

## TRAVELING WAVES FOR A NONLOCAL DISPERSAL SIR MODEL WITH STANDARD INCIDENCE

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**ABSTRACT.** This paper is concerned with traveling wave solutions of a nonlocal dispersal SIR epidemic model with standard incidence. We show that our results on existence and nonexistence of traveling wave solutions are determined by the basic reproduction number of the corresponding ordinary differential model and the minimal wave speed. These threshold dynamics are proved by constructing an invariant cone and applying Schauder's fixed point theorem on this cone and the Laplace transform. The main difficulties are the lack of an occurrence of a regularizing effect and the loss of the order-preserving property of this model.

**1. Introduction.** The current paper is concerned with the existence and nonexistence of traveling wave solutions of the following nonlocal dispersal SIR model with standard incidence:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t}S(x,t) = d_1(J * S(x,t) - S(x,t)) - \frac{\beta S(x,t)I(x,t)}{S(x,t)+I(x,t)}, \\ \frac{\partial}{\partial t}I(x,t) = d_2(J * I(x,t) - I(x,t)) + \frac{\beta S(x,t)I(x,t)}{S(x,t)+I(x,t)} - \gamma I(x,t), \\ \frac{\partial}{\partial t}R(x,t) = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t), \end{cases}$$

where  $S$ ,  $I$  and  $R$  denote the sizes of the susceptible, infected and removal individuals, respectively. The infection rate  $\beta$  and the removal rate  $\gamma$  are positive numbers.  $d_i > 0$  ( $i = 1, 2, 3$ ) are dispersal rates for the susceptible, infected and removal individuals, respectively.  $J * S(x,t)$ ,  $J * I(x,t)$  and  $J * R(x,t)$  are the standard convolutions with

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space invariable  $x$ . Throughout this paper, we assume that the nonlocal dispersal convolution kernel  $J$  is a smooth function on  $\mathbf{R}$  and satisfies:

$$(J) \quad \begin{aligned} J \in C^1(\mathbf{R}), \quad & J(x) = J(-x) \geq 0, \\ \int_{\mathbf{R}} J dy = 1 \quad & \text{and} \quad J \text{ is compactly supported.} \end{aligned}$$

It is well known that disease propagation in space is relevant to the so-called traveling waves, solutions of the form  $(S(x+ct), I(x+ct), R(x+ct))$  for which  $c$  is called the wave speed. For applications to disease control and prevention, it is important to determine whether traveling waves exist and what the propagation speed  $c$  is. Moreover, it is found that the first and second equations of system (1.1) form a closed system, it suffices to consider a two-dimensional system for  $S$  and  $I$ .

System (1.1) is the nonlocal counterpart of the following SIR disease outbreak model with the standard incidence

$$(1.2) \quad \begin{cases} \frac{\partial}{\partial t} S = d_1 \frac{\partial^2 S}{\partial x^2} - \frac{\beta SI}{S+I}, \\ \frac{\partial}{\partial t} I = d_2 \frac{\partial^2 I}{\partial x^2} + \frac{\beta SI}{S+I} - \gamma I, \\ \frac{\partial}{\partial t} R = d_3 \frac{\partial^2 R}{\partial x^2} + \gamma I. \end{cases}$$

This model was considered by Wang et al. [35]. They showed that if  $R_0 := \beta/\gamma > 1$ , then for each  $c > c_0 = 2\sqrt{d_2(\beta - \gamma)}$ , the system

$$(1.3) \quad \begin{cases} \frac{\partial}{\partial t} S = d_1 \frac{\partial^2 S}{\partial x^2} - \frac{\beta SI}{S+I}, \\ \frac{\partial}{\partial t} I = d_2 \frac{\partial^2 I}{\partial x^2} + \frac{\beta SI}{S+I} - \gamma I \end{cases}$$

