JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 26, Number 3, Fall 2014

ASYMPTOTICALLY TYPED SOLUTIONS TO A SEMILINEAR INTEGRAL EQUATION

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ABSTRACT. In this paper, we investigate the existence of μ-pseudo almost automorphic solutions to the semilinear integral equation $x(t) = \int_{-\infty}^{t} a(t-s)[Ax(s) + f(s, x(s))]\, ds$, $t \in \mathbf{R}$ in a Banach space $\mathbf{X}$, where $a \in L^1(\mathbf{R}_+)$, A is the generator of an integral resolvent family of linear bounded operators defined on the Banach space $\mathbf{X}$, and $f : \mathbf{R} \times \mathbf{X} \to \mathbf{X}$ is a μ-pseudo almost automorphic function. The main results are proved by using integral resolvent families combined with the theory of μ-pseudo almost automorphic functions.

1. Introduction. In this paper, we are mainly concerned with the existence of μ-pseudo almost automorphic mild solutions to the following semilinear integral equations such as

$$x(t) = \int_{-\infty}^{t} a(t-s)[Ax(s) + f(s, x(s))]\, ds, \quad t \in \mathbf{R}, \tag{1}$$

where $a \in L^1(\mathbf{R}_+)$, $A : D(A) \subseteq \mathbf{X} \to \mathbf{X}$ is the generator of an integral resolvent family defined on a complex Banach space $\mathbf{X}$, and $f : \mathbf{R} \times \mathbf{X} \to \mathbf{X}$ is a μ-pseudo almost automorphic function satisfying some suitable conditions given later.

The concept of almost automorphy was first introduced in the literature by Bochner **[7, 8]**; it is an important generalization of the classical almost periodicity. For more details on these topics, we refer the reader to **[1, 2, 3, 4, 11, 12, 15, 14, 13, 23, 24, 25]** and the references therein. Since then, almost automorphy has become

2010 AMS *Mathematics subject classification.* Primary 34K14, 60H10, 35B15, 34F05.

Keywords and phrases. μ-pseudo almost automorphic function, semilinear integral equations, integral resolvent family, fixed point.

This work was supported by NSF of China (No. 11361032) and the Program for Longyuan Youth Innovative Talents of Gansu Province of China (No. 2014-4-80).

Received by the editors on June 19, 2013, and in revised form on December 19, 2013.

DOI:10.1216/JIE-2014-26-3-323

one of the most attractive topics in the qualitative theory of evolution equations, and there have been several interesting, natural and powerful generalizations of the classical almost automorphic functions. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [**26**]. Liang, Xiao and Zhang in [**19, 31**] presented the concept of pseudo almost automorphy. In [**27**], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to investigate the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Blot et al. introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [**6**], which generalizes that of pseudo-almost automorphic functions. Xia and Fan presented the notation of Stepanov-like weighted pseudo almost automorphic functions in [**30**]. Zhang, Chang and N'Guérékata investigated some properties and new ergodic theorems of Stepanov-like weighted pseudo almost automorphic functions in [**33, 34**]. Recently, Blot, Cieutat and Ezzinbi in [**5**] applied the measure theory to define an ergodic function, and they investigated many powerful properties of μ-pseudo almost automorphic functions, and the classical theories of pseudo almost automorphy and weighted pseudo almost automorphy become particular cases of μ-pseudo almost automorphy.

In recent years, Cuevas and Lizama [**10**] studied the existence and uniqueness of almost automorphic solutions to equation (1). In [**17**], Henríquez and Lizama investigated the existence and regularity of compact almost automorphic solutions to semilinear integral equation (1). The authors investigated the existence of pseudo-almost automorphic solutions to equation (1) in [**36**]. And, in [**35**], the existence of weighted pseudo almost automorphic solutions to equation (1) with S^p -weighted pseudo almost automorphic coefficients was investigated. However, to the best of our knowledge, the existence of μ-pseudo almost automorphic solutions to equation (1) is an untreated topic in the literature. Motivated by the above-mentioned works [**10, 17, 36, 35**], the main purpose of this paper is to investigate the existence results of μ-pseudo almost automorphic solutions to problem (1) by using integral resolvent families combined with the theory of μ-pseudo almost automorphic functions. Our main results can be seen as a generalization of the classical results on almost automorphic, pseudo almost automorphic and weighted pseudo almost automorphic solutions in [**10, 17, 36, 35**].

The rest of this paper is organized as follows. In Section 2, we introduce some basic definitions, lemmas and preliminary results which will be used throughout this paper. In Section 3, we prove some existence results of μ-pseudo almost automorphic mild solutions for the semilinear integral equation (1).

2. Preliminaries. In this section, we fix some basic definitions, notations, lemmas and preliminary facts which will be used in the sequel. Throughout the paper, the notation $(\mathbf{X}, \|\cdot\|)$ is a complex Banach space and $BC(\mathbf{R}, \mathbf{X})$ denotes the Banach space of all bounded continuous functions from $\mathbf{R}$ to $\mathbf{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbf{R}} \|f(t)\|$. Furthermore, we denote by $\mathfrak{B}(\mathbf{X})$ the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{X}$ endowed with the operator topology, and the notation $\rho(A)$ stands for the resolvent set of A.

Throughout this work, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbf{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbf{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbf{R}(a < b)$.

