

AN EXISTENCE RESULT FOR A CLASS OF NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

KHALID LATRACH

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ABSTRACT. We consider a general nonlinear functional integral equation, and we prove the existence of solutions of this equation in the space of Lebesgue integrable functions on \mathbb{R}^+ . Our analysis uses a recent version of Krasnosel'skii's fixed point theorem (Theorem 2.6) and the concept of the measure of weak noncompactness. In the appendix, we give an extension of Theorem 2.6 to expansive mappings.

1. Introduction. Nonlinear integral equations have a lot of applications in engineering, mechanics, economics, mathematical physics and so on (see, for example, [4, 5, 10, 14, 28] and the references therein). The typical equations which are often investigated are the Hammerstein integral equation and its generalization, the Urysohn integral equation (cf., [3, 4, 5, 7, 10, 14, 15, 16, 22, 23, 26, 28]),

$$(1.1) \quad \psi(t) = f(t, \psi(t)) + \int_I \zeta(t, s, \psi(s)) ds, \quad t \in I,$$

where I is an interval in \mathbb{R} (bounded or not) and $f : I \times \mathbb{R} \rightarrow \mathbb{R}$, $\zeta : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions while $\psi(\cdot)$ is an unknown function. In the case where I is bounded, the theory of (1.1) is well understood in the L^1 setting (see, for example, [12, 21, 25, 27, 31]). However, the case of unbounded interval I seems to be more difficult. Actually, in [6], an example of the measure of weak noncompactness on $L^1(\mathbb{R}^+)$ was constructed by Banaś and Knap. It furnished an adequate tool to settle such problems on unbounded intervals of \mathbb{R} . So, during the last decade, much work was done for a variety of nonlinear integral

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equations on the space $L^1(I)$ where I is an unbounded interval of \mathbb{R} (see, for example, [4, 5, 7, 10, 23, 28]).

Motivated by these works, we consider the following quite general nonlinear functional integral equation

$$(1.2) \quad \psi(t) = f_1(t, \psi(t)) + f_2\left(t, \int_0^t \zeta_1(t, s, \psi(s)) ds, \int_0^{+\infty} \zeta_2(t, s, \psi(s)) ds\right),$$

where f_1, f_2, ζ_1 and ζ_2 are measurable functions while $\psi(\cdot)$ belongs to the space $L^1(\mathbb{R}^+)$. The goal of this paper is to discuss the solvability of equation (1.2) on the space $L^1(\mathbb{R}^+)$ under general hypotheses on the functions f_1, f_2, ζ_1 and ζ_2 . We will require that these functions satisfy the Carathéodory conditions and some other technical assumptions. Our analysis is based on a recent version of Krasnosel'skii's fixed point theorem established in [22] (cf., Theorem 2.6) and uses the concept of the measure of weak noncompactness on $L_1(\mathbb{R}^+)$. Our result seems to be new and covers many previous ones present in the literature, in particular, those obtained in [4, 5, 7, 10, 23, 26, 28].

2. Preliminaries. This section is devoted to recalling some notions and results needed in the sequel. Let \mathbb{R} be the set of real numbers, \mathbb{R}^+ the interval $[0, +\infty)$, and let

$$X := L^1(\mathbb{R}^+, dx)$$

be the space of Lebesgue integrable functions on the set \mathbb{R} endowed with the standard norm

$$\|\psi\| = \int_0^{+\infty} |\psi(t)| dt.$$

We recall some facts concerning superposition operators required below. Let I be an interval of \mathbb{R} . A function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy the Carathéodory conditions on $I \times \mathbb{R}$ if

$$\begin{cases} t \longrightarrow f(t, x) & \text{is measurable on } I \text{ for all } x \in \mathbb{R}, \\ x \longrightarrow f(t, x) & \text{is continuous on } \mathbb{R} \text{ for almost all } t \in I. \end{cases}$$

If f satisfies the Carathéodory conditions, then we can define the superposition (or Nemytskii) operator generated by f, \mathcal{N}_f , on the set of measurable real functions $\psi(\cdot)$ by $(\mathcal{N}_f\psi)(t) = f(t, \psi(t))$ for all $t \in I$. In

L^p -spaces, the superposition operator has been extensively investigated (see, for example, [2, 21] and the bibliography therein). We recall the following result which states a basic fact for the theory of these operators in L^1 -spaces.

Lemma 2.1. [2, 21]. *Let I be an interval of \mathbb{R} , and let $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying the Carathéodory conditions. The superposition operator \mathcal{N}_f maps continuously the space $L^1(I)$ into itself if and only if*

$$|f(t, x)| \leq a(t) + b|x|, \quad \text{almost everywhere in } t, \text{ for all } x \in \mathbb{R},$$

where b is a nonnegative constant and $a(\cdot)$ is a function of $L^1_+(I)$ where $L^1_+(I)$ stands for positive cone of the space $L^1(I)$.

This theorem is valid for both bounded and unbounded intervals. It was proved for bounded intervals by Krasnosel'skii [20]. A proof of this result for unbounded intervals appeared in [2].

Notation. Let f be a function defined on a set I , and let J be a nonempty subset of I . By $F|_J$, we denote the restriction of f to J . For any subset D of \mathbb{R} , by $\mu(D)$, we denote the Lebesgue measure of D .

We also need the following result due to Scorza-Drăgoni concerning the structure of functions satisfying the Carathéodory conditions (see [9]).