has a traveling wave solution  $(S(x+ct), I(x+ct))$  and no traveling wave solutions for  $0 < c < c_0$ . On the other hand, there are no traveling wave solutions for  $R_0 = \beta/\gamma \leq 1$ . The proof is mainly based on that of Wang and Wu [38] and several earlier studies, see [13, 20, 23, 36, 37]. Here,  $R_0 = \beta/\gamma$ , which is the so-called basic reproduction number calculated from the corresponding ordinary differential system of (1.3) at the initial disease-free equilibrium  $(S_0, 0, 0)$ , completely determines the transmission dynamics and epidemic potential: if  $R_0 > 1$ ,  $I(t)$  increases to its maximum firstly and then decreases to zero and hence an epidemic takes place; if  $R_0 < 1$ , then  $I(t)$  decreases to zero and the epidemic does not occur.

Systems (1.1) and (1.3) describe the transmission of communicable disease. Classically, the random (or local) dispersal operator is common in modeling the diffusion of the species; it suggests that the population at location  $x$  can only be influenced by the variation of the population near the same location  $x$ , see [1, 11–15, 19, 24, 29, 38, 40, 42]. However, the random (or local) dispersal operator such as the Laplace operator can only influence a species' immediate neighborhood for ecological and epidemiological models. One method in overcoming these problems with the Laplace operator is to describe these models connecting spatial migration by a nonlocal dispersal operator, see [15, 17, 18, 23]. For traveling wave solutions of nonlocal dispersal problems, we refer to [3–8, 10, 21, 25–28, 30–32, 34, 43–49] and references therein. In addition, one refers to [2, 9, 16–18, 33] to understand the development of this type of work.

In the present paper, we focus on the existence and nonexistence of traveling wave solutions of system (1.1). Note that system (1.1) does not satisfy the mixed quasimonotone or exponential mixed quasimonotone conditions and the method in [22, 27, 28] may not be used directly to consider the existence of traveling wave solutions any longer. It is known that Schauder's fixed theorem has been used to prove the existence of nontrivial traveling wave solutions by constructing a suitable invariant set in the past years, see [13, 20, 23, 38]. Inspired by [12, 13, 39], we want to prove the existence of nontrivial traveling wave solutions by constructing an invariant cone in a large bounded domain with the initial functions being defined on and applying Schauder's fixed theorem on this cone, then passing to the unbounded domain by a limiting argument. But, we find that a uniform a priori bound of solutions of nonlocal dispersal problem (1.1) is more difficult to be obtained than those in [39]. In particular, we must point out that the exact boundary behavior of susceptible  $S(\xi)$  at  $\xi = +\infty$  is more difficult to be obtained than the local SIR model (1.3). The main reason is that we do not know whether  $S(\xi)$  is monotone or non-monotone due to the effect of nonlocal dispersal compared to problem (1.3) in [35].

In summary, we shall prove that, if  $R_0 = \beta/\gamma > 1$ , then there exists a critical velocity  $c_*$  such that, for each  $c > c_*$ , system (1.1) admits nontrivial traveling wave solutions with wave speed  $c$  and no nontrivial traveling wave solutions of (1.1) for  $0 < c < c_*$ , and, if  $R_0 = \beta/\gamma \leq 1$ , no nontrivial traveling wave solutions exist for every speed  $c \geq 0$ .

This paper is organized as follows. In Section 2, we prove the existence of nontrivial traveling wave solutions by Schauder's fixed theorem. The nonexistence of traveling waves is obtained in Section 3 by applying the two-sided Laplace transform, which is first introduced by Carr and Chmaj [5] and further used by Wang et al. [36–38].