Definition 2.1. [8] A continuous function $f : \mathbf{R} \to \mathbf{X}$ is said to be almost automorphic if, for every sequence of real numbers $\{s'_n\}_{n\in\mathbf{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbf{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbf{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbf{R}$. The collection of all such functions will be denoted by $AA(\mathbf{R}, \mathbf{X})$.

Define

$$PAA_0(\mathbf{R}, \mathbf{X}) := \left\{\phi \in BC(\mathbf{R}, \mathbf{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0\right\}.$$

In the same way, we define $PAA_0(\mathbf{R} \times \mathbf{X}, \mathbf{X})$ as the collection of jointly continuous functions $f : \mathbf{R} \times \mathbf{X} \to \mathbf{X}$ which belong to $BC(\mathbf{R} \times \mathbf{X}, \mathbf{X}))$

and satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma, x)\|\, d\sigma = 0$$

uniformly in the compact subset of $\mathbf{X}$.

Definition 2.2. [**18, 32**] A continuous function $f : \mathbf{R} \to \mathbf{X}$ (respectively $\mathbf{R} \times \mathbf{X} \to \mathbf{X}$) is called pseudo-almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbf{R}, \mathbf{X})$(respectively $AA(\mathbf{R}\times\mathbf{X}, \mathbf{X})$) and $\phi \in PAA_0(\mathbf{R}, \mathbf{X})$ (respectively $PAA_0(\mathbf{R}\times\mathbf{X}, \mathbf{X})$). Denote by $PAA(\mathbf{R}, \mathbf{X})$ (respectively $PAA(\mathbf{R} \times \mathbf{X}, \mathbf{X})$) the set of all such functions.

Definition 2.3. [**5**] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbf{R} \to \mathbf{X}$ is said to be μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t), d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbf{R}, \mathbf{X}, \mu)$.

Definition 2.4. [**5**] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbf{R} \to \mathbf{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbf{R}, \mathbf{X})$ and $\phi \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbf{R}, \mathbf{X}, \mu)$.

Obviously, we have $AA(\mathbf{R}, \mathbf{X}) \subset PAA(\mathbf{R}, \mathbf{X}, \mu) \subset BC(\mathbf{R}, \mathbf{X})$.

Lemma 2.5. [**5,** Proposition 2.13] *Let $\mu \in \mathcal{M}$. Then $(\varepsilon(\mathbf{R}, \mathbf{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.6. [**5,** Theorem 4.1] *Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbf{R}, \mathbf{X}, \mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbf{R}, \mathbf{X})$ and $\phi \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$. If $PAA(\mathbf{R}, \mathbf{X}, \mu)$ is translation invariant, then $\{g(t) : t \in \mathbf{R}\} \subset \overline{\{f(t) : t \in \mathbf{R}\}}$, (the closure of the range of f).*

Lemma 2.7. [**5,** Theorem 2.14] *Let $\mu \in \mathcal{M}$ and I be the bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbf{R}, \mathbf{X})$. Then the following assertions are equivalent.*

(i) $f \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$.
(ii) $\lim r \to +\infty \frac{1}{\mu([-r,r]\setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| \, d\mu(t) = 0$.
(iii) *For any* $\varepsilon > 0$, $\lim r \to +\infty \frac{\mu(\{t\in[-r,r]\setminus I : \|f(t)\| > \varepsilon\})}{\mu([-r,r]\setminus I)} = 0$.

Lemma 2.8. [**5,** Theorem 4.7] *Let* $\mu \in \mathcal{M}$. *Assume that* $PAA(\mathbf{R}, \mathbf{X}, \mu)$ *is translation invariant. Then the decomposition of a* μ*-pseudo almost automorphic function in the form* $f = g + \phi$ *where* $g \in AA(\mathbf{R}, \mathbf{X})$ *and* $\phi \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$ *is unique.*

Lemma 2.9. [**5,** Theorem 4.9] *Let* $\mu \in \mathcal{M}$. *Assume that* $PAA(\mathbf{R}, \mathbf{X}, \mu)$ *is translation invariant. Then* $(PAA(\mathbf{R}, \mathbf{X}, \mu), \|\cdot\|_\infty)$ *is a Banach space.*

We recall that the Laplace transform of a function $f \in L^1_{\mathrm{loc}}(\mathbf{R}_+, \mathbf{X})$ is given by

$$\mathcal{L}(f)(\lambda) := \widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) \, dt, \quad \operatorname{Re} \lambda > \omega,$$

where the integral is absolutely convergent for $\operatorname{Re} \lambda > \omega$. In order to establish an operator theoretical approach to equation (1), we recall the following definition.

Definition 2.10. [**22**] Let A be a closed linear operator with domain $D(A) \subseteq \mathbf{X}$. We say that A is the generator of an integral resolvent if there exist $\omega \geq 0$ and a strongly continuous function $S : \mathbf{R}_+ \to \mathfrak{B}(\mathbf{X})$ such that $\{1/\widehat{a}(\lambda) : \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$ and

$$\left(\frac{1}{\widehat{a}(\lambda)} I - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x \, dt, \quad \operatorname{Re} \lambda > \omega, \ x \in \mathbf{X}.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

Now, we establish several relations between the integral resolvent family and its generator. The following result is a direct consequence of [**20,** Proposition 3.1, Lemma 2.2].