Lemma 2.2. *Let I be a bounded interval, and let $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying the Carathéodory conditions. Then, for each $\varepsilon > 0$, there exists a closed subset D_ε of the interval I such that $\mu(I \setminus D_\varepsilon) \leq \varepsilon$ and $f|_{D_\varepsilon \times \mathbb{R}}$ is continuous.*

Let us now recall some facts about the Volterra linear operator on the space X . Denote by Π the set

$$\Pi = \{(t, s) : 0 \leq s \leq t, s, t \in \mathbb{R}\},$$

and let κ be the function defined from Π into \mathbb{R} measurable with respect to both variables. Let K be the operator defined for all $\psi \in \mathcal{X}$ by

$$(K\psi)(t) = \int_0^t \kappa(t, s)\psi(s) ds, \quad t \geq 0.$$

The operator K is called the *Volterra integral operator*. The following lemma gives a fundamental fact concerning this class of operators (see, for example, [31]).

Lemma 2.3. *If the Volterra integral operator K transforms the space \mathcal{X} into itself, then it is continuous.*

Let Z be an infinite dimensional Banach space with the norm $\|\cdot\|$ and the zero element 0 . Let $\mathcal{B}(Z)$ denote the family of all nonempty bounded subsets of Z , and let $\mathcal{W}(Z)$ be the subset of $\mathcal{B}(Z)$ consisting of all weakly compact subsets of Z . For every real $r > 0$, we denote by B_r the closed ball of Z centered at 0 with radius r .

Recall that the concept of a measure of weak noncompactness has been introduced by De Blasi [11] as follows:

$$\begin{cases} w : \mathcal{B}(Z) \longrightarrow \mathbb{R}^+ \\ w(M) := \inf \{ r > 0 : \text{there exists } Y \in \mathcal{W}(Z) \text{ such that } M \subset Y + B_r \}. \end{cases}$$

In [1], Appell and De Pascale showed in $L_1(a, b)$ where $[a, b]$ is an interval of \mathbb{R} that $w(\cdot)$ possesses the following simple form:

$$w(M) = \limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\psi \in M} \left[\int_D |\psi(t)| dt : \mu(D) \leq \varepsilon \right] \right\}$$

for all bounded subsets M of $L^1(a, b)$.

In [13], Dieudonné established the following characterization of weakly compact subsets of $L^1(\Omega)$ where Ω is an arbitrary subset of \mathbb{R}^n , $n \geq 1$. In the special case where $n = 1$ and $\Omega = \mathbb{R}^+$, this result can be stated as follows:

Lemma 2.4. *A bounded subset M of \mathcal{X} is relatively weakly compact if and only if*

- (i) for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\mu(D) \leq \delta$, then $\int_D |f(t)| dt < \varepsilon$ for all $f \in M$,
- (ii) for any $\varepsilon > 0$, there exists $T > 0$ such that $\int_T^{+\infty} |f(t)| dt \leq \varepsilon$ for all $f \in M$.

Using this lemma, Banaś and Knap [6] constructed a measure of weak noncompactness on the space X in the following way:

$$\phi(M) = \phi_1(M) + \phi_2(M) \quad \text{for all } M \in \mathcal{B}(X),$$

where

$$\phi_1(M) = \limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\psi \in M} \left(\int_D |\psi(t)| dt : \text{meas}(D) < \varepsilon \right) \right\},$$

and

$$\phi_2(M) = \lim_{T \rightarrow +\infty} \left\{ \sup_{\psi \in M} \left(\int_T^{+\infty} |\psi(t)| dt \right) \right\}.$$

For completeness, we recall some useful properties of $\phi(\cdot)$ needed below (for the proofs we refer to [6]).

Lemma 2.5. *Letting $M_1, M_2 \in \mathcal{B}(X)$, then we have:*

- (i) $M_1 \subseteq M_2$ implies $\phi(M_1) \leq \phi(M_2)$.
- (ii) $\phi(\text{co}(M_1)) = \phi(M_1)$.
- (iii) $\phi(M_1 + M_2) \leq \phi(M_1) + \phi(M_2)$.
- (iv) If $(M_n)_{n \geq 1}$ is a sequence of nonempty, weakly closed subsets of X with $M_1 \in \mathcal{B}(X)$ and $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ with $\lim_{n \rightarrow \infty} w(M_n) = 0$, then $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$ and $\phi(\bigcap_{n=1}^{\infty} M_n) = 0$.

Let Z be a Banach space, and let T be a nonlinear operator from $D(T) \subset Z$ into Z . The following condition is required in the sequel.

- (A) If $(x_n)_{n \in \mathbb{N}} \subseteq D(T)$ is a weakly convergent sequence in Z , then $(Tx_n)_{n \in \mathbb{N}}$ has a strongly convergent subsequence in Z .

The proof of our existence result will use the following fixed point theorem established in [22].

Theorem 2.6. *Let M be a nonempty bounded closed convex subset of a Banach space Z . Suppose that $A : M \rightarrow Z$ and $B : M \rightarrow Z$ such that:*

- (i) *A is continuous and satisfies (A),*
- (ii) *there exists $\gamma \in [0, 1[$ such that $w(A(S) + B(S)) \leq \gamma w(S)$ for all $S \subset M$,*
- (iii) *B is a strict contraction,*
- (iv) *$A(M) + B(M) \subseteq M$.*

Then there is $x \in M$ satisfying the equation $x = A(x) + B(x)$.

Remark 2.7.