**2. Existence of traveling waves.** In this section, we shall consider the existence of traveling wave solutions of system (1.1). Since the third equation of (1.1) is relatively independent, we only consider the first two equations. Let  $\xi = x + ct$ . Then the traveling wave equation of system (1.1) is as follows:

$$(2.1) \quad \begin{cases} cS'(\xi) = d_1(J * S(\xi) - S(\xi)) - \frac{\beta S(\xi) I(\xi)}{S(\xi) + I(\xi)}, \\ cI'(\xi) = d_2(J * I(\xi) - I(\xi)) + \frac{\beta S(\xi) I(\xi)}{S(\xi) + I(\xi)} - \gamma I(\xi). \end{cases}$$

Assume that the initial disease-free equilibrium is  $(S_0, 0)$ . We intend to find solutions  $(S(\xi), I(\xi))$  which are nonnegative and satisfy the following boundary conditions:

$$(2.2) \quad S(-\infty) = S_0, \quad \lim_{\xi \rightarrow +\infty} S(\xi) := S_\infty < S_0, \quad I(\pm\infty) = 0.$$

Define a function as follows:

$$f(\lambda, c) = d_2 \left( \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} dy - 1 \right) - c\lambda + \beta - \gamma.$$

Then, we have the following result.

**Lemma 2.1.** *Suppose that  $R_0 := \beta/\gamma > 1$ . There exists  $c_* > 0$  and  $\lambda_*$  such that*

$$\frac{\partial f(\lambda, c)}{\partial \lambda} \Big|_{(\lambda_*, c_*)} = 0 \quad \text{and} \quad f(\lambda_*, c_*) = 0.$$

Furthermore, the following alternatives hold.

- (i) If  $c > c_*$ , the equation  $f(\lambda, c) = 0$  has two positive roots,  $\lambda_1(c)$  and  $\lambda_2(c)$ , with  $0 < \lambda_1(c) < \lambda_* < \lambda_2(c) < \hat{\lambda}$  for some  $\hat{\lambda} \in (0, +\infty]$ , and  $f(\cdot, c) < 0$  in  $(\lambda_1(c), \lambda_2(c))$  and  $f(\cdot, c) > 0$  in  $(0, \lambda_1(c)) \cup (\lambda_2(c), \hat{\lambda})$ .

(ii) If  $c < c_*$ , we have  $f(\lambda, c) > 0$  for all  $\lambda \in (0, \hat{\lambda})$ .

*Proof.* By direct calculation, we know  $f(0, c) > 0$  and  $f(\lambda, +\infty) = -\infty$  for  $\lambda > 0$ . In addition,

$$\frac{\partial f(\lambda, c)}{\partial c} = -\lambda < 0 \quad \text{for any } \lambda > 0, \quad \frac{\partial f(0, c)}{\partial \lambda} = -c < 0$$

and

$$\frac{\partial^2 f(\lambda, c)}{\partial \lambda^2} = d_2 \int_{-\infty}^{+\infty} J(x)x^2 e^{-\lambda x} dx > 0.$$

Then, by simple analysis theory, we can obtain these results.  $\square$

In the following, we intend to show the existence of traveling wave solutions of (2.1) when  $R_0 > 1$ . Denote  $\lambda_i(c)$  by  $\lambda_i$ , and fix  $c > c_*$ . Define functions as follows:

$$\begin{aligned} S_+(\xi) &= S_0, & S_-(\xi) &= \max\{S_0(1 - \sigma e^{\alpha \xi}), 0\}, \\ I_+(\xi) &= e^{\lambda_1 \xi}, & I_-(\xi) &= \max\{e^{\lambda_1 \xi}(1 - M e^{\eta \xi}), 0\}, \end{aligned}$$

where  $\sigma$ ,  $\alpha$ ,  $M$  and  $\eta$  are all positive constants. Then we have the following results.

**Lemma 2.2.** Suppose  $\alpha < \lambda_1$  is sufficiently small. Then, the function  $S_-(\xi)$  satisfies

$$(2.3) \quad cS'(\xi) \leq d_1(J * S(\xi) - S(\xi)) - \beta I_+(\xi)$$

for any  $\xi < (1/\alpha) \ln(1/\sigma)$  and  $\sigma > 1$  large enough.