Lemma 2.11. *Let* $S(t)$ *be the integral resolvent family on* $\mathbb{X}$ *with generator* A. *Then the following properties hold*:

(b-1) *$S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.*

(b-2) *Let $x \in D(A)$ and $t \geq 0$. Then*

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)x\,ds.$$

(b-3) *Let $x \in \mathbf{X}$ and $t \geq 0$. Then $\int_0^t a(t-s)AS(s)x\,ds \in D(A)$ and*

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)x\,ds.$$

In particular, $S(0) = a(0)I$.

For more on integral resolvent families and related issues, we refer the reader to [**20, 21, 28, 29**].

Now, we recall a useful compactness criterion.

Let $h : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbf{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(\mathbf{X}) = \left\{ u \in C(\mathbf{R}, \mathbf{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0 \right\}.$$

Endowed with the norm $\|u\|_h = \sup_{t\in\mathbf{R}} \|u(t)\|/h(t)$, it is a Banach space (see [**17**]).

Lemma 2.12. [**17**] *A subset $K \subseteq C_h(\mathbf{X})$ is a relatively compact set if it verifies the following conditions*:

(c-1) *The set $K(t) = \{u(t) : u \in K\}$ is relatively compact in $\mathbf{X}$ for each $t \in \mathbf{R}$.*

(c-2) *The set K is equicontinuous.*

(c-3) *For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \leq \varepsilon h(t)$ for all $u \in K$ and all $|t| > L$.*

Lemma 2.13. [**16**] (Leray-Schauder alternative theorem). *Let D be a closed convex subset of a Banach space $\mathbf{X}$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

3. Main results. This section is mainly focused upon some existence results of μ-pseudo almost automorphic solutions to problem (1).

Theorem 3.1. *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbf{R} \times \mathbf{X}, \mathbf{X}, \mu)$. Assume that*

(a1) *$f(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbf{X}$ uniformly in $t \in \mathbf{R}$.*
(a2) *$g(t, x)$ is uniformly continuous on any bounded subset $Q \subset \mathbf{X}$ uniformly in $t \in \mathbf{R}$.*

Then the function is defined by $F(\cdot) := f(\cdot, \phi(\cdot)) \in PAA(\mathbf{R}, \mathbf{X}, \mu)$ if $\phi \in PAA(\mathbf{R}, \mathbf{X}, \mu)$.

Proof. The main proof of this theorem is conducted similarly as that of [**9,** Theorem 3.1]. For completeness and readability, we give the detailed proof here. Let $f = g + h$ with $g \in AA(\mathbf{R} \times \mathbf{X}, \mathbf{X})$, $h \in \varepsilon(\mathbf{R} \times \mathbf{X}, \mathbf{X}, \mu)$, and $\phi = u + v$, with $u \in AA(\mathbf{R}, \mathbf{X})$, and $v \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$.

Now we define

$$\begin{aligned} F(t) &= g(t, u(t)) + f(t, \phi(t)) - g(t, u(t)) \\ &= g(t, u(t)) + f(t, \phi(t)) - f(t, u(t)) + h(t, u(t)). \end{aligned}$$

Let us rewrite

$$G(t) = g(t, u(t)),\ \Phi(t) = f(t, \phi(t)) - f(t, u(t)),\ H(t) = h(t, u(t)).$$

Thus, we have $F(t) = G(t) + \Phi(t) + H(t)$. In view of [**19,** Lemma 2.2], $G(t) \in AA(\mathbf{R}, \mathbf{X})$. Next we prove that $\Phi(t) \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$. Clearly, $\Phi(t) \in BC(\mathbf{R}, \mathbf{X})$. For Φ to be in $\varepsilon(\mathbf{R}, \mathbf{X}, \mu)$, it is enough to show that

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\Phi(t)\| \, d\mu(t) = 0.$$

By Lemma 2.6, $u(\mathbf{R}) \subset \overline{\phi(\mathbf{R})}$ is a bounded set. From assumption (a1) with $Q = \overline{\phi(\mathbf{R})}$, we conclude that, for each $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $t \in \mathbf{R}$,

$$\|\phi - u\| \le \delta \Longrightarrow \|f(t, \phi(t)) - f(t, u(t))\| \le \varepsilon.$$

Denote by the following set $A_{r,\varepsilon} = \{t \in [-r,r] : \|f(t)\| > \varepsilon\}$. Thus, we obtain

$$\begin{aligned} A_{r,\varepsilon}(\Phi) &= A_{r,\varepsilon}(f(t,\phi(t)) - f(t,u(t))) \subset A_{r,\delta}(\phi(t) - u(t)) \\ &= A_{r,\delta}(v). \end{aligned}$$

Therefore, the following inequality holds

$$\frac{\mu\{t \in [-r,r] : \|f(t,\phi(t)) - f(t,u(t))\| > \varepsilon\}}{\mu([-r,r])} \leq \frac{\mu\{t \in [-r,r] : \|\phi(t) - u(t)\| > \delta\}}{\mu([-r,r])}.$$

Since $\phi(t) = u(t) + v(t)$ and $v \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$, Lemma 2.7 yields that, for the above-mentioned δ, we have

$$\lim_{r\to\infty} \frac{\mu\{t \in [-r,r] : \|\phi(t) - u(t)\| > \delta\}}{\mu([-r,r])} = 0,$$

and then we obtain

$$\lim_{r\to\infty} \frac{\mu\{t \in [-r,r] : \|f(t,\phi(t)) - f(t,u(t))\| > \varepsilon\}}{\mu([-r,r])} = 0. \tag{2}$$

From Lemma 2.7 and the relation (2), we draw a conclusion that $\Phi(t) \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$.