- (i) Evidently, this result remains valid if the condition $A(M) + B(M) \subseteq M$ is replaced by the weaker one $(x = Bx + Ay, y \in M) \Rightarrow x \in M$ (cf., [8]).
- (ii) In Theorem 2.6, the operator B is assumed to be a strict contraction. In fact, this result remains valid if B belongs to other classes of operators. In the case where B is a separate contractions [24] (it is a large class of functions which contains strictly the class of large contractions and strict contractions), this result was proved in [29]. An extension of this result to expansive mappings is given in the appendix.

3. Existence result. In this section, we are concerned with the solvability of the functional integral equation (1.2). To do so, we will impose suitable conditions on the functions involved in that equation. Namely, we assume

- (i) $f_1 : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, and there are a function $0 < \alpha(\cdot) \in \mathbf{X}$ and a constant $0 \leq \beta$ such that $|f_1(t, x)| \leq \alpha(t) + \beta|x|$ for $t \in \mathbb{R}^+$ and for $x \in \mathbb{R}$;
- (ii) f_1 is a strict contraction with respect to the second variable;
- (iii) $f_2 : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, and there are a constant $\tau > 0$ and a function $0 \leq \rho \in \mathbf{X}$ such that $|f_2(t, x, y)| \leq \rho(t) + \tau(|x| + |y|)$ for $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$;
- (iv) $\zeta_1 : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$, $t \rightarrow \zeta_1(t, s, x)$ is measurable on \mathbb{R}^+ for all $(s, x) \in \mathbb{R}^+ \times \mathbb{R}$, and the function $(s, x) \rightarrow \zeta(t, s, x)$ is continuous on the set $\mathbb{R}^+ \times \mathbb{R}$ for almost all $t \in \mathbb{R}^+$;

- (v) $|\zeta_1(t, s, x)| \leq \kappa_1(t, s) + \kappa_2(t, s)|x|$ for $(t, s, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ where $\kappa_i : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $i = 1, 2$, satisfy the Carathéodory conditions; the linear operator K defined by

$$(K\psi)(t) = \int_0^t \kappa_2(t, s)\psi(s) ds$$

transforms \mathbf{X} into itself and $\int_0^t \kappa_1(t, s) ds \in \mathbf{X}$;

- (vi) $t \rightarrow \zeta_2(t, s, x)$ is measurable on \mathbb{R}^+ for all $(s, x) \in \mathbb{R}^+ \times \mathbb{R}$, and the function $(s, x) \rightarrow \zeta_2(t, s, x)$ is continuous on the set $\mathbb{R}^+ \times \mathbb{R}$ for all $t \in \mathbb{R}^+$;
- (vii) $|\zeta_2(t, s, x)| \leq \kappa_3(t, s)[\eta(s) + \nu|x|]$ for $(s, t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, where ν is a nonnegative constant, $0 \leq \eta(\cdot) \in \mathbf{X}$ and $\kappa_3 : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a measurable function such that the linear operator

$$(\widehat{K}\psi)(t) := \int_0^{+\infty} \kappa_3(t, s)\psi(s) ds$$

transforms \mathbf{X} into itself and is bounded;

- (viii) $\gamma := \beta + \tau(\|K\| + \nu\|\widehat{K}\|) < 1$.

Remark 3.1. It should be noticed that in condition (viii), $\|\kappa\|$ (respectively, $\|\widehat{K}\|$) denotes the norm of the operator K (respectively, \widehat{K}). The boundedness of K (respectively, \widehat{K}) follows from Lemma 2.3 (respectively, hypothesis (vii)).

Now we are ready to state and prove our existence result.

Theorem 3.2. *Assume that the hypotheses (i)–(viii) are satisfied. Then Problem (1.2) has at least one solution $\psi \in \mathbf{X}$.*

Proof. Defining on \mathbf{X} the operators

$$(\Pi\psi)(t) := \int_0^t \zeta_1(t, s, \psi(s)) ds,$$

$$(\Xi\psi)(t) := \int_0^{+\infty} \zeta_2(t, s, \psi(s)) ds,$$

we see that the problem (1.2) may take the form

$$\psi(t) = (\mathcal{N}_{f_1}\psi)(t) + (\mathcal{H}\psi)(t),$$

where \mathcal{N}_{f_1} is the superposition operator generated by the function f_1 and \mathcal{H} is defined by

$$(\mathcal{H}\psi)(t) = f_2(t, (\Pi\psi)(t), (\Xi\psi)(t)).$$

The proof uses Theorem 2.6 and will be derived in several steps.

Step 1. Continuity of the operators \mathcal{N}_{f_1} and \mathcal{H} . The continuity of \mathcal{N}_{f_1} follows from hypothesis (i) and Lemma 2.1.

Now we shall prove the continuity of the operators Π and Ξ . To do so, let $(\psi_n)_n$ be a sequence in \mathbb{X} which converges to $\varphi \in \mathbb{X}$. According to [18, Theorem 6.11], there exist a function $\varphi(\cdot) \in \mathbb{X}$ and a subsequence $(\psi_{n_k})_k$ which converges to ψ almost everywhere on \mathbb{R}^+ such that

$$(3.1) \quad |\psi_{n_k}(s)| \leq \varphi(s) \quad \text{almost everywhere in } \mathbb{R}^+.$$

Since $\psi_{n_k} \rightarrow \psi$ almost everywhere on \mathbb{R}^+ , the use of the continuity of ζ_1 with respect to the third variable gives:

$$\zeta_1(t, s, \psi_{n_k}(s)) \longrightarrow \zeta_1(t, s, \psi(s)) \quad \text{for almost all } s, t \in \mathbb{R}^+.$$