*Proof.* If  $\xi < (1/\alpha) \ln(1/\sigma)$ , then  $S_-(\xi) = S_0(1 - \sigma e^{\alpha \xi})$ . We need to verify that

$$cS'_-(\xi) \leq d_1(J * S_-(\xi) - S_-(\xi)) - \beta I_+(\xi).$$

That is,

$$-c\sigma\alpha S_0 e^{\alpha \xi} - \sigma d_1 S_0 e^{\alpha \xi} \left( 1 - \int_{-\infty}^{+\infty} J(x) e^{-\alpha x} dx \right) + \beta e^{\lambda_1 \xi} \leq 0.$$

Since  $\xi < (1/\alpha) \ln(1/\sigma)$ , it is sufficient to show that

$$-c\sigma\alpha S_0 - \sigma d_1 S_0 \left( 1 - \int_{-\infty}^{+\infty} J(x) e^{-\alpha x} dx \right) + \beta \sigma^{(\alpha - \lambda_1)/\alpha} \leq 0.$$

Keep  $\sigma = 1/\alpha$ , and let  $\alpha$  be small enough. The above inequality holds, which implies that (2.3) holds.  $\square$

**Lemma 2.3.** *Suppose  $\eta$  is small enough such that  $\eta < \min\{\alpha, \lambda_2 - \lambda_1\}$  and the constant  $M$  is large enough. Then,  $I_-(\xi)$  satisfies*

$$(2.4) \quad cI'(\xi) \leq d_2(J * I(\xi) - I(\xi)) + \frac{\beta S_-(\xi) I(\xi)}{S_-(\xi) + I(\xi)} - \gamma I(\xi)$$

for any  $\xi < (1/\eta) \ln(1/M)$ .

*Proof.* If  $\xi < (1/\eta) \ln(1/M)$ , then  $I_-(\xi) = e^{\lambda_1 \xi} (1 - M e^{\eta \xi})$ . When  $\xi \geq (1/\alpha) \ln(1/\sigma)$ ,  $S_-(\xi) = 0$ . In this case, we only need to verify that

$$(2.5) \quad cI'_-(\xi) \leq d_2(J * I_-(\xi) - I_-(\xi)) - \gamma I_-(\xi).$$

That is,

$$e^{\lambda_1 \xi} (\beta - f(\lambda_1, c)) \leq M e^{(\eta + \lambda_1) \xi} (\beta - f(\eta + \lambda_1, c)),$$

where  $f(\lambda, c)$  is defined as above. Note that  $f(\lambda_1, c) = 0$  and  $f(\eta + \lambda_1, c) < 0$  because  $\eta < \min\{\lambda_1, \lambda_2 - \lambda_1\}$  as  $\alpha < \lambda_1$ . Thus, following  $\xi \geq (1/\alpha) \ln(1/\sigma)$ , it is sufficient to verify that

$$\beta \leq M e^{\eta/\alpha \ln 1/\sigma} (\beta - f(\eta + \lambda_1, c)).$$

Consequently, we have that inequality (2.5) holds if we take

$$M \geq \frac{\beta \sigma^{\eta/\alpha}}{\beta - f(\eta + \lambda_1, c)}.$$

When  $\xi < (1/\alpha) \ln(1/\sigma)$ ,  $S_-(\xi) = S_0(1 - \sigma e^{\alpha \xi})$ . Then, we need to show that

$$cI'_-(\xi) \leq d_2(J * I_-(\xi) - I_-(\xi)) + \frac{\beta S_-(\xi) I_-(\xi)}{S_-(\xi) + I_-(\xi)} - \gamma I_-(\xi).$$

The above inequality holds if we show

$$cI'_-(\xi) \leq d_2(J*I_-(\xi) - I_-(\xi)) + (\beta - \gamma)I_-(\xi) - \frac{\beta(S_0 - S_-(\xi) + I_+(\xi))I_-(\xi)}{S_0 + I_+(\xi)}.$$

Applying the definition of  $f(\lambda, c)$  to obtain the above inequality, it is sufficient to verify

$$\frac{\beta(S_0 - S_-(\xi) + I_+(\xi))I_-(\xi)}{S_0 + I_+(\xi)} \leq -Mf(\lambda_1 + \eta, c)e^{(\lambda_1 + \eta)\xi}.$$

That is,

$$\begin{aligned} e^{\lambda_1\xi}(S_0\sigma e^{\alpha\xi} + e^{\lambda_1\xi})(1 - Me^{\eta\xi}) \\ \leq -\frac{M}{\beta}f(\lambda_1 + \eta, c)e^{(\lambda_1 + \eta)\xi}(S_0 + I_+(\xi)). \end{aligned}$$

It is sufficient to verify

$$S_0\sigma e^{(\alpha-\eta)\xi} + e^{(\lambda_1-\eta)\xi} \leq -\frac{MS_0}{\beta}f(\lambda_1 + \eta, c).$$

Since  $\xi < (1/\eta)\ln(1/M)$  and  $f(\lambda_1 + \eta, c) < 0$ , we need only to show

$$\sigma S_0 M^{-\alpha/\eta} + M^{-\lambda_1/\eta} \leq -\frac{S_0}{\beta}f(\lambda_1 + \eta, c)$$

due to  $\eta < \alpha$ . Note that the above inequality holds if we choose  $M$  large enough. So, we end our proof.  $\square$

Let  $M$  and  $\eta$  be defined as in Lemma 2.3. Fix  $0 < \delta_0 < \max\{(d_2 + \gamma)/\lambda_1, c_*\}$  and set  $\lambda_0 = (d_2 + \gamma)/\delta_0$ . Denote  $\xi_0 = (1/\eta)\ln(\lambda_0 + \lambda_1)/[M(\lambda_0 + \lambda_1 + \eta)]$ . We define a new function as follows:

$$\psi_-(\xi) = \begin{cases} e^{\lambda_1\xi}(1 - Me^{\eta\xi}) & \text{if } \xi \leq \xi_0, \\ \frac{\eta}{\lambda_0 + \lambda_1 + \eta} \left(\frac{\lambda_0 + \lambda_1}{M(\lambda_0 + \lambda_1 + \eta)}\right)^{(\lambda_0 + \lambda_1)/\eta} e^{-\lambda_0\xi} & \text{if } \xi > \xi_0. \end{cases}$$

Then, we have the following result.

**Lemma 2.4.** *The function  $\psi_-(\xi)$  satisfies:*

$$(2.6) \quad c\psi'_-(\xi) \leq d_2(J * \psi_-(\xi) - \psi_-(\xi)) + \frac{\beta S_-(\xi)\psi_-(\xi)}{S_-(\xi) + \psi_-(\xi)} - \gamma\psi_-(\xi).$$

*Proof.* Denote  $\varepsilon = [\eta/(\lambda_0 + \lambda_1 + \eta)][(\lambda_0 + \lambda_1)/M(\lambda_0 + \lambda_1 + \eta)]^{(\lambda_0 + \lambda_1)/\eta}$ , and define a function as follows

$$f(\xi) = \varepsilon + M e^{(\lambda_0 + \lambda_1 + \eta)\xi} - e^{(\lambda_0 + \lambda_1)\xi}.$$

We claim that  $f(\xi) \geq 0$  for any  $\xi \geq \xi_0$ . Obviously,  $f(\xi_0) = 0$ . Thus, we need only to verify that  $f'(\xi) > 0$  for any  $\xi > \xi_0$ . Since

$$\xi > \xi_0 = \frac{1}{\eta} \ln \frac{\lambda_0 + \lambda_1}{M(\lambda_0 + \lambda_1 + \eta)},$$

we have

$$e^{\eta\xi} > \frac{\lambda_0 + \lambda_1}{M(\lambda_0 + \lambda_1 + \eta)}.$$

Then, we can obtain that

$$M(\lambda_0 + \lambda_1 + \eta)e^{(\lambda_0 + \lambda_1 + \eta)\xi} > (\lambda_0 + \lambda_1)e^{(\lambda_0 + \lambda_1)\xi}.$$

Consequently, we get

$$\begin{aligned} f'(\xi) &= M(\lambda_0 + \lambda_1 + \eta)e^{(\lambda_0 + \lambda_1 + \eta)\xi} \\ &\quad - (\lambda_0 + \lambda_1)e^{(\lambda_0 + \lambda_1)\xi} > 0 \end{aligned}$$

for any  $\xi > \xi_0$ . Thus,  $f(\xi) \geq 0$  for any  $\xi \geq \xi_0$ .

Note that this claim implies that  $\varepsilon e^{-\lambda_0\xi} \geq e^{\lambda_1\xi}(1 - M e^{\eta\xi})$  for any  $\xi \geq \xi_0$ .

If  $\xi \leq \xi_0$ , then  $\psi_-(\xi) = e^{\lambda_1\xi}(1 - M e^{\eta\xi})$ . Now, we need to verify that  $\psi_-(\xi)$  satisfies inequality (2.6). It is noticed that

$$\begin{aligned} J * \psi_-(\xi) &= \int_{-\infty}^{\xi_0} J(\xi - y)e^{\lambda_1 y}(1 - M e^{\eta y}) dy \\ &\quad + \varepsilon \int_{\xi_0}^{+\infty} J(\xi - y)e^{-\lambda_0 y} dy \\ &\geq \int_{-\infty}^{+\infty} J(\xi - y)e^{\lambda_1 y}(1 - M e^{\eta y}) dy. \end{aligned}$$

Thus, it is sufficient to verify

$$\begin{aligned} c\psi'_-(\xi) &\leq d_2 \int_{-\infty}^{+\infty} J(\xi - y)e^{\lambda_1 y}(1 - M e^{\eta y}) dy - d_2\psi_-(\xi) \\ &\quad + \frac{\beta S_-(\xi)\psi_-(\xi)}{S_-(\xi) + \psi_-(\xi)} - \gamma\psi_-(\xi) \end{aligned}$$

for any  $\xi \leq \xi_0$ . It follows from the proof of Lemma 2.3 that the above inequality holds. Hence, (2.6) holds.

If  $\xi > \xi_0$ , then  $\psi_-(\xi) = \varepsilon e^{-\lambda_0 \xi}$ . We need only prove that

$$(2.7) \quad -c\lambda_0\varepsilon e^{-\lambda_0\xi} \leq d_2 J * \psi_-(\xi) + \frac{\varepsilon\beta S_-(\xi)e^{-\lambda_0\xi}}{S_-(\xi) + \varepsilon e^{-\lambda_0\xi}} - (d_2 + \gamma)\varepsilon e^{-\lambda_0\xi}.$$

It is sufficient to show

$$-c\lambda_0 \leq \varepsilon^{-1}d_2 e^{\lambda_0\xi} J * \psi_-(\xi) + \frac{\beta S_-(\xi)}{S_0 + I_+(\xi)} - (d_2 + \gamma).$$

Indeed,

$$c\lambda_0 = \frac{c(d_2 + \gamma)}{\delta_0} > \frac{c(d_2 + \gamma)}{c_*} > (d_2 + \gamma).$$

Thus, (2.7) holds. This completes the proof.  $\square$

Now, we establish the existence of traveling wave solutions of (1.1) for  $R_0 > 1$  ( $\beta > \gamma$ ). Take  $X > \max\{(1/\alpha)\ln\sigma, (1/\eta)\ln[M(\lambda_1 + \eta)/\lambda_1]\}$ , and let

$$\Gamma_X = \left\{ (\phi(\cdot), \varphi(\cdot)) \in C([-X, X], \mathbf{R}^2) \mid \begin{array}{l} \phi(-X) = S_-(-X), \quad \varphi(-X) = \psi_-(-X), \\ S_-(\xi) \leq \phi(\xi) \leq S_0, \quad \psi_-(\xi) \leq \varphi(\xi) \leq I_+(\xi) \\ \text{for any } \xi \in [-X, X]. \end{array} \right\}.$$

For any  $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$ , define

$$\tilde{\phi}(\xi) = \begin{cases} \phi(X), & \xi > X, \\ \phi(\xi), & |\xi| \leq X, \\ S_-(\xi), & \xi < -X, \end{cases}$$

$$\tilde{\varphi}(\xi) = \begin{cases} \varphi(X), & \xi > X, \\ \varphi(\xi), & |\xi| \leq X, \\ \psi_-(\xi), & \xi < -X. \end{cases}$$

Then, we consider the following initial value problems

$$(2.8) \quad \begin{aligned} cS'(\xi) &= d_1 \int_{-\infty}^{+\infty} J(y) \tilde{\phi}(\xi - y) dy \\ &\quad - \left( d_1 + \frac{\beta \varphi(\xi)}{\phi(\xi) + \varphi(\xi)} \right) S(\xi), \end{aligned}$$

$$(2.9) \quad \begin{aligned} cI'(\xi) &= d_2 \int_{-\infty}^{+\infty} J(y) \tilde{\varphi}(\xi - y) dy \\ &\quad + \left( \frac{\beta \phi(\xi)}{\phi(\xi) + \varphi(\xi)} - d_2 - \gamma \right) I(\xi) \end{aligned}$$

with

$$(2.10) \quad S(-X) = S_-(-X), \quad I(-X) = \psi_-(-X).$$

By the ODE theory, we have that (2.8)–(2.10) admit a unique solution  $(S_X(\xi), I_X(\xi))$  satisfying  $S_X(\cdot) \in C^1([-X, X])$  and  $I_X(\cdot) \in C^1([-X, X])$ . Then, we define an operator  $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \Gamma_X \rightarrow C([-X, X])$  by  $S_X(\xi) = \mathcal{G}_1[\phi, \varphi](\xi)$  and  $I_X(\xi) = \mathcal{G}_2[\phi, \varphi](\xi)$  for  $\xi \in [-X, X]$ .

**Theorem 2.5.** *The operator  $\mathcal{G}$  maps  $\Gamma_X$  into  $\Gamma_X$  and is completely continuous.*

*Proof.* Obviously, 0 is a sub-solution of (2.8). Then, by the strong maximum principle, we know  $S_X(\xi) > 0$ . According to the definition of  $\tilde{\phi}(\xi)$ , we have

$$\begin{aligned} d_1 \int_{-\infty}^{+\infty} J(\xi - x) \tilde{\phi}(x) dx - \left( d_1 + \frac{\beta \varphi(\xi)}{\phi(\xi) + \varphi(\xi)} \right) S_0 \\ \leq -\frac{\beta S_0 \psi_-(\xi)}{S_0 + I_+(\xi)} \leq 0, \end{aligned}$$

which implies that  $S_0$  is a super-solution of (2.8). Taking  $X_1 = 1/\alpha \ln 1/\sigma$ ,  $S_-(\xi) = S_0(1 - \sigma e^{\alpha \xi})$  holds for any  $\xi \in [-X, X_1]$ . According to Lemma 2.2, we have

$$\begin{aligned} cS'_-(\xi) - d_1 \int_{-\infty}^{+\infty} J(\xi - x)\tilde{\phi}(x)dx \\ + \left(d_1 + \frac{\beta\varphi(\xi)}{\phi(\xi) + \varphi(\xi)}\right)S_-(\xi) \\ \leq cS'_-(\xi) - d_1(J * S_-(\xi)) + \beta I_+(\xi) \leq 0, \end{aligned}$$

for any  $\xi \in [-X, X_1]$ . In addition,  $S_X(-X) = S_-(-X)$ , by the maximum principle, we have  $S_X(\xi) \geq S_-(\xi)$  for any  $\xi \in [-X, X_1]$ . Consequently, it is obtