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ESSENTIAL NORM OF A VOLTERRA-TYPE INTEGRAL OPERATOR FROM HARDY SPACES TO SOME ANALYTIC FUNCTION SPACES

JIZHEN ZHOU AND XIANGLING ZHU

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ABSTRACT. In this paper, we obtain some estimates of essential norm of the Volterra-type integral operator T_g, where

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta,$$

from Hardy spaces to the BMOA space, Besov spaces, Bergman spaces and Bloch-type spaces.

1. Introduction. The space of all analytic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane is denoted by $H(\mathbb{D})$. Let $0 < p < \infty$. The Bergman space, denoted by A^p, is the space of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. The Hardy space H^p consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

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As usual, H^∞ denotes the space of bounded analytic function. We say that an $f \in H(\mathbb{D})$ belongs to the BMOA space, if

$$\|f\|_*^2 = \sup_{I \subseteq \partial\mathbb{D}} \frac{1}{|I|} \int_I |f(\zeta) - f_I|^2 \frac{d\zeta}{2\pi} < \infty,$$

where $f_I = (1/|I|)\int_I f(\zeta)(d\zeta/2\pi)$. It is well known that BMOA is a Banach space under the norm $\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*$. From **[4]**, we have $\|f\|_*$ is comparable with $\sup_{w\in\mathbb{D}} \|f \circ \sigma_w - f(w)\|_{H^2}$, where $\sigma_w(z) = (w - z)/(1 - \overline{w}z)$ is a Möbius transformation of $\mathbb{D}$. We say that an $f \in H(\mathbb{D})$ belongs to the VMOA space if

$$\lim_{|w|\to 1} \|f \circ \sigma_w - f(w)\|_{H^2} = 0.$$

For $\alpha > 0$, we say that an $f \in H(\mathbb{D})$ belongs to Bloch-type space $\mathcal{B}^\alpha$ if

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

An $f \in H(\mathbb{D})$ belongs to the little Bloch-type space $\mathcal{B}_0^\alpha$ if

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

Let $p > 1$. The Besov space $\mathcal{B}_p$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_p}^p = \int |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

Let $0 < p,\ s < \infty,\ -2 < q < \infty$. An $f \in H(\mathbb{D})$ is said to belong to the space $F(p, q, s)$ if, see **[19]**,

$$\|f\|_{p,q,s}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) < \infty.$$

An $f \in H(\mathbb{D})$ belongs to the space $F_0(p, q, s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) = 0.$$

The $F(p, q, s)$ space becomes a Banach space with the norm $\|f\|_{F(p,q,s)} = |f(0)| + \|f\|_{p,q,s}$. $F(p, q, s)$ is called general function space since it can get many function spaces by taking special parameters of p, q, s. For example, $F(2, 1, 0) = H^2$, $F(p, p, 0) = A^p$, $F(2, 0, 1) =$ BMOA and $F(p,q,s) = \mathcal{B}^{(q+2)/p}$ for $s > 1$. We denote $F(p, p\alpha - 2, 1)$ and $F_0(p, p\alpha - 2, 1)$ by BMOA_p^α and $\mathrm{BMOA}_{p,0}^\alpha$, respectively.

Let X and Y be two Banach spaces. The essential norm of a bounded linear operator T between X and Y is defined as follows.

$$\|T\|_e^{X\to Y} = \inf\{||T - K||^{X\to Y} : K \text{ is compact}\},$$

where $\|\cdot\|^{X\to Y}$ is the operator norm. It is easy to see that $\|T\|_e^{X\to Y} = 0$ if and only if T is compact. For two Banach spaces X and Y with $Y \subset X$, if $f \in X$, then the distance of f to the space Y is defined by

$$\mathrm{dist}_X(f, Y) = \inf_{h\in Y} \|f - h\|_X.$$

For any $g \in H(\mathbb{D})$, the Volterra-type integral operator T_g is defined as follows:

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \qquad z \in \mathbb{D},\ f \in H(\mathbb{D}).$$

The operator T_g was introduced by Pommerenke [**14**], and he proved that T_g is bounded on the Hardy space H^2 if and only if g belongs to BMOA. In [**1**], Aleman and Siskakis showed that Pommerenke's boundedness characterization is valid on each H^p for $1 \le p < \infty$ and that T_g is compact on H^p if and only if $g \in$ VMOA. The boundedness and compactness of the operator T_g on some holomorphic spaces, as well as its extension on the unit ball, were investigated, for example, [**1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22**] (and related references therein).