Next, the use of the estimate (3.1) together with assumption (v) yields

$$|\zeta_1(t, s, \psi_{n_k}(s))| \leq \kappa_1(t, s) + \kappa_2(t, s)\psi_{n_k}(s).$$

Accordingly, applying the Lebesgue dominated convergence theorem we get

$$(\Pi\psi_{n_k})(t) \longrightarrow (\Pi\psi)(t) \quad \text{for almost all } t \in \mathbb{R}^+.$$

By hypothesis (v), it is clear that

$$|(\Pi\psi_{n_k})(t)| \leq \int_0^t \kappa_1(t, s) ds + \int_0^t \kappa_2(t, s)\varphi(s) ds$$

for almost all $t \in \mathbb{R}^+$. Since, again by (v),

$$\int_0^{+\infty} \int_0^t \kappa_1(t, s) ds dt + \int_0^{+\infty} \int_0^t \kappa_2(t, s)\varphi(s) ds dt < +\infty,$$

the use of Lebesgue's dominated convergence theorem gives

$$\|\Pi\psi_{n_k} - \Pi\psi\|_{\mathbb{X}} \longrightarrow 0.$$

Since any sequence $(\psi_n)_n$ of \mathbb{X} converging to ψ in \mathbb{X} has a converging subsequence $(\psi_{n_k})_k$ such that $\Pi\psi_{n_k} \rightarrow \Pi\psi$ in \mathbb{X} , we infer that Π is continuous.

Similar reasoning proves that Ξ is a continuous operator from X into itself.

The continuity of \mathcal{H} follows from the continuity of Π, Ξ , hypotheses (iii)–(vii) and Lemma 2.1.

Step 2. There exists a real number $r > 0$ such that $\mathcal{N}_{f_1}(B_r) + \mathcal{H}(B_r) \subseteq B_r$. Indeed, if $\psi, \varphi \in X$, then we have

$$\begin{aligned} \|\mathcal{N}_{f_1}\psi + \mathcal{H}\varphi\| &\leq \int_0^{+\infty} |f_1(t, \psi(t))| dt \\ &\quad + \int_0^{+\infty} |f_2(t, (\Pi\varphi)(t), (\Xi\varphi)(t))| dt \\ &= \int_0^{+\infty} |f_1(t, \psi(t))| dt \\ &\quad + \int_0^{+\infty} \left| f_2\left(t, \int_0^t \zeta_1(t, s, \varphi(s)) ds, \right. \right. \\ &\quad \quad \quad \left. \left. \int_0^{+\infty} \zeta_2(t, s, \varphi(s)) ds\right) dt \\ &\leq \|\alpha\| + \beta\|\psi\| + \|\rho\| + \tau \int_0^{+\infty} |(\Pi\psi)(t)| dt \\ &\quad + \tau \int_0^{+\infty} |(\Xi\psi)(t)| dt. \end{aligned}$$

If $\|\kappa_1\| = \int_0^{+\infty} \int_0^t \kappa_1(t, s) ds dt$, then easy calculations give

$$\int_0^{+\infty} |(\Pi\psi)(t)| dt \leq \|\kappa_1\| + \|K\|\|\varphi\|,$$

and

$$\begin{aligned} \int_0^{+\infty} |(\Xi\psi)(t)| dt &\leq \int_0^{+\infty} \int_0^{+\infty} \kappa_3(t, s)\eta(s) ds dt \\ &\quad + \nu \int_0^{+\infty} \int_0^{+\infty} \kappa_3(t, s)|\varphi(s)| ds dt \\ &\leq \|\widehat{K}\|(\|\eta\| + \nu\|\varphi\|). \end{aligned}$$

Consequently,

$$(3.2) \quad \|\mathcal{N}_{f_1}\psi + \mathcal{H}\varphi\| \leq \|\alpha\| + \beta\|\psi\| + \|\rho\|$$

$$\begin{aligned}
& + \tau(\|\kappa_1\| + \|K\|\|\varphi\|) \\
& + \tau\|\widehat{K}\|(\|\eta\| + \nu\|\varphi\|) \\
\leq & \|\alpha\| + \|\rho\| + \tau(\|\kappa_1\| + \|\widehat{K}\|\|\eta\|) \\
& + (\beta + \tau\|K\| + \tau\nu\|\widehat{K}\|).
\end{aligned}$$

Let r be the real defined by

$$(3.3) \quad r := \frac{\|\alpha\| + \|\rho\| + \tau(\|\kappa_1\| + \|\widehat{K}\|\|\eta\|)}{1 - (\beta + \tau\|K\| + \tau\nu\|\widehat{K}\|)}.$$

It is clear that $r > 0$ (use condition (viii)). Now the estimate (3.2) implies that $\mathcal{N}_{f_1}(B_r) + \mathcal{H}(B_r) \subset B_r$.

Step 3. There exists $\gamma \in [0, 1)$ such that $\phi(\mathcal{N}_{f_1}(S) + \mathcal{H}(S)) \leq \gamma \phi(S)$ for all bounded subset S of \mathbf{X} .