Finally, we have only to show that $H(t) = h(t,u(t)) \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$. We have the set $u([-r,r])$ is compact since u is continuous on $\mathbf{R}$ as almost automorphic functions. So, the function g belongs to $AA(\mathbf{R} \times \mathbf{X}, \mathbf{X})$, and g is uniformly continuous on $[-r,r] \times u([-r,r])$. Then it follows from (a1) that $h(t,x)$ is uniformly continuous with $x \in u([-r,r])$ uniformly in $t \in [-r,r]$. Thus, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that, for $x_1,\ x_2 \in u([-r,r])$ with $\|x_1 - x_2\| < \delta$, we have

$$\|h(t,x_1) - h(t,x_2)\| < \frac{\varepsilon}{2}, \quad \text{for all } t \in [-r,r]. \tag{3}$$

On the other hand, since the set $u([-r,r])$ is compact, there exist finite balls O_k with $\beta_k \in u([-r,r])$, $k = 1,\ldots,m$, and radius δ given above, such that $u([-r,r]) \subset \bigcup_{k=1}^{m} O_k$. Then the sets $U_k := \{t \in [-r,r] : u(t) \in O_k\}$, $k = 1,\ldots,m$ are open in $[-r,r]$ and $[-r,r] = \bigcup_{k=1}^{m} U_k$.

Define V_k by

$$V_1 = U_1, \qquad V_k = U_k - \cup_{i=1}^{k-1} U_i, \quad 2 \le k \le m.$$

Then it is obvious that $V_i \cap V_j = \emptyset$, if $i \neq j$, $1 \le i, j \le m$. So we get

$$\begin{aligned}\Lambda :&= \{t \in [-r,r] : \|H(t)\| \ge \varepsilon\}\\ &= \{t \in [-r,r] : \|h(t,u(t))\| \ge \varepsilon\}\\ &\subset \bigcup_{k=1}^{m} \{t \in V_k : \|h(t,u(t)) - h(t,\beta_k)\| + \|h(t,\beta_k)\| \ge \varepsilon\}\\ &\subset \bigcup_{k=1}^{m} \left(\left\{ t \in V_k : \|h(t,u(t)) - h(t,\beta_k)\| \ge \frac{\varepsilon}{2} \right\} \right.\\ &\left. \bigcup \left\{ t \in V_k : \|h(t,\beta_k)\| \ge \frac{\varepsilon}{2} \right\} \right).\end{aligned}$$

It follows from relation (3) that

$$\left\{ t \in V_k : \|h(t,u(t)) - h(t,\beta_k)\| \ge \frac{\varepsilon}{2} \right\} = \emptyset, \quad k = 1, \dots, m.$$

Thus, if we set $A_{r,(\varepsilon/2)}(h_k) := A_{r,(\varepsilon/2)}(h(t,\beta_k))$, then $A_{r,\varepsilon}(H) \subset \bigcup_{k=1}^{m} A_{r,(\varepsilon/2)}(h_k)$ and

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|H(t)\| \, d\mu(t) \le \sum_{k=1}^{m} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h_k(t)\| \, d\mu(t).$$

And, since $h \in \varepsilon(\mathbf{R} \times \mathbf{X}, \mathbf{X}, \mu)$, we have

$$\lim_{r \to \infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h_k(t)\| \, d\mu(t) = 0, \quad k = 1, \dots, m.$$

It follows that $\lim_{r\to\infty} 1/\mu([-r,r]) \int_{[-r,r]} \|H(t)\| \, d\mu(t) = 0$. According to Lemma 2.7, we deduce that $H(t) = h(t,u(t)) \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$. This completes the proof. □

From the above theorem, we have the following result.

Corollary 3.2. *Let $\mu \in \mathcal{M}$. Suppose that $f = g + h \in PAA((\mathbf{R}, \mathbf{X}, \mu))$ with $g \in AA(\mathbf{R}, \mathbf{X})$, $h \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$, and both f and g are Lipschitizian with $x \in \mathbf{X}$ uniformly in $t \in \mathbf{R}$. Then the function defined by $F(\cdot) := f(\cdot, \phi(\cdot)) \in PAA((\mathbf{R}, \mathbf{X}, \mu))$ if $\phi \in PAA((\mathbf{R}, \mathbf{X}, \mu))$.*

Definition 3.3. [**17**] Let A be the generator of a resolvent family $\{S(t)\}_{t\geq 0}$. A continuous function $x:\mathbf{R}\to\mathbf{X}$ satisfying the integral equation

$$x(t)=\int_{-\infty}^{t} S(t-s)f(s)\,ds, \quad \text{for all } t\in\mathbf{R},$$

is called a mild solution on $\mathbf{R}$ to equation (1).

First, we list the following basic assumptions:

(H1) For strongly continuous functions, $S:[0,\infty)\to\mathfrak{B}(\mathbf{X})$, there exists $\phi\in L^1(\mathbf{R}_+)$ such that $\|S(t)\|\leq\phi(t)$ for all $t\in\mathbf{R}_+$.

