 0, \tag{7.14}$$

$$(v_h) \subset C[0,1], \qquad \|v_h\|_\infty \le 1,\ h = 1/n, \quad \Longrightarrow (\mathcal{T}_h v_h) \text{ is relatively compact in } C[0,1]. \tag{7.15}$$

We observe that the sets $\{\mathcal{A}(t,\cdot) : 0 \le t \le 1\}$ and $\{\mathcal{B}(t,\cdot) : 0 \le t \le 1\}$ are relatively compact in $C[-\delta, 1+\delta]$, with a fixed $\delta > 0$. Therefore, we obtain by Lemma 5.2, for a fixed $v \in C[0,1]$ extended by $v(s) = v(0)$ for $-\delta \le s \le 0$ and $v(s) = v(1)$ for $1 \le s \le 1+\delta$, that

$$\sup_{0\le t\le 1} \max_{0\le s\le 1} \left|\mathcal{A}(t,s)\, v(s) - P_{h,m}\left(\mathcal{A}(t,s)v(s)\right)\right| \longrightarrow 0 \quad \text{as } n \to \infty, \tag{7.16}$$

$$\sup_{0\le t\le 1} \max_{0\le s\le 1} \left|\mathcal{B}(t,s)v(s) - P_{h,m}\left(\mathcal{B}(t,s)v(s)\right)\right| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{7.17}$$

Further, we have

$$\int_0^1 |t-s|^{-\nu}\, ds \le \frac{2}{1-\nu}, \quad 0 \le t \le 1,\ 0 < \nu < 1. \tag{7.18}$$

Therefore,

$$\begin{aligned}
\|\mathcal{T}_h v - \mathcal{T} v\|_\infty = \sup_{0\le t\le 1} \Bigg| \int_0^1 & \left[|t-s|^{-\nu} P_{h,m}(\mathcal{A}(t,s)v(s)) \right.\\
&\left. + P_{h,m}(\mathcal{B}(t,s)v(s))\right] ds \\
&- \int_0^1 \left[|t-s|^{-\nu} \mathcal{A}(t,s) + \mathcal{B}(t,s)\right] v(s)\, ds \Bigg| \\
\le \left(\frac{2}{1-\nu}\right) & \sup_{0\le t\le 1} \max_{0\le s\le 1} \left|\mathcal{A}(t,s)v(s) - P_{h,m}(\mathcal{A}(t,s)v(s))\right| \\
&+ \sup_{0\le t\le 1} \max_{0\le s\le 1} \left|\mathcal{B}(t,s)v(s) - P_{h,m}(\mathcal{B}(t,s)v(s))\right|.
\end{aligned}$$

This, together with (7.16) and (7.17), yields (7.14).

The proof of (7.15) can be built with the help of the Arzelà-Ascoli theorem.

Due to the condition $\mathcal{N}(I-T)=\{0\}$, $\mathcal{N}(I-\mathcal{T})=\{0\}$ also. As is well known, see **[2, 3, 17, 36, 37]**, relations (7.14), (7.15) and $\mathcal{N}(I-\mathcal{T})=\{0\}$ imply that equation (7.10) (equation (3.6)) has a unique solution $v\in C[0,1]$, and there exists an $n_0\in\mathbb{N}$ such that, for $n\ge n_0$, equation (7.11) (equation (7.7)) has a unique solution $v_h\in C[0,1]$ and

$$\|v-v_h\|_\infty \le c\,\|\mathcal{T}v-\mathcal{T}_hv\|_\infty\,,\quad n\ge n_0, \tag{7.19}$$

with a constant $c>0$ independent of n (on $h=1/n$). Convergence (7.8) is a consequence of (7.14).

Next, we establish estimate (7.9). For solutions u and v of equations (1.1) and (3.6), we have $v(t)=u(\varphi(t))$ and $u\in C^{m,\theta_0,\theta_1}(0,1)$ by Theorem 2.3. In order to prove (7.9), it remains to show that (see 7.19):

$$\|\mathcal{T}v-\mathcal{T}_hv\|_\infty \le ch^m\,\|f\|_{C^{m,\theta_0,\theta_1}(0,1)}\,,\quad n\ge n_1. \tag{7.20}$$

We have, see (7.12) and (7.13),

$$(\mathcal{T}v)(t)-(\mathcal{T}_hv)(t)=\int_0^1 |t-s|^{-\nu}\,(I-P_{h,m})\,(\mathcal{A}(t,s)v(s))\,ds + \int_0^1 (I-P_{h,m})(\mathcal{B}(t,s)v(s))\,ds,\quad 0\le t\le 1.$$