Let $\varepsilon > 0$ be an arbitrary real, and take a nonempty set $D \subset \mathbb{R}^+$ such that $\mu(D) \leq \varepsilon$. Let S be a bounded subset of \mathbf{X} . For $\varphi, \psi \in S$, we have

$$\begin{aligned}
\int_D |(\mathcal{N}_{f_1}\varphi)(t) + (\mathcal{H}\psi)(t)| dt & \leq \int_D |f_1(t, \varphi(t))| dt \\
& + \int_D \left| f_2\left(t, \int_0^t \zeta_1(t, s, \varphi(s)) ds, \right. \right. \\
& \quad \left. \left. \int_0^{+\infty} \zeta_2(t, s, \varphi(s)) ds \right) dt \\
& \leq \int_D \alpha(t) dt + \beta \int_D |\varphi(t)| dt + \int_D \rho(t) dt \\
& + \tau \int_D \left| \int_0^t \zeta_1(t, s, \psi(s)) ds \right| dt \\
& + \tau \int_D \left| \int_0^{+\infty} \zeta_2(t, s, \psi(s)) ds \right| dt \\
& \leq \int_D \alpha(t) dt + \beta \int_D |\varphi(t)| dt + \int_D \rho(t) dt \\
& + \tau \int_D \int_0^t \kappa_1(t, s) ds dt \\
& + \tau \int_D \int_0^{+\infty} \kappa_2(t, s) |\varphi(s)| ds dt
\end{aligned}$$

$$\begin{aligned}
 & + \tau \int_D \int_0^{+\infty} \kappa_3(t, s) \eta(s) \, ds \, dt \\
 & + \int_D \int_0^{+\infty} \kappa_3(t, s) |\psi(s)| \, ds \, dt \\
 \leq & \int_D \alpha(t) \, dt + \beta \int_D |\varphi(t)| \, dt + \int_D \rho(t) \, dt \\
 & + \tau \int_D \int_0^t \kappa_1(t, s) \, ds \, dt \\
 & + \tau \|\widehat{K}\| \int_D |\eta(t)| \, dt \\
 & + \tau (\|K\| + \nu \|\widehat{K}\|) \int_D |\psi(t)| \, dt.
 \end{aligned}$$

Next, using the fact that sets consisting in one element are weakly compact we infer that

$$\begin{aligned}
 (3.4) \quad \vartheta_1(\mathcal{N}_{f_1}(S) + \mathcal{H}(S)) & \leq (\beta + \tau(\|K\| + \nu\|\widehat{K}\|))\vartheta_1(S) \\
 & = \gamma\vartheta_1(S).
 \end{aligned}$$

On the other hand, for $T \in \mathbb{R}^+$, we have $\int_T^{+\infty} |(\mathcal{N}_{f_1}\varphi)(t) + (\mathcal{H}\psi)(t)| \, dt$ is sufficiently small for T sufficiently large. So, using hypotheses (iii), (v) and (vii) for $\varphi, \psi \in S$, we have:

$$\begin{aligned}
 \int_T^{+\infty} |(\mathcal{N}_{f_1}\varphi)(t) + (\mathcal{H}\psi)(t)| \, dt & \leq \int_T^{+\infty} \alpha(t) \, dt \\
 & + \beta \int_T^{+\infty} |\varphi(t)| \, dt + \int_T^{+\infty} \rho(t) \, dt \\
 & + \tau \int_T^{+\infty} \int_0^t \kappa_1(t, s) \, ds \, dt \\
 & + \int_T^{+\infty} \int_0^t \kappa_2(t, s) |\psi(s)| \, ds \, dt \\
 & + \tau \int_T^{+\infty} \int_0^{+\infty} \kappa_3(t, s) \eta(s) \, ds \, dt \\
 & + \nu \int_T^{+\infty} \int_0^{+\infty} \kappa_3(t, s) |\psi(s)| \, ds \, dt \\
 & \leq \int_T^{+\infty} \alpha(t) \, dt + \beta \int_T^{+\infty} |\varphi(t)| \, dt
 \end{aligned}$$

$$\begin{aligned}
 & + \int_T^{+\infty} \rho(t) dt + \tau \int_T^{+\infty} \int_0^t \kappa_1(t, s) ds dt \\
 & + \tau \|\widehat{K}\| \int_T^{+\infty} \eta(t) dt + \tau (\|K\| + \nu \|\widehat{K}\|) \\
 & \quad \times \int_T^{+\infty} |\psi(t)| dt.
 \end{aligned}$$

According to the definition of ϕ_2 , we infer that

$$(3.5) \quad \phi_2(\mathcal{H}(S)) \leq (\beta + \tau(\|K\| + \nu\|\widehat{K}\|))\phi_2(S) = \gamma\phi_2(S).$$

Therefore, (3.4) and (3.5) imply

$$(3.6) \quad \phi(\mathcal{N}_{f_1}(S) + \mathcal{H}(S)) \leq \gamma\phi(S) \quad \text{for all } S \in \mathcal{B}(X)$$

Step 4. There exists a nonempty convex weakly compact subset M_∞ of B_r such that $\mathcal{N}_{f_1}(M_\infty) + \mathcal{H}(M_\infty) \subseteq M_\infty$. Indeed, set

$$M^0 = B_r \quad \text{and} \quad M^{n+1} = \overline{\text{co}}(\mathcal{N}_{f_1}(M^n) + \mathcal{H}(M^n)) \quad n = 1, 2, \dots,$$

where $\overline{\text{co}}(A)$ denotes the closed convex hull of the set A . It is clear that family $(M^n)_n$ consists of nonempty closed convex subset of X satisfying $M^{n+1} \subset M^n$, $n = 1, 2, \dots$. So, for all $n \in \mathbb{N} \setminus \{0\}$, we have

$$\phi(\mathcal{N}_{f_1}(M^n) + \mathcal{H}(M^n)) \leq \gamma\phi(M^n) \leq \dots \leq \gamma^n\phi(M^0).$$

Using the fact that $\gamma < 1$, we get $\lim_{n \rightarrow +\infty} \phi(M^n) = 0$. Set

$$M_\infty = \bigcap_0^{+\infty} M^n.$$

Applying Lemma 2.5 (iv), we conclude that M_∞ is a nonempty convex weakly compact subset of X ; moreover,

$$(3.7) \quad \mathcal{N}_{f_1}(M_\infty) + \mathcal{H}(M_\infty) \subseteq M_\infty.$$

Step 5. \mathcal{H} satisfies the condition (A) on the set M_∞ . The proof of this step is given in the appendix.

Step 6. Existence of a solution to problem (1.2). Indeed, let r be the real defined by (3.3) and restrict the problem (1.2) to the set $M_\infty \subset B_r$. It is clear that \mathcal{N}_{f_1} is a strict contraction on M_∞ (use

hypothesis (ii)). Now, using the results of Steps 1, 2, 3 and 5, one sees that the operators \mathcal{N}_{f_1} and \mathcal{H} satisfy the hypotheses of Theorem 2.6. Accordingly, problem (1.2) has at least a solution in $M_\infty \subset B_r$. \square

Remark 3.3.

- (i) Note that, as in many recent works (see, for example, [19, 10, 23]), instead of investigating equation (1.2), we can consider the equation

$$\psi(t) = f_1(t, \psi(\phi_1(t))) + f_2\left(t, \int_0^{\phi_2(t)} \zeta_1(t, s, \psi(\phi_3(s))) ds, \int_0^{+\infty} \zeta_2(t, s, \psi(\phi_4(s))) ds\right)$$

where $\phi_i, i = 1, 2, 3, 4$, are functions from \mathbb{R}^+ into \mathbb{R}^+ . Our existence result remains valid. It suffices to suppose that ϕ_2 is increasing, ϕ_i are absolutely continuous functions, $\lim_{t \rightarrow +\infty} \phi_i(t) = +\infty$, and there exist positive constants $\alpha_i > 0$ such that $\phi'_i(t) \geq \alpha_i$ for all $t \in \mathbb{R}^+, i = 1, 2, 3, 4$.

- (ii) Our existence result holds true if we suppose that the function f_1 is a large contraction, a separate contraction or expansive and onto with respect to the second variable (see Theorem 4.2).
- (iii) Finally, we note that Theorem 3.2 remains valid if the operator Π is defined by

$$(\Pi\psi)(t) = \int_0^{+\infty} \zeta_1(t, s, \psi(s)) ds.$$

4. Appendix. The proof of Step 5 uses Scorza-Dragoni’s theorem (Lemma 2.2), and it is an adaptation to our case of the arguments used in [4, 5] (see also [23, 28]). It seems that these arguments were first used by Emmanuele [15, 17].

Proof of Step 5. \mathcal{H} satisfies condition (A) on the set M_∞ . Indeed, let $\varepsilon > 0$, and consider an arbitrary sequence $(\psi_n)_n$ of M_∞ . Since M_∞ is weakly compact, by Lemma 2.4, there exists $\tau > 0$ such that

$$\int_\tau^{+\infty} |\psi_n(t)| dt \leq \frac{\varepsilon}{6}.$$

Since $\mathcal{H}(M_\infty) \subset M_\infty$, we infer that

$$(4.1) \quad \int_\tau^{+\infty} |\mathcal{H}(\psi_n(t))| dt \leq \frac{\varepsilon}{6}.$$

Consider the functions

$$\begin{aligned} f_2|_{[0,\tau] \times \mathbb{R}^+ \times \mathbb{R}}, & \quad \zeta_1|_{[0,\tau] \times \mathbb{R}^+ \times \mathbb{R}}, \\ \zeta_2|_{[0,\tau] \times \mathbb{R}^+ \times \mathbb{R}}, & \quad \eta|_{[0,\tau]}, \end{aligned}$$

and

$$\kappa_i|_{[0,\tau] \times \mathbb{R}^+}, \quad i = 1, 2, 3.$$

According to Lusin's theorem and Lemma 2.2, there exists a subset D_ε of \mathbb{R}^+ such that $\mu([0, \tau] \setminus D_\varepsilon) \leq \varepsilon$ and

$$\begin{aligned} f_2|_{D_\varepsilon \times \mathbb{R}^+ \times \mathbb{R}}, & \quad \zeta_1|_{D_\varepsilon \times \mathbb{R}^+ \times \mathbb{R}}, \\ \zeta_2|_{D_\varepsilon \times \mathbb{R}^+ \times \mathbb{R}}, & \quad \zeta|_{D_\varepsilon \times \mathbb{R}^+} \end{aligned}$$

and

$$\kappa_i|_{D_\varepsilon \times \mathbb{R}^+}, \quad i = 1, 2, 3,$$

are continuous. Set

$$\Lambda_i := \sup_{(t,s) \in D_\varepsilon \times [0,\tau]} \kappa_i(t, s), \quad i = 1, 2, 3.$$

For $\psi \in B_r$ and $t \in D_\varepsilon$, we have

$$\begin{aligned} (4.2) \quad \int_0^t |\zeta_1(t, s, \psi(t))| ds & \leq \int_0^t \kappa_1(t, s) ds \\ & \quad + \int_0^t \kappa_2(t, s) |\psi(s)| ds \\ & \leq \tau \sup_{(t,s) \in D_\varepsilon \times [0,\tau]} \kappa_1(t, s) \\ & \quad + \sup_{(t,s) \in D_\varepsilon \times [0,\tau]} \kappa_2(t, s) \times \int_0^t |\psi(s)| ds \\ & \leq \tau \Lambda_1 + r \Lambda_2 \end{aligned}$$

and

$$\begin{aligned}
 (4.3) \quad \int_0^{+\infty} |\zeta_2(t, s, \psi(t))| ds &\leq \int_0^{+\infty} \kappa_3(t, s)\eta(s) ds \\
 &\quad + \int_0^{+\infty} \kappa_3(t, s)\nu|\psi(s)| ds \\
 (4.4) \quad &\leq \Lambda_3 \left[\sup_{s \in D_\varepsilon} \eta(s)\tau + \nu r \right].
 \end{aligned}$$

Set

$$M_\varepsilon := \tau \Lambda_1 + r \Lambda_2 \quad \text{and} \quad \widehat{M}_\varepsilon := \Lambda_3 \left[\sup_{s \in D_\varepsilon} \eta(s)\tau + \nu r \right].$$

Let us take (arbitrary reals) $t_1, t_2 \in D_\varepsilon$ and suppose that $t_1 < t_2$. Using the uniform continuity of $\zeta_1|_{D_\varepsilon \times [0, \tau] \times [-M_\varepsilon, M_\varepsilon]}$, we can write:

$$\begin{aligned}
 &\left| \int_0^{t_2} \zeta_1(t_2, s, \psi_n(s)) ds - \int_0^{t_1} \zeta_1(t_2, s, \psi_n(s)) ds \right. \\
 &\leq \left| \int_0^{t_2} \zeta_1(t_2, s, \psi_n(s)) ds - \int_0^{t_2} \zeta_1(t_1, s, \psi_n(s)) ds \right| \\
 &\quad + \left| \int_{t_1}^{t_2} \zeta_1(t_2, s, \psi_n(s)) ds \right| \\
 &\leq \left| \int_0^\tau \zeta_1(t_2, s, \psi_n(s)) ds - \int_0^\tau \zeta_1(t_1, s, \psi_n(s)) ds \right| \\
 &\quad + \left| \int_{t_1}^{t_2} \zeta_1(t_2, s, \psi_n(s)) ds \right| \\
 &\leq \int_{[0, \tau] \setminus D_\varepsilon} |\zeta_1(t_2, s, \psi_n(s))| ds \\
 &\quad + \int_{[0, \tau] \setminus D_\varepsilon} |\zeta_1(t_1, s, \psi_n(s))| ds \\
 &\quad + \left| \int_{D_\varepsilon} \zeta_1(t_2, s, \psi_n(s)) ds - \int_{D_\varepsilon} \zeta_1(t_1, s, \psi_n(s)) ds \right| \\
 &\quad + \int_{t_1}^{t_2} \kappa_1(t_2, s) ds + \int_{t_1}^{t_2} \kappa_2(t_2, s)|\psi_n(s)| ds \\
 &\leq \phi^{\zeta_1}(|t_2 - t_1|) + 2\Lambda_1|[0, \tau] \setminus D_\varepsilon|
 \end{aligned}$$

$$\begin{aligned}
 &+ 2\Lambda_2 \int_{[0, \tau] \setminus D_\varepsilon} |\psi_n(s)| \, ds + \Lambda_1 |t_2 - t_1| \\
 &+ \Lambda_2 \int_{t_1}^{t_2} |\psi_n(s)| \, ds \\
 \leq &\varphi^{\zeta_1}(|t_2 - t_1|) + 2\Lambda_1 |[0, \tau] \setminus D_\varepsilon| + 2\Lambda_2 r |[0, \tau] \setminus D_\varepsilon| \\
 &+ \Lambda_1 |t_2 - t_1| + \Lambda_2 \int_{t_1}^{t_2} |\psi_n(s)| \, ds
 \end{aligned}$$

where $\varphi^{\zeta_1}(|t_2 - t_1|)$ denotes the modulus of continuity of ζ_1 on the set $D_\varepsilon \times [0, \tau] \times [-M_\varepsilon, M_\varepsilon]$. Observe that the weak compactness of the set $\{\psi_n : n \in \mathbb{N}\}$ together with Lemma 2.4 implies that the numbers $\int_{t_1}^{t_2} |\psi_n(s)| \, ds$ are arbitrary small provided the number $t_2 - t_1$ is small enough. This proves that the right hand side of the previous inequality tends to zero independent of n as $t_2 - t_1$ tends to zero. Therefore, using (4.2), we conclude that the sequence

$$\left(\int_0^t |\zeta_1(t, s, \psi_n(s))| \, ds \right)_{n \in \mathbb{N}}$$

is equibounded and equicontinuous in the space $C(D_\varepsilon)$ (the space of all continuous function on D_ε). Similar reasoning using (4.3) and the same arguments as above proves that the sequence

$$\left(\int_0^{+\infty} |\zeta_2(t, s, \psi_n(s))| \, ds \right)_{n \in \mathbb{N}}$$

is equibounded and equicontinuous in the space $C(D_\varepsilon)$. Next, using the fact that $f_2|_{D_\varepsilon \times [-M_\varepsilon, M_\varepsilon] \times [-\widehat{M}_\varepsilon, \widehat{M}_\varepsilon]}$ is uniformly continuous, we infer that the sequence $(\mathcal{H}(\psi_n))_{n \in \mathbb{N}}$ is equibounded on D_ε . Applying the Arzèla-Ascoli theorem we see that the set $\{\mathcal{H}(\psi_n), n = 1, 2, \dots\}$ forms a relatively compact set in the space $C(D_\varepsilon)$.

It should be noticed that the above results do not depend on the choice of ε . Therefore, we can construct a sequence $(D_{1/k})_{k \in \mathbb{N}}$ of closed subsets of interval $[0, \tau]$ such that

$$\mu([0, \tau] \setminus D_{1/k}) \longrightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

and the set $\{\mathcal{H}(\psi_n), n = 1, 2, \dots\}$ is relatively compact in every space $C(D_{1/k})$. Passing to a subsequence, if necessary, we can assume that

$(\mathcal{H}(\psi_n))_n$ is a Cauchy sequence in each space $C(D_{1/k})$, $k = 1, 2, \dots$

Next, using the fact that $\mathcal{H}(M_\infty)$ is relatively weakly compact (because M_∞ is weakly compact and $\mathcal{H}(M_\infty) \subset M_\infty$), we can choose $\delta > 0$ such that, for each closed subset D of the interval $[0, \tau]$ satisfying $\mu([0, \tau] \setminus D) \leq \delta$, we have

$$(4.5) \quad \int_{[0, \tau] \setminus D} |\mathcal{H}(\psi(s))| ds \leq \frac{\varepsilon}{6}$$

for any $x \in M_\infty$. Keeping in mind the fact that $(\mathcal{H}(\psi_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in each space $C(D_{1/k})$, we can choose an integer k_0 such that $\mu([0, \tau] \setminus D_{1/k_0}) \leq \delta$ and, for arbitrary integers $n, m \geq k_0$, the following inequality holds:

$$(4.6) \quad |(\mathcal{H}\psi_n)(t) - (\mathcal{H}\psi_m)(t)| \leq \frac{\varepsilon}{3\mu(D_{1/k_0})}$$

for any $t \in D_{1/k_0}$. Combining (4.1), (4.5) and (4.6), we get

$$\begin{aligned} \|\mathcal{H}(\psi_n) - \mathcal{H}(\psi_m)\| &= \int_0^{+\infty} |\mathcal{H}(x_n)(t) - \mathcal{H}(x_m)(t)| dt \\ &= \int_\tau^{+\infty} |\mathcal{H}(\psi_n)(t) - \mathcal{H}(\psi_m)(t)| dt \\ &\quad + \int_{D_{1/k_0}} |\mathcal{H}(\psi_n)(t) - \mathcal{H}(\psi_m)(t)| dt \\ &\quad + \int_{[0, \tau] \setminus D_{1/k_0}} |\mathcal{H}(\psi_n)(t) - \mathcal{H}(x_m)(t)| dt \\ &\leq \varepsilon. \end{aligned}$$

This proves that $(\mathcal{H}(\psi_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{X} . The completeness of \mathbb{X} ensures that \mathcal{H} satisfies the condition (A). \square

In the remainder of this appendix we extend Theorem 2.6 to the class of expansive mappings.

Let (Z, d) be a metric space, and let M be a subset of Z . A mapping $B : M \rightarrow Z$ is said to be expansive, if there exists a constant $\mu > 1$ such that

$$d(Bx, By) \geq \mu d(x, y), \quad \text{for all } x, y \in M.$$

The following result gives an important property of expansive mappings.

Lemma 4.1. [30, Theorem 2.1]. *Let M be a nonempty closed subset of a Banach space Z , and let B be a function from M into Z . If B is expansive and onto, then B possesses a unique fixed point belonging to M .*

The following statement is an improvement of Theorem 2.6.

Theorem 4.2. *Let M be a nonempty bounded closed convex subset of a Banach space Z . Suppose that $A : M \rightarrow Z$ and $B : M \rightarrow Z$ satisfy conditions (a), (b) and (d) of Theorem 2.6. Further, if B is expansive and onto, then there is $x \in M$ such $x = A(x) + B(x)$.*

Proof. For any $z \in M$, the map π from M into Z defined by $\pi(x) = B(x) + z$ satisfies $\|\pi(x) - \pi(x')\| = \|B(x) - B(x')\| \geq \varrho\|x - x'\|$. So, it is expansive and onto. Thus, Lemma 4.1 shows that the equation $x = B(x) + z$ has a unique solution $u(z)$ in M . Therefore, u defines a bijective function from M into itself. Next, let $y_1, y_2 \in M$. Using

$$B(u(y_1)) + y_1 = u(y_1) \quad \text{and} \quad B(u(y_2)) + y_2 = u(y_2),$$

we arrive at the inequality

$$\|B(u(y_1)) - B(u(y_2))\| \leq \|y_1 - y_2\| + \|u(y_1) - u(y_2)\|.$$

Since B is expansive, there exists a constant $\varrho > 1$ such that

$$\|B(u(y_1)) - B(u(y_2))\| \geq \varrho\|u(y_1) - u(y_2)\|,$$

and, consequently,

$$\|u(y_1) - u(y_2)\| \leq \frac{1}{\varrho - 1}\|y_1 - y_2\|.$$

This implies that the map $u : M \rightarrow M$ is continuous. Using the continuity of A we deduce that $u \circ A$ is continuous. The remainder of the proof is now similar to that of [22, Theorem 2.1]. \square

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UNIVERSITÉ BLAISE PASCAL (CLERMONT II), LABORATOIRE DE MATHÉMATIQUES,
CNRS UMR 6620, UMR 6620 - CNRS, COMPLEXE DES CÉZEAUX, BP 80026,
63117 AUBIÈRE, FRANCE

Email address: Khalid.Latrach@math.univ-bpclermont.fr