(H2) $f\in PAA(\mathbf{R}\times\mathbf{X},\mathbf{X},\mu)$ and there exists a positive number L_f such that

$$\|f(t,x)-f(t,y)\|\leq L_f\|x-y\|$$

for all $t\in\mathbf{R}$ and each x, $y\in\mathbf{X}$.

(H3) $f\in PAA(\mathbf{R}\times\mathbf{X},\mathbf{X},\mu)$, and there exists a nonnegative function $L_f\in L^1(\mathbf{R})$ such that

$$\|f(t,x)-f(t,y)\|\leq L_f(t)\|x-y\|$$

for all $t\in\mathbf{R}$ and each x, $y\in\mathbf{X}$.

Lemma 3.4. *Let $\mu\in\mathcal{M}$. Let $\{S(t)\}_{t\geq 0}\subset\mathfrak{B}(\mathbf{X})$ be a strongly continuous family of bounded linear operators that satisfies assumption* (H1). *If $f:\mathbf{R}\to\mathbf{X}$ is a μ-pseudo almost automorphic function, and $F(t)$ is given by*

$$F(t)=\int_{-\infty}^{t} S(t-s)f(s)\,ds, \quad t\in\mathbf{R},$$

then $F\in PAA(\mathbf{R},\mathbf{X},\mu)$.

Proof. Since $f\in PAA(\mathbf{R},\mathbf{X},\mu)$, we have by definition that $f=g+h$, where $g\in AA(\mathbf{R},\mathbf{X})$ and $h\in\varepsilon(\mathbf{R},\mathbf{X},\mu)$. Then

$$F(t)=\int_{-\infty}^{t} S(t-s)g(s)\,ds+\int_{-\infty}^{t} S(t-s)h(s)\,ds=G(t)+H(t),$$

where $G(t)=\int_{-\infty}^{t} S(t-s)g(s)\,ds$ and $H(t)=\int_{-\infty}^{t} S(t-s)h(s)\,ds$. From the proof of Cuevas [**10,** Lemma 3.1], it follows that $t\to G(t)$ is almost automorphic. To complete the proof, we show that $H(t) \in \varepsilon(\mathbf{R}, \mathbf{X}, \mu)$. For $r > 0$, we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|H(t)\| \, d\mu(t) \\
&\qquad \leq \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{-\infty}^{t} \|S(t-s)\| \|h(s)\| \, ds \, d\mu(t) \\
&\qquad \leq \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{0}^{\infty} \|S(s)\| \|h(t-s)\| \, ds \, d\mu(t) \\
&\qquad \leq \int_{0}^{\infty} \phi(s) \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t-s)\| \, d\mu(t) \right) ds.
\end{aligned}$$

Now, using the fact that the space $\varepsilon(\mathbf{R}, \mathbf{X}, \mu)$ is translation invariant, it follows that $t \to h(t-s)$ belongs to $\varepsilon(\mathbf{R}, \mathbf{X}, \mu)$ for each $s \in \mathbf{R}$. Moreover, since $\phi(s)$ is integrable in $[0, \infty)$, using the Lebesgue dominated convergence theorem, it follows that

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|H(t)\| d\mu(t) = 0.$$

The proof is now complete. □

Theorem 3.5. *Let $\mu \in \mathcal{M}$. Assume that A generates an integral resolvent family $\{S(t)\}_{t \geq 0}$ and that* (H1)–(H2) *hold. If $L_f \leq \|\phi\|_{L^1(\mathbf{R}_+)}^{-1}$, then equation* (1) *has a unique μ-pseudo almost automorphic mild solution.*

Proof. We define the nonlinear operator $\Lambda : PAA(\mathbf{R}, \mathbf{X}, \mu) \to PAA(\mathbf{R}, \mathbf{X}, \mu)$ by

$$(\Lambda x)(t) := \int_{-\infty}^{t} S(t-s) f(s, x(s)) \, ds, \quad t \in \mathbf{R}.$$

Given $x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$, it follows from Theorem 3.1 that the function $s \to f(s, x(s))$ is in $PAA(\mathbf{R}, \mathbf{X}, \mu)$. Now, by Lemma 3.4, we have that $\Lambda x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$. Hence, Λ is well defined. Now it suffices to show that the operator Λ has a unique fixed point in $PAA(\mathbf{R}, \mathbf{X}, \mu)$. For this, let x and y be in $PAA(\mathbf{R}, \mathbf{X}, \mu)$. We have

$$\|\Lambda x - \Lambda y\|_{\infty} = \sup_{t \in \mathbf{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s, x(s)) - f(s, y(s))] \, ds \right\|$$

$$\begin{aligned} &\leq L_f \sup_{t\in\mathbf{R}} \int_0^\infty \|S(s)\|\|x(t-s)-y(t-s)\|\,ds \\ &\leq L_f\|x-y\|_\infty \int_0^\infty \phi(s)\,ds \\ &= L_f\|\phi\|_{L^1(\mathbf{R}_+)}\|x-y\|_\infty. \end{aligned}$$

This implies that Λ is a contraction, so by the Banach contraction principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Λ in $PAA(\mathbf{R},\mathbf{X},\mu)$ such that $\Lambda x = x$. It is clear that x is a μ-pseudo almost automorphic mild solution of equation (1). The proof is complete. □

An immediate consequence of Theorem 3.5 and [**17,** Corollary 3.5] is the following result for the scalar equation.

Corollary 3.6. *Let $\mu \in \mathcal{M}$, and let $\rho > 0$ be a real number. Suppose $a \in L^1(\mathbf{R}_+)$ is a positive, nonincreasing and log-convex function. If $f : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ is a μ-pseudo almost automorphic function and satisfies the Lipschitz condition*

$$|f(t,x)-f(t,y)| \leq L|x-y|, \quad \textit{for all } t,\ x,\ y \in \mathbf{R}.$$

Then there is $S_\rho \in L^1(\mathbf{R}_+)\cap C(\mathbf{R}_+)$ satisfying the linear equation

$$S_\rho(t) = a(t) - \rho\int_0^t a(t-s)S_\rho(s)\,ds.$$

Moreover, if $L < \|S_\rho\|^{-1}_{L^1(\mathbf{R}_+)}$, then the semilinear equation

$$x(t) = \int_{-\infty}^t a(t-s)[-\rho x(s) + f(s,x(s))]\,ds, \quad t \in \mathbf{R},$$

has a unique μ-pseudo almost automorphic mild solution.

A differential Lipschitz condition is considered in the following result. Recall that an integral resolvent family $\{S(t)\}_{t\geq 0} \subset \mathfrak{B}(\mathbb{X})$ is said to be uniformly bounded if there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$.

Theorem 3.7. *Let $\mu \in \mathcal{M}$. Assume that A generates a uniformly bounded integral resolvent family $\{S(t)\}_{t\geq 0}$ and that* (H1) *and* (H3) *hold. Then equation* (1) *has a unique μ-pseudo almost automorphic mild solution.*

Proof. Consider the nonlinear operator Λ given by

$$(\Lambda x)(t) := \int_{-\infty}^{t} S(t-s) f(s, x(s))\, ds, \quad t \in \mathbf{R}.$$

Let $x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$, and by Corollary 3.2, it follows that the function $s \to f(s, x(s))$ is in $PAA(\mathbf{R}, \mathbf{X}, \mu)$. Moreover, from Lemma 3.4, we infer that $\Lambda x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$, that is, Λ maps $PAA(\mathbf{R}, \mathbf{X}, \mu)$ into itself.

Next, we prove that the operator Λ has a unique fixed point in $PAA(\mathbf{R}, \mathbf{X}, \mu)$. Indeed, for each $t \in \mathbf{R}$, x, $y \in PAA(\mathbf{R}, \mathbf{X}, \mu)$, we have

$$\begin{aligned} \|(\Lambda x)(t) - (\Lambda y)(t)\| &\le \int_{-\infty}^{t} \|S(t-s)[f(s, x(s)) - f(s, y(s))]\|\, ds \\ &\le M \int_{-\infty}^{t} L_f(s) \|x(s) - y(s)\|\, ds \\ &\le M \|x - y\|_{\infty} \int_{-\infty}^{t} L_f(s)\, ds, \end{aligned}$$

and

$$\begin{aligned} \|(\Lambda^2 x)(t) - (\Lambda^2 y)(t)\| &\le M \int_{-\infty}^{t} L_f(s) \|(\Lambda x)(s) - (\Lambda y)(s)\|\, ds \\ &\le M^2 \|x - y\|_{\infty} \int_{-\infty}^{t} L_f(s) \int_{-\infty}^{s} L_f(\sigma)\, d\sigma\, ds \\ &\le \frac{M^2}{2} \|x - y\|_{\infty} \left(\int_{-\infty}^{t} L_f(s)\, ds \right)^2. \end{aligned}$$

Inducting on n in the same way, we get

$$\begin{aligned} \|(\Lambda^n x)(t) - (\Lambda^n y)(t)\| &\le \frac{M^n}{(n-1)!} \|x - y\|_{\infty} \\ &\quad \times \left[\int_{-\infty}^{t} L_f(s) \left(\int_{-\infty}^{s} L_f(\sigma)\, d\sigma \right)^{n-1} ds \right] \\ &\le \frac{M^n}{n!} \|x - y\|_{\infty} \left(\int_{-\infty}^{t} L_f(s)\, ds \right)^n. \end{aligned}$$

Therefore,

$$\|\Lambda^n x - \Lambda^n y\|_\infty \le \frac{(M\|L_f\|_{L^1(\mathbf{R})})^n}{n!}\|x-y\|_\infty.$$

Since $(M\|L_f\|_{L^1(\mathbf{R})})^n/n! < 1$ for n sufficiently large, by the contraction mapping theorem, we conclude that Λ has a unique fixed point $x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$. It is clear that the fixed point is the μ-pseudo almost automorphic mild solution of equation (1). This ends the proof. □

We next study the existence of μ-pseudo almost automorphic mild solutions of equation (1) when the perturbation f is not Lipschitz continuous. For that, we require the following assumptions:

(H4) $f \in PAA(\mathbf{R} \times \mathbf{X}, \mathbf{X}, \mu)$, and $f(t,x)$ is uniformly continuous in any bounded subset $K \subset \mathbf{X}$ uniformly for $t \in \mathbf{R}$.

(H5) There exists a continuous nondecreasing function $W_f : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,x)\| \le W_f(\|x\|) \quad \text{for all } t \in \mathbf{R} \text{ and } x \in \mathbf{X}.$$

Remark 3.8. Condition (H4) is applied in [**1**] to consider weighted pseudo-almost periodic solutions to a semilinear fractional differential equation, see [**1,** Remark 3.4].

Now, we are ready to state another main result.

Theorem 3.9. *Let $\mu \in \mathcal{M}$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\ge 0}$ that satisfies assumption* (H1)*, let $f : \mathbf{R} \times \mathbf{X} \to \mathbf{X}$ be a function that satisfies assumptions (H4) and (H5), and the following additional conditions*:

(i) *For each $r \ge 0$, the function $t \to \int_{-\infty}^t \phi(t-s)W(rh(s))\,ds$ belongs to $BC(\mathbf{R})$. We set*

$$\beta(r) = \left\| \int_{-\infty}^t \phi(t-s)W(rh(s))\,ds \right\|_h.$$

(ii) *For each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that, for every* u, $v \in C_h(\mathbf{X})$, $\|u - v\|_h \leq \delta$ *implies that*

$$\int_{-\infty}^{t} \phi(t-s)\|f(s,u(s)) - f(s,v(s))\| \, ds \leq \varepsilon$$

for all $t \in \mathbf{R}$.

(iii) $\liminf_{\xi\to\infty}(\xi/\beta(\xi)) > 1$.

(iv) *For all* a, $b \in \mathbf{R}$, $a < b$, *and* $r > 0$, *the set* $\{f(s,x) : a \leq s \leq b, x \in C_h(\mathbf{X}), \|x\|_h \leq r\}$ *is relatively compact in* $\mathbf{X}$.

Then equation (1) has a μ*-pseudo almost automorphic mild solution.*

Proof. We define the nonlinear operator $\Lambda : C_h(\mathbf{X}) \to C_h(\mathbf{X})$ by

$$(\Lambda x)(t) := \int_{-\infty}^{t} S(t-s)f(s,x(s))\, ds, \quad t \in \mathbf{R}.$$

We will show that Λ has a fixed point in $PAA(\mathbf{R}, \mathbf{X}, \mu)$. For the sake of convenience, we divide the proof into several steps.

(I) For $x \in C_h(\mathbf{X})$, we have that

$$\begin{aligned}\|(\Lambda x)(t)\| &\leq \int_{-\infty}^{t} \phi(t-s)W_f(\|x(s)\|)\, ds \\ &\leq \int_{-\infty}^{t} \phi(t-s)W_f(\|x\|_h h(s))\, ds.\end{aligned}$$

It follows from condition (i) that Λ is well defined.

(II) The operator Λ is continuous. In fact, for $\varepsilon > 0$, we take $\delta > 0$ involved in condition (ii). If x, $y \in C_h(\mathbf{X})$ and $\|x - y\|_h \leq \delta$, then

$$\begin{aligned}&\|(\Lambda x)(t) - (\Lambda y)(t)\| \\ &\leq \int_{-\infty}^{t} \phi(t-s)\|f(s,x(s)) - f(s,y(s))\|\, ds \leq \varepsilon,\end{aligned}$$

which shows the assertion.

(III) We will show that Λ is completely continuous. We set $B_r(\mathbf{X})$ for the closed ball with center at 0 and radius r in the space $\mathbf{X}$. Let $V = \Lambda(B_r(C_h(\mathbf{X})))$ and $v = \Lambda(x)$ for $x \in B_r(C_h(\mathbf{X}))$. First, we will prove that $V(t)$ is a relative subset of $\mathbf{X}$ for each $t \in \mathbf{R}$. It follows from condition (i) that the function $s \to \phi(s)W_f(rh(t-s))$ is integrable on $[0,\infty)$. Hence, for $\varepsilon > 0$, we can choose $a \geq 0$ such that $\int_a^\infty \phi(s)W_f(rh(t-s))\,ds \leq \varepsilon$. Since

$$v(t) = \int_0^a S(s)f(t-s,x(t-s))\,ds + \int_a^\infty S(s)f(t-s,x(t-s))\,ds$$

and

$$\left\| \int_a^\infty S(s)f(t-s,x(t-s))\,ds \right\| \leq \int_a^\infty \phi(s)W_f(rh(t-s))\,ds \leq \varepsilon,$$

we get $v(t) \in \overline{ac_0(K)} + B_\varepsilon(\mathbf{X})$, where $c_0(K)$ denotes the convex hull of K and $K = \{S(s)f(\xi,x) : 0 \leq s \leq a,\ t-a \leq \xi \leq t,\ \|x\|_h \leq r\}$. Just as the proofs in **[17,** Theorem 4.9(iii)], using the strong continuous argument of $S(\cdot)$ and property (iv) of f, we infer that K is a relatively compact set, and $V(t) \subseteq \overline{ac_0(K)} + B_\varepsilon(\mathbf{X})$, which establishes our assertion.

Second, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(s) &= \int_0^s S(\sigma)f(t+s-\sigma,x(t+s-\sigma))\,d\sigma \\ &\quad + \int_0^a [S(\sigma+s)-S(\sigma)]f(t-\sigma,x(t-\sigma))\,d\sigma \\ &\quad + \int_a^\infty [S(\sigma+s)-S(\sigma)]f(t-\sigma,x(t-\sigma))\,d\sigma. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned}
&\bigg\| \int_0^s S(\sigma) f(t+s-\sigma, x(t+s-\sigma))\, d\sigma \\
&\quad + \int_a^\infty [S(\sigma+s) - S(\sigma)] f(t-\sigma, x(t-\sigma))\, d\sigma \bigg\| \\
&\le \int_0^s \phi(\sigma) W_f(rh(t+s-\sigma))\, d\sigma \\
&\quad + \int_a^\infty [\phi(\sigma+s) + \phi(\sigma)]) W_f(rh(t-\sigma))\, d\sigma \\
&\le \frac{\varepsilon}{2}
\end{aligned}$$

for $s \le \delta_1$. Moreover, since $\{f(t-\sigma, x(t-\sigma)) : 0 \le \sigma \le a, x \in B_r(C_h(\mathbf{X}))\}$ is a relatively compact set and $S(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[S(\sigma+s) - S(\sigma)] f(t-\sigma, x(t-\sigma))\| \le \varepsilon/2a$ for $s \le \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \le \varepsilon$ for s small enough and independent of $x \in B_r(C_h(\mathbf{X}))$.

Finally, applying condition (i), it is easy to see that

$$\frac{\|v(t)\|}{h(t)} \le \frac{1}{h(t)} \int_{-\infty}^t \phi(t-s) W_f(rh(s))\, ds \longrightarrow 0, \quad |t| \to \infty,$$

and this convergence is independent of $x \in B_r(C_h(\mathbf{X}))$. Hence, by Lemma 2.12, V is a relatively compact set in $C_h(\mathbf{X})$.

(IV) Let us now assume that $x^\lambda(\cdot)$ is a solution of equation $x^\lambda = \lambda\Lambda(x^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned}
\|x^\lambda(t)\| &= \lambda \bigg\| \int_{-\infty}^t S(t-s) f(s, x^\lambda(s)) \bigg\| \\
&\le \int_{-\infty}^t \phi(t-s) W_f(\|x^\lambda\|_h h(s))\, ds \\
&\le \beta(\|x^\lambda\|_h) h(t).
\end{aligned}$$

Hence, we get

$$\frac{\|x^\lambda\|_h}{\beta(\|x^\lambda\|_h)} \le 1,$$

and combined with condition (iii), we conclude that the set $\{x^\lambda : x^\lambda = \lambda\Lambda(x^\lambda), \lambda \in (0,1)\}$ is bounded.

(V) It follows from assumption (H4) and Theorem 3.1 that the function $t \to f(t, x(t))$ belongs to $PAA(\mathbf{R}, \mathbf{X}, \mu)$, whenever $x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$. Moreover, from Lemma 3.4 we infer that $\Lambda(PAA(\mathbf{R}, \mathbf{X}, \mu)) \subset PAA(\mathbf{R}, \mathbf{X}, \mu)$ and noting that $PAA(\mathbf{R}, \mathbf{X}, \mu)$ is a closed subspace of $C_h(\mathbf{X})$, consequently, we can consider $\Lambda : PAA(\mathbf{R}, \mathbf{X}, \mu) \to PAA(\mathbf{R}, \mathbf{X}, \mu)$. Using properties (I)–(III), we deduce that this map is completely continuous. Applying Lemma 2.13 we can infer that Λ has a fixed point $x \in PAA(\mathbf{R}, \mathbf{X}, \mu)$, which finishes the proof. □

Corollary 3.10. *Let $\mu \in \mathcal{M}$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption* (H1). *Let $f : \mathbf{R}\times\mathbf{X} \to \mathbf{X}$ be a function that satisfies assumption* (H4) *and the Hölder type condition*

$$\|f(t,x) - f(t,y)\| \leq C\|x-y\|^\alpha, \ 0 < \alpha < 1,$$

for all $t \in \mathbf{R}$ and x, $y \in \mathbf{X}$, where $C > 0$ is a constant. Moreover, assume the following conditions:

(a) $f(t,0) = q$.

(b) $\sup_{t\in\mathbf{R}} \int_{-\infty}^t \phi(t-s)h(s)^\alpha\, ds = C_2 < \infty$.

(c) *For all a, $b \in \mathbf{R}$, $a < b$, and $r > 0$, the set $\{f(s,x) : a \leq s \leq b, x \in \mathbf{X}, \|x\|_h \leq r\}$ is relatively compact in $\mathbf{X}$.*

Then equation (1) *has a μ-pseudo almost automorphic mild solution.*

Proof. Let $C_0 = \|q\|$, $C_1 = C$. We take $W_f(\xi) = C_0 + C_1\xi^\alpha$. Then condition (H4) is satisfied. It follows from (b), we can see that function f satisfies (i) in Theorem 3.7. To verify (ii), note that for each $\varepsilon > 0$ there is $0 < \delta^\alpha < \varepsilon/(C_1C_2)$ such that for every u, $v \in C_h(\mathbf{X})$, $\|u-v\|_h \leq \delta$ implies that $\int_{-\infty}^t \phi(t-s)\|f(s,u(s)) - f(s,v(s))\|\, ds \leq \varepsilon$ for all $t \in \mathbf{R}$. On the other hand, the hypothesis (iii) in the statement of Theorem 3.7 can be easily verified using the definition of W_f. This completes the proof. □

Acknowledgments. The authors would like to thank the referees for valuable suggestions to improve this manuscript.

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