Therefore,

$$|(\mathcal{T}v)(t)-(\mathcal{T}_hv)(t)|\le \left|\int_0^1 |t-s|^{-\nu}\,(I-P_{h,m})\,(\mathcal{A}(t,s)v(s))\,ds\right| + \left|\int_0^1 (I-P_{h,m})(\mathcal{B}(t,s)v(s))\,ds\right|,\quad 0\le t\le 1. \tag{7.21}$$

We estimate the first integral on the right hand side of the inequality (7.21) by dividing the integration into four subintervals: $[0,mh]$, $[mh,1/2]$, $[1/2,1-mh]$ and $[1-mh,1]$, where $mh\le 1/2$, or equivalently, $n\ge 2m$. Thus, first, we estimate

$$(7.22)\quad \left| \int_0^{mh} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))\, ds \right|$$
$$\leq \Big(1 + \|P_{h,m}\|_{\mathcal{L}(C[0,1],C[0,1])}\Big) \max_{0\leq s\leq mh} |\mathcal{A}(t,s)|\, \|v\|_\infty \int\limits_0^{mh} |t-s|^{-\nu}\, ds,$$

with $0 \leq t \leq 1$. It follows from (7.2) by $\delta_0 := (1-\nu-\lambda_0)p_0 - m > 0$ that

$$\max_{0\leq s\leq mh} |\mathcal{A}(t,s)| \leq c \max_{0\leq s\leq mh} s^{m-(1-\nu)+\delta_0} \leq c(mh)^{m-(1-\nu)+\delta_0},$$

where $0 \leq t \leq 1$. Since

$$\int\limits_0^{mh} |t-s|^{-\nu}\, ds \leq \frac{2m^{1-\nu}}{1-\nu} h^{1-\nu}, \quad 0 \leq t \leq 1,$$

we now obtain

$$\max_{0\leq s\leq mh} |\mathcal{A}(t,s)| \int\limits_0^{mh} |t-s|^{-\nu}\, ds \leq c_1 h^m, \quad 0 \leq t \leq 1,$$

with a constant $c_1 = c_1(m,\nu,\delta_0) > 0$ which is independent of $h = 1/n$. This, together with (5.11), (7.22) and $\|v\|_\infty = \|u\|_\infty \leq \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}$ yields
(7.23)

$$\left| \int_0^{mh} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))\, ds \right| \leq c_2 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)},$$

where $0 \leq t \leq 1$, and c_2 is a positive constant which does not depend on $h = 1/n$.

On the subinterval $[mh, 1/2]$, we use (5.10) to estimate

$$|(I - P_{h,m})\mathcal{A}(t,s)v(s)| \leq \vartheta_m h^m \left|\partial_s^m \left[\mathcal{A}(t,s)v(s)\right]\right|$$

for $0 \leq t \leq 1$, $mh \leq s \leq 1/2$. Using (7.3), we get for $0 \leq t \leq 1$ that

$$\left| \int_{mh}^{1/2} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))\, ds \right|$$

$$\begin{aligned}(7.24)\qquad &\le c_3 h^m \int_0^{1/2} |t-s|^{-\nu}\, s^{-(1-\nu)+\delta_0} ds\, \|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \\ &\le c_4 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}\,,\end{aligned}$$

with some positive constants c_3 and c_4 which are independent of $h = 1/n$.

In a similar manner, we obtain, for $0 \le t \le 1$,

$$(7.25)\qquad \left| \int_{1/2}^{1-mh} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))\, ds \right| \le c_5 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}\,,$$

$$(7.26)\qquad \left| \int_{1-mh}^{1} |t-s|^{-\nu} (I - P_{h,m})(\mathcal{A}(t,s)v(s))\, ds \right| \le c_6 h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)}\,,$$

where c_5 and c_6 are constants which do not depend upon $h = 1/n$.

Due to estimates (7.23)–(7.26) and (2.4), we finally obtain that

$$\begin{aligned}(7.27)\qquad &\left| \int_0^1 |t-s|^{-\nu} (I - P_{h,m}) (\mathcal{A}(t,s)v(s))\, ds \right| \\ &\le c h^m \|u\|_{C^{m,\theta_0,\theta_1}(0,1)} \\ &\le c' h^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}\,, \quad 0 \le t \le 1,\end{aligned}$$

with some constants c and c' independent of $h = 1/n$.

In order to estimate the second integral on the right hand side of inequality (7.21), we use (5.10), (2.4) and (7.6) and obtain

$$\left| \int_0^1 (I - P_{h,m})(\mathcal{B}(t,s)v(s))\, ds \right| \le c h^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}\,, \quad 0 \le t \le 1,$$

with a constant c independent of $h = 1/n$. This, together with (7.21) and (7.27), proves (7.20) and completes the proof of Theorem 7.9. □

Remark 7.3. With respect to $u_h(x) := v_h(\varphi^{-1}(x))$, estimate (7.9) reads, for sufficiently large n, as

$$\max_{0\le x\le 1} |u(x) - u_h(x)| = \max_{0\le t\le 1} |v(t) - v_h(t)| \le ch^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)} .$$

Remark 7.4. An advantage of the product integration method (7.7), compared to the collocation method (6.3), is that the number of integrals which must be numerically computed are, respectively, of order $2mn$ and mn^2, see subsection 7.2.

7.2. Matrix form of the method. Let us derive the matrix form of the product interpolation method (7.7). This method is of Nyström type; the solution v_h of equation (7.7) is uniquely determined by its knot values $v_h(ih)$, $i = 0, \dots, n$, through the Nyström extension, derived from (7.7) with the aid of (5.4) and (5.7),

(7.28)

$$\begin{aligned} v_h(t) = {} & \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \sum_{k\in\mathbb{Z}_m} \mathcal{B}\left(t, (j+k)h\right) v_h\left((j+k)h\right) L_{k,m}(ns-j)\, ds \\ & + \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} |t-s|^{-\nu} \\ & + \sum_{k\in\mathbb{Z}_m} \mathcal{A}\left(t, (j+k)h\right) v_h\left((j+k)h\right) L_{k,m}(ns-j)\, ds + f_\varphi(t), \\ & \qquad 0 \le t \le 1. \end{aligned}$$

An algebraic system of linear equations is obtained with respect to the grid values $v_h(ih)$, $i = 0, \dots, n$, by collocating (7.28) at the points $t = ih$:

(7.29)

$$\begin{aligned} v_h(ih) = \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \Big\{ & \mathcal{A}\left(ih, (j+k)h\right) \int_{jh}^{(j+1)h} |ih-s|^{-\nu} L_{k,m}(ns-j)\, ds \\ & + \mathcal{B}(ih, (j+k)h) \int_{jh}^{(j+1)h} L_{k,m}(ns-j)\, ds \Big\} \\ & \cdot v_h((j+k)h) + f_\varphi(ih), \quad i = 0, \dots, n. \end{aligned}$$

We extend $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$ with respect to s outside $[0,1]$ by the zero value; thus,

$$\mathcal{A}(ih,(j+k)h=0,$$
$$\mathcal{B}(ih,(j+k)h=0 \quad \text{for } j+k\le 0 \text{ and } j+k\ge n;$$

therefore, on the right hand side of (7.29), the values $v_h(lh)$ with $l\le 0$ and $l\ge n$ are actually not exploited. Occurring here (in 7.29), the integrals depend on the difference $i-j$; with the change of variables $ns-j=\sigma$, we see that

$$\int_{jh}^{(j+1)h} |ih-s|^{-\nu}\, L_{k,m}(ns-j)\,ds = h^{1-\nu}\int_0^1 |i-j-\sigma|^{-\nu}\, L_{k,m}(\sigma)\,d\sigma.$$

System (7.29) then takes the form

$$v_h(ih) = h^{1-\nu}\sum_{j=0}^{n-1}\sum_{k\in\mathbb{Z}_m}\{\mathcal{A}(ih,(j+k)h)\alpha_{i-j,k} + \mathcal{B}(ih,(j+k)h)\beta_k\}\, v_h((j+k)h) + f_\varphi(ih),$$
$$i=0,\dots,n,$$

or, collecting on the right hand side the coefficients by $v_h((j+k)h)$ with fixed $j+k=l$,

$$v_h(ih) = \sum_{l=1}^{n-1} c_{i,l}v_h(lh) + f_\varphi(ih), \quad i=0,\dots,n, \tag{7.30}$$

where

(7.31)
$$c_{i,l} = h^{1-\nu}\left[\mathcal{A}(ih,lh)\sum_{\{k\in\mathbb{Z}_m:0\le l-k\le n-1\}}\alpha_{i-l+k,k} + \mathcal{B}(ih,lh)\sum_{k\in\mathbb{Z}_m}\beta_k\right],$$
$$i=0,\dots,n,\quad l=1,\dots,n-1,$$

$$\alpha_{i',k} := \int_0^1 |i'-\sigma|^{-\nu} L_{k,m}(\sigma)\,d\sigma, \tag{7.32}$$
$$i' = -n+1,\dots,n,\ k\in\mathbb{Z}_m,$$

and

$$\beta_k := h^\nu \int_0^1 L_{k,m}(\sigma)\, d\sigma, \quad k \in \mathbb{Z}_m. \tag{7.33}$$

Note that $\mathcal{A}(ih, lh) = 0$ for $l \le 0$ and $l \ge n$.

Having found the solution $\{v_h(ih)\}$, $i = 0, \dots, n$, of system (7.30), we can use (7.28) to find the solution at any point $t \in [0,1]$.

Remark 7.5. We can also find an approximate solution $\widetilde{v}_h(t)$ by (6.12):

$$\widetilde{v}_h(t) = \sum_{k\in\mathbb{Z}_m} L_{k,m}(nt - j) \begin{cases} v_h(0) & \text{for } j + k \le 0, \\ v_h((j+k)h) & \text{for } 1 \le j + k \le n - 1, \\ v_h(1) & \text{for } j + k \ge n, \end{cases}$$

where $L_{k,m}$, $k \in \mathbb{Z}_m$, are the Lagrange fundamental polynomials defined in (5.3) and $0 \le t \le 1$. With the conditions of Theorem 7.2 (ii), the error estimate of order $O(h^m)$ also remains for $\widetilde{v}_h$:

$$\|v - \widetilde{v}_h\|_\infty \le ch^m \|f\|_{C^{m,\theta_0,\theta_1}(0,1)}, \quad n \ge n_1.$$

8. Numerical example. Here we numerically test the convergence behavior of the proposed algorithms. We will solve equation (1.1) with $a = 1$, $b = 0$ and $\nu = 1/2$:

$$u(x) = \int_0^1 |x - y|^{-1/2} u(y)\, dy + f(x), \quad 0 \le x \le 1. \tag{8.1}$$

We put

$$u(x) = 1 + x^{1/2} + (1 - x)^{1/2}$$

as the solution of (8.1); it corresponds to the free term

(8.2)

$$f(x) = 1 - \frac{\pi}{2} - 2x^{1/2} - 2(1-x)^{1/2} - x \log\left(1 + (1-x)^{1/2}\right) - (1-x)\log\left(1 + x^{1/2}\right) + \frac{1}{2} x \log x + \frac{1}{2}(1-x)\log(1-x).$$

First, we perform in (8.1) the change of variables $x = \varphi(t)$, $y = \varphi(s)$, where φ is given by (3.2), with $p_0 = p_1 = p \in \mathbb{N}$. As a result, we get the equation

$$v(t) = \int_0^1 |t-s|^{-1/2}\,\Phi(t,s)^{-1/2}\varphi'(s)v(s)\,ds + f_\varphi(t), \quad 0 \le t \le 1, \tag{8.3}$$

where $f_\varphi(t) = f(\varphi(t))$, with f defined by (8.2), $\Phi(t,s)$ given by (3.10) and $v(t) = u(\varphi(t))$ the function for which we look.

In order to solve equation (8.3) by the collocation method (6.3), we need to assemble the system (6.11). The parameters p_0 and p_1 in the definition of φ must be greater than $m/(1-1/2) = 2m$ to achieve the expected convergence order $O(n^{-m})$ of our method. Thus, we have to take $p \ge 2m+1$ (see Theorem 3.2 with $\lambda_0 = \lambda_1 = 0$).

In Tables 1–4, the errors

$$\epsilon_{m,n,p} := \max_{0\le i\le n} |v(ih) - v_h(ih)| \tag{8.4}$$

are presented. Here, v is the exact solution of equation (8.3), and v_h is the approximate solution to v obtained by method (6.3). Moreover, in Tables 1–4, the quotients $\epsilon_{m,n/2,p}/\epsilon_{m,n,p}$ for different values of m, n and $p = 2m+1$ are presented. Due to Theorem 6.1, the expected limit value of $\epsilon_{m,n/2,p}/\epsilon_{m,n,p}$ is 2^m.

TABLE 1. $m = 2$, $p = 5$.

n	$\epsilon_{2,n,5}$	$(\epsilon_{2,n/2,5})/(\epsilon_{2,n,5})$
4	4.25E-02	
8	2.05E-02	2.07
16	8.02E-03	2.55
32	2.63E-03	3.05
64	7.58E-04	3.47
128	2.03E-04	3.73
256	5.29E-05	3.84
512	1.36E-05	3.89

TABLE 2. $m = 3$, $p = 7$.

n	$\epsilon_{3,n,7}$	$(\epsilon_{3,n/2,7})/(\epsilon_{3,n,7})$
4	2.06E-01	
8	2.95E-02	6.98
16	3.95E-03	7.48
32	5.47E-04	7.23
64	7.16E-05	7.63
128	9.17E-06	7.81
256	1.17E-06	7.86
512	1.48E-07	7.89

TABLE 3. $m = 4$, $p = 9$.

n	$\epsilon_{4,n,9}$	$(\epsilon_{4,n/2,9})/(\epsilon_{4,n,9})$
4	1.94E-02	
8	1.01E-02	2.07
16	1.31E-03	7.74
32	1.09E-04	12.01
64	7.62E-06	14.27
128	5.07E-07	15.04
256	3.27E-07	1.55
512	3.55E-07	0.92

TABLE 4. $m = 5$, $p = 11$.

n	$\epsilon_{5,n,11}$	$(\epsilon_{5,n/2,11})/(\epsilon_{5,n,11})$
4	8.33E-02	
8	1.99E-02	10.4
16	8.38E-04	23.7
32	2.89E-05	28.9
64	9.31E-07	31.1
128	3.69E-07	2.52
256	3.60E-07	1.02
512	1.48E-07	7.89

In order to solve equation (8.3) by the product integration method (7.7), we need to assemble the system (7.30). In Tables 5–8, the errors (8.4) and the quotients $\epsilon_{m,n/2,p}/\epsilon_{m,n,p}$ for different values of m, n and $p = 2m+1$ are presented. Due to Theorem 7.2, the expected limit value of $\epsilon_{m,n/2,p}/\epsilon_{m,n,p}$ is 2^m.

TABLE 5. $m = 2$, $p = 5$.

n	$\epsilon_{2,n,5}$	$(\epsilon_{2,n/2,5})/(\epsilon_{2,n,5})$
4	1.07E-01	
8	3.23E-01	3.31
16	1.15E-01	2.81
32	3.66E-02	3.14
64	1.03E-02	3.54
128	2.74E-03	3.77
256	7.07E-04	3.87
512	1.80E-04	3.92

TABLE 6. $m = 3$, $p = 7$.

n	$\epsilon_{3,n,7}$	$(\epsilon_{3,n/2,7})/(\epsilon_{3,n,7})$
4	2.94E-00	
8	3.41E-01	8.65
16	5.10E-02	6.68
32	7.34E-03	6.95
64	9.53E-04	7.70
128	1.22E-04	7.82
256	1.55E-05	7.86
512	1.97E-06	7.88

TABLE 7. $m = 4$, $p = 9$.

n	$\epsilon_{4,n,9}$	$(\epsilon_{4,n/2,9})/(\epsilon_{4,n,9})$
4	1.00E-00	
8	2.49E-01	4.02
16	3.02E-02	8.28
32	2.57E-03	11.74
64	1.79E-04	14.34
128	1.19E-05	15.06
256	7.71E-07	15.42
512	3.31E-07	2.33

TABLE 8. $m = 5$, $p = 11$.

n	$\epsilon_{5,n,11}$	$(\epsilon_{5,n/2,11})/(\epsilon_{5,n,11})$
4	2.33E-00	
8	8.91E-01	8.65
16	1.99E-02	44.8
32	7.25E-04	27.4
64	2.19E-05	33.0
128	7.06E-07	31.1
256	3.71E-07	1.90
512	1.97E-06	7.88

In all cases, the `Fortran` in-built package was used for the numerical results. We see from Tables 1–2 and 5–6 that the obtained numerical results are in quite good accordance with the theoretical results. However, it follows from Tables 3–4 and 7–8 that, for greater n, the actual convergence order sometimes is not achieved. Therefore, a further study connected with the realization of the proposed algorithms in practice is needed. Also, other test examples and a comparison with other existing methods are of great interest. We plan to study these questions in a separate paper in the future.

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