Recently, many researchers have also been interested in the study of the essential norm of various operators. Laitila, Miihkinen and Nieminen [**7**] studied the essential norm of the operator T_g on the Hardy space. Liu, Lou and Xiong [**13**] studied the essential norm of the operator T_g on the Bloch space and some other spaces. Zhuo and Ye studied the essential norm of the operator T_g from Morrey spaces to the Bloch space [**22**].

Zhao [**20**] obtained some characterizations of the operator T_g from Hardy spaces to some other analytic function spaces. Therefore, it is also interesting to study the essential norm of the operator T_g on these spaces. The main purpose of this paper is to obtain some estimates for the essential norm of the operator T_g from Hardy spaces H^p to the BMOA space, Besov spaces, Bergman spaces and Bloch-type spaces.

Throughout the paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Essential norms of T_g. In this section, we will state our main results and proofs. For this purpose, we need some useful lemmas as follows.

Lemma 2.1 ([**20**]). *Let $g \in H(\mathbb{D})$, $p \geq 1$, $\alpha > 0$ and $\alpha - 1/p > 0$. Then the following statements hold.*

(i) $T_g : H^p \to \mathrm{BMOA}$ $(p > 1)$ *is bounded if and only if* $g \in \mathcal{B}^{1-1/p}$, $T_g : H^p \to \mathrm{BMOA}$ *is compact if and only if* $g \in \mathcal{B}_0^{1-1/p}$.
(ii) $T_g : H^p \to \mathcal{B}^\alpha$ *is bounded if and only if* $g \in \mathcal{B}^{\alpha-1/p}$, $T_g : H^p \to \mathcal{B}^\alpha$ *is compact if and only if* $g \in \mathcal{B}_0^{\alpha-1/p}$.
(iii) $T_g : H^p \to B_p$ $(p > 1)$ *is bounded if and only if* $g \in \mathrm{BMOA}_p^{1-1/p}$, $T_g : H^p \to B_p$ *is compact if and only if* $g \in \mathrm{BMOA}_{p,0}^{1-1/p}$.
(iv) $T_g : H^p \to A^p$ *is bounded if and only if* $g \in \mathrm{BMOA}_p^{1+1/p}$, $T_g : H^p \to A^p$ *is compact if and only if* $g \in \mathrm{BMOA}_{p,0}^{1+1/p}$.

Remark 2.2. When $p = 1$, from [**20,** Theorem 11], we see that $T_g : H^1 \to \mathrm{BMOA}$ is bounded if and only if $g' \in H^\infty$. $T_g : H^1 \to \mathrm{BMOA}$ is compact if and only if g is a constant.

The next lemma can be proved similarly as [**18**]. For the completeness of this paper, we include the proof here.

Lemma 2.3. *If $\alpha > 0$ and $g \in \mathcal{B}^\alpha$, then*

$$\mathrm{dist}_{\mathcal{B}^\alpha}(g, \mathcal{B}_0^\alpha) \approx \limsup_{|z|\to 1}(1 - |z|^2)^\alpha |g'(z)|.$$

Proof. Denote by $f_r(z) = f(rz)$ for $0 < r < 1$. For any given $g \in \mathcal{B}^\alpha$, then $g_r \in \mathcal{B}_0^\alpha$ and $\|g_r\|_{\mathcal{B}^\alpha} \lesssim \|g\|_{\mathcal{B}^\alpha}$. For any given $\delta \in (0, 1)$, it is easy to see that

$$\lim_{r\to 1} \sup_{|z|\leq\delta} (1 - |z|^2)|g'(z) - g_r'(z)| = 0,$$

which implies

$$\begin{aligned}\operatorname{dist}_{\mathcal{B}^\alpha}(g,\mathcal{B}^\alpha_0) &= \inf_{f\in\mathcal{B}^\alpha_0}\|g-f\|_{\mathcal{B}^\alpha} \le \lim_{r\to 1}\|g-g_r\|_{\mathcal{B}^\alpha}\\ &= \lim_{r\to 1}\sup_{|z|>\delta}(1-|z|^2)^\alpha|g'(z)-rg'(rz)|\\ &\quad + \lim_{r\to 1}\sup_{|z|\le\delta}(1-|z|^2)|g'(z)-g_r'(z)|\\ &\le \sup_{|z|>\delta}(1-|z|^2)^\alpha|g'(z)| + \lim_{r\to 1}\sup_{|z|>\delta}(1-|z|^2)^\alpha|rg'(rz)|.\end{aligned}$$

Since δ is arbitrary, we have $\operatorname{dist}_{\mathcal{B}^\alpha}(g,\mathcal{B}^\alpha_0) \lesssim \limsup_{|z|\to 1}(1-|z|^2)^\alpha|g'(z)|$.

On the other hand, for any $f\in\mathcal{B}^\alpha_0$,

$$\begin{aligned}\|g-f\|_{\mathcal{B}^\alpha} &\ge \limsup_{|z|\to 1}(1-|z|^2)|g'(z)-f'(z)|\\ &= \limsup_{|z|\to 1}(1-|z|^2)^\alpha|g'(z)|.\end{aligned}$$

This yields

$$\operatorname{dist}_{\mathcal{B}^\alpha}(g,\mathcal{B}^\alpha_0) = \inf_{f\in\mathcal{B}^\alpha_0}\|g-f\|_{\mathcal{B}^\alpha} \ge \limsup_{|z|\to 1}(1-|z|^2)^\alpha|g'(z)|,$$

as desired. The proof is complete. □

Lemma 2.4. ([**7,** Lemma 3]). *Suppose* $g\in\mathrm{BMOA}$. *Then*

$$\begin{aligned}\operatorname{dist}_{\mathrm{BMOA}}(g,\mathrm{VMOA}) &\approx \limsup_{r\to 1}\|g-g_r\|_{\mathrm{BMOA}}\\ &\approx \limsup_{|a|\to 1}\|g\circ\sigma_a-g(a)\|_{H^2}.\end{aligned}$$

Here $g_r(z)=g(rz)$ *with* $0<r<1$.

Lemma 2.5. *Let* $p>0$ *and* $\alpha>0$. *If* $g\in\mathrm{BMOA}^\alpha_p$, *then*

$$\begin{aligned}&\operatorname{dist}_{\mathrm{BMOA}^\alpha_\mathrm{p}}(g,\mathrm{BMOA}^\alpha_{\mathrm{p},0})\\ &\qquad\approx \limsup_{|a|\to 1}\int_{\mathbb{D}}|g'(z)|^2(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z).\end{aligned}$$

Proof. Denote by $f_r(z)=f(rz)$ for $0<r<1$. For any given $g\in\mathrm{BMOA}^\alpha_p$, then $g_r\in\mathrm{BMOA}^\alpha_{p,0}$ and $\|g_r\|_{\mathrm{BMOA}^\alpha_\mathrm{p}}\lesssim\|g\|_{\mathrm{BMOA}^\alpha_\mathrm{p}}$. Let

$\delta \in (0,1)$. We choose $a \in (0,\delta)$. Then $\sigma_a(z)$ lies in a compact subset of $\mathbb{D}$. So

$$\lim_{r\to 1}\sup_{z\in\mathbb{D}} |g'(\sigma_a(z)) - rg'(r\sigma_a(z))| = 0.$$

Making a change of variables, we have

$$\begin{aligned}
&\lim_{r\to 1}\sup_{|a|\le\delta}\int_{\mathbb{D}} |g'(z) - g_r'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z)\\
&\quad= \lim_{r\to 1}\sup_{|a|\le\delta}\int_{\mathbb{D}} |g'(\sigma_a(z)) - g_r'(\sigma_a(z))|^p(1-|z|^2)^{p\alpha-1}|\sigma_a'(z)|^{p\alpha}dA(z)\\
&\quad= \lim_{r\to 1}\sup_{|a|\le\delta}\sup_{z\in\mathbb{D}} |g'(\sigma_a(z)) - g_r'(\sigma_a(z))|^p\int_{\mathbb{D}}(1-|z|^2)^{p\alpha-1}|\sigma_a'(z)|^{p\alpha}dA(z) = 0.
\end{aligned}$$

By the definition of distance, we obtain

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{BMOA}_p^\alpha}(g, \mathrm{BMOA}_{p,0}^\alpha) = \inf_{f\in \mathrm{BMOA}_{p,0}^\alpha}\|g-f\|_{\mathrm{BMOA}_p^\alpha}\\
&\quad\le \lim_{r\to 1}\|g-g_r\|_{\mathrm{BMOA}_p^\alpha}\\
&\quad= \lim_{r\to 1}\sup_{|a|>\delta}\int_{\mathbb{D}} |g'(z) - g_r'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z)\\
&\qquad+ \lim_{r\to 1}\sup_{|a|\le\delta}\int_{\mathbb{D}} |g'(z) - g_r'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z)\\
&\quad\lesssim \sup_{|a|>\delta}\int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z)\\
&\qquad+ \lim_{r\to 1}\sup_{|a|>\delta}\int_{\mathbb{D}} |g_r'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z).
\end{aligned}$$

Denote by $\psi_{r,a}(z) = \sigma_{ra}\circ r\sigma_a(z)$. Then $\psi_{r,a}$ is an analytic self-map of $\mathbb{D}$ and $\psi_{r,a}(0) = 0$. Making a change of variable of $z = \sigma_a(z)$ and applying Littlewood's subordination theorem (see **[3,** Theorem 1.7]), we have

$$\begin{aligned}
&\int_{\mathbb{D}} |g_r'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_a(z)|^2)\,dA(z)\\
&\quad= \int_{\mathbb{D}} |g_r'(\sigma_a(z))|^p(1-|\sigma_a(z)|^2)^{p\alpha}(1-|z|^2)^{-1}dA(z)\\
&\quad\le \int_{\mathbb{D}} |g'\circ\sigma_{ra}\circ\psi_{r,a}(z)|^p(1-|\sigma_{ra}\circ\psi_{r,a}(z)|^2)^{p\alpha}(1-|z|^2)^{-1}dA(z)
\end{aligned}$$

$$\leq \int_{\mathbb{D}} |g' \circ \sigma_{ra} \circ \psi_{r,a}(z)|^p (1-|\sigma_{ra} \circ \psi_{r,a}(z)|^2)^{p\alpha} (1-|z|^2)^{-1} dA(z)$$
$$\leq \int_{\mathbb{D}} |g' \circ \sigma_{ra}(z)|^p (1-|\sigma_{ra}(z)|^2)^{p\alpha} (1-|z|^2)^{-1} dA(z)$$
$$\leq \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p\alpha-2} (1-|\sigma_{ra}(z)|^2)\, dA(z).$$

Since δ is arbitrary, we get

$$\text{(2.1)} \quad \operatorname{dist}_{\mathrm{BMOA}_{\mathrm{p}}^{\alpha}}(g, \mathrm{BMOA}_{p,0}^{\alpha}) \lesssim \limsup_{|a|\to 1} \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p\alpha-2} (1-|\sigma_a(z)|^2)\, dA(z).$$

On the other hand, for any $f \in \mathrm{BMOA}_p^{\alpha}$,

$$\operatorname{dist}_{\mathrm{BMOA}_{\mathrm{p}}^{\alpha}}(g, \mathrm{BMOA}_{p,0}^{\alpha}) = \inf_{f \in \mathrm{BMOA}_{p,0}^{\alpha}} \|g-f\|_{\mathrm{BMOA}_{\mathrm{p}}^{\alpha}}$$
$$\gtrsim \limsup_{|a|\to 1} \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p\alpha-2} (1-|\sigma_a(z)|^2)\, dA(z),$$

which, together with equation (2.1), implies the desired result. The proof is complete. □

Theorem 2.6. *Let $g \in H(\mathbb{D})$ and $p > 1$. Suppose that $T_g : H^p \to$ BMOA is bounded. Then*

$$\|T_g\|_e^{H^p \to \mathrm{BMOA}} \approx \limsup_{|z|\to 1} (1-|z|^2)^{1-1/p} |g'(z)|.$$

Proof. First, we prove the upper estimate for the essential norm of T_g. For each $h \in \mathcal{B}_0^{1-1/p}$, the operator $T_h : H^p \to$ BMOA is compact by Lemma 2.1. Moreover, by the linearity of T_g respect to g, we have

$$\|T_g\|_e^{H^p \to \mathrm{BMOA}} \leq \|T_g - T_h\|^{H^p \to \mathrm{BMOA}} = \|T_{g-h}\|^{H^p \to \mathrm{BMOA}} \lesssim \|g-h\|_{\mathcal{B}^{1-1/p}}.$$

Hence,

$$\text{(2.2)} \quad \|T_g\|_e^{H^p \to \mathrm{BMOA}} \lesssim \inf_{h \in \mathcal{B}_0^{1-1/p}} \|g-h\|_{\mathcal{B}^{1-1/p}} = \operatorname{dist}_{\mathcal{B}^{1-1/p}}(g, \mathcal{B}_0^{1-1/p})$$
$$\approx \limsup_{|z|\to 1} (1-|z|^2)^{1-1/p} |g'(z)|.$$

For any $a \in \mathbb{D}$, we define

$$f_a(z) = \left[\frac{1-|a|^2}{(1-\overline{a}z)^2}\right]^{1/p}. \tag{2.3}$$

Taking $z = \mathrm{re}^{i\theta}$ and the Poisson integral formula gives the following:

$$\begin{aligned}
\|f_a\|_{H^p} &= \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f_a(\mathrm{re}^{i\theta})|^p d\theta\right)^{1/p} \\
&= \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|a|^2}{|1-\overline{a}\mathrm{re}^{i\theta}|^2}\,d\theta\right)^{1/p} \\
&= \sup_{0<r<1}\left(\frac{1-|a|^2}{1-|ar|^2}\right)^{1/p} = 1.
\end{aligned}$$

In the meantime, we have $|f_a(a)|(1-|a|^2)^{1/p} = 1$. Since $f_a \to 0$ weakly in H^p as $|a| \to 1$, we have $\|Kf_a\|_{\mathrm{BMOA}} \to 0$ as $|a| \to 1$ for any compact operator $K : H^p \to \mathrm{BMOA}$. Moreover,

$$\|T_g - K\|^{H^p\to\mathrm{BMOA}} \geq \|(T_g - K)f_a\|_{\mathrm{BMOA}} \geq \|T_g f_a\|_{\mathrm{BMOA}} - \|Kf_a\|_{\mathrm{BMOA}}.$$

Therefore,

$$\|T_g - K\|^{H^p\to\mathrm{BMOA}} \geq \limsup_{|a|\to 1}\|(T_g - K)f_a\|_{\mathrm{BMOA}} \geq \limsup_{|a|\to 1}\|T_g f_a\|_{\mathrm{BMOA}},$$

which implies that

$$\|T_g\|_e^{H^p\to\mathrm{BMOA}} \geq \limsup_{|a|\to 1}\|T_g f_a\|_{\mathrm{BMOA}}.$$

In addition, by **[19,** Lemma 2.9], we have

$$\begin{aligned}
\|T_g f_a\|_{\mathrm{BMOA}} &= \sqrt{\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}|(T_g f_a)'(z)|^2(1-|\sigma_b(z)|)\,dA(z)} \\
&\geq \sqrt{\int_{\mathbb{D}}|(T_g f_a)'(z)|^2(1-|\sigma_a(z)|)\,dA(z)} \\
&= \sqrt{\int_{\mathbb{D}}|g'(z)|^2(1-|z|^2)^{-2/p}(1-|\sigma_a(z)|^2)^{1+2/p}\,dA(z)} \\
&\gtrsim (1-|a|^2)^{1-1/p}|g'(a)|.
\end{aligned}$$

Therefore,

$$\|T_g\|_e^{H^p\to \mathrm{BMOA}} \geq \limsup_{|a|\to 1} \|T_g f_a\|_{\mathrm{BMOA}} \gtrsim \limsup_{|a|\to 1}(1-|a|^2)^{1-1/p}|g'(a)|.$$

Then inequality (2.2) combined with the last inequality gives the desired result. The proof is complete. □

Remark 2.7. When $p = 1$, from Remark 2.2 and the definition of the essential norm operator we see that $\|T_g\|_e^{H^1\to \mathrm{BMOA}} = 0$.

Theorem 2.8. *Let $g \in H(\mathbb{D})$, $p \geq 1$ and $\alpha > 1/p$. Suppose that $T_g : H^p \to \mathcal{B}^\alpha$ is bounded. Then*

$$\|T_g\|_e^{H^p\to \mathcal{B}^\alpha} \approx \limsup_{|z|\to 1}(1-|z|^2)^{\alpha-1/p}|g'(z)|.$$

Proof. The upper estimate for the essential norm of T_g is similar to the proof of Theorem 2.6. We omit the details of the proof.

Now we only give the proof for the lower estimate. For any $a \in \mathbb{D}$, we choose the test function f_a which is defined in equation (2.3). Since $f_a \to 0$ weakly in H^p as $|a| \to 1$, we have

$$\|K f_a\|_{\mathcal{B}^\alpha} \longrightarrow 0 \quad \text{as } |a| \to 1,$$

for any compact operator $K : H^p \to \mathcal{B}^\alpha$. Thus,

$$\begin{aligned}\|T_g - K\|^{H^p\to \mathcal{B}^\alpha} &\geq \limsup_{|a|\to 1} \|(T_g - K) f_a\|_{\mathcal{B}^\alpha} \\ &\geq \limsup_{|a|\to 1} \|T_g f_a\|_{\mathcal{B}^\alpha} - \limsup_{|a|\to 1} \|K f_a\|_{\mathcal{B}^\alpha}.\end{aligned}$$

Note that

$$\begin{aligned}\|T_g f_a\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}} |(T_g f_a)'(z)|(1-|z|^2)^\alpha \geq |(T_g f_a)'(z)|(1-|a|^2)^\alpha \\ &= |g'(a)|(1-|a|^2)^{\alpha-1/p}.\end{aligned}$$

The last inequality gives

$$\|T_g\|_e^{H^p\to \mathcal{B}^\alpha} \geq \limsup_{|a|\to 1} \|T_g f_a\|_{\mathcal{B}^\alpha} \gtrsim \limsup_{|a|\to 1}(1-|a|^2)^{\alpha-1/p}|g'(a)|.$$

The proof is complete. □

Theorem 2.9. *Let $g \in H(\mathbb{D})$ and $p > 1$. Suppose that $T_g : H^p \to \mathcal{B}_p$ is bounded. Then,*

$$\|T_g\|_e^{H^p \to \mathcal{B}_p} \approx \limsup_{|a|\to 1} \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-3}(1-|\sigma_a(z)|^2)\, dA(z).$$

Proof. First, we consider the upper estimate for the essential norm of T_g. Indeed, for each $h \in \mathrm{BMOA}_{p,0}^{1-1/p}$, the operator T_h is compact from H^p to $\mathcal{B}_p$ by Lemma 2.1. Moreover,

(2.4)
$$\|T_g\|_e^{H^p \to \mathcal{B}_p} \le \|T_g - T_h\|^{H^p \to \mathcal{B}_p} = \|T_{g-h}\|^{H^p \to \mathcal{B}_p} \lesssim \|g-h\|_{\mathrm{BMOA}_p^{1-1/p}}.$$

Hence, by Lemma 2.4 and inequality (2.4) we have

(2.5)
$$\begin{aligned}\|T_g\|_e^{H^p \to \mathcal{B}_p} &\lesssim \inf_{h \in \mathrm{BMOA}_{p,0}^{1-1/p}} \|g-h\|_{\mathrm{BMOA}_\mathrm{p}^{1-1/\mathrm{p}}} \\ &= \mathrm{dist}_{\mathrm{BMOA}_\mathrm{p}^{1-1/\mathrm{p}}}(g, \mathrm{BMOA}_{\mathrm{p},0}^{1-1/\mathrm{p}}) \\ &\approx \limsup_{|a|\to 1} \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-3}(1-|\sigma_a(z)|^2)\, dA(z).\end{aligned}$$

Let f_a be defined as in equation (2.3). Since $f_a \to 0$ weakly in H^p as $|a| \to 1$, we have $\|Kf_a\|_{\mathcal{B}_p} \to 0$ as $|a| \to 1$ for any compact operator $K : H^p \to \mathcal{B}_p$. In addition,

$$\|T_g - K\|^{H^p \to \mathcal{B}_p} \ge \|(T_g - K)f_a\|_{\mathcal{B}_p} \ge \|T_g f_a\|_{\mathcal{B}_p} - \|Kf_a\|_{\mathcal{B}_p},$$

we have

$$\|T_g - K\|^{H^p \to \mathcal{B}_p} \ge \limsup_{|a|\to 1} \|(T_g - K)f_a\|_{\mathcal{B}_p} \ge \limsup_{|a|\to 1} \|T_g f_a\|_{\mathcal{B}_p}.$$

Since

$$\begin{aligned}\|T_g f_a\|_{\mathcal{B}_p} &= \int_{\mathbb{D}} |(T_g f_a)'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-3}(1-|\sigma_a(z)|)\, dA(z),\end{aligned}$$

we get

$$(2.6)\quad \|T_g\|_e^{H^p\to\mathcal{B}_p} \gtrsim \limsup_{|a|\to 1}\int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^{p-3}(1-|\sigma_a(z)|^2)\,dA(z).$$

Then, inequality (2.5) combined with inequality (2.6) gives the desired result. The proof is complete. □

Remark 2.10. When $p=1$, the definition of Besov space is completely different than the case of $p>1$. The analytic Besov space $\mathcal{B}_1$ is defined to be the set of all $f\in H(\mathbb{D})$ which can be written as

$$f(z)=\sum_{n=1}^{\infty} a_n\sigma_{\lambda_n}(z)$$

for $\{a_n\}$ in l^1 and $\lambda_n\in\mathbb{D}$. The norm of $\mathcal{B}_1$ is defined by

$$\|f\|_{\mathcal{B}_1}=\inf\left\{\sum_{n=1}^{\infty}|a_n| : f(z)=\sum_{n=1}^{\infty} a_n\sigma_{\lambda_n}(z)\right\}.$$

It is obvious that $\mathcal{B}_1\subset H^\infty\subset \mathrm{BMOA}$. From Remarks 2.2 and 2.7 we see that $T_g : H^1\to\mathcal{B}_1$ is compact if and only if g is a constant. Moreover, $\|T_g\|_e^{H^1\to\mathrm{BMOA}}=0$.

Similarly to the proof of Theorem 2.9, we immediately get the following result. We omit the proof here. □

Theorem 2.11. *Let $g\in H(\mathbb{D})$ and $p\geq 1$. Suppose that $T_g : H^p\to A^p$ is bounded. Then*

$$\|T_g\|_e^{H^p\to A^p}\approx \limsup_{|a|\to 1}\int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^{p-1}(1-|\sigma_a(z)|^2)\,dA(z).$$

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School of Sciences, Anhui University of Science and Technology, Huainan, Anhui 232001, China
Email address: **hope189@163.com**

Department of Mathematics, Jiaying University, 514015, Meizhou, Guangdong, China
Email address: **jyuzxl@163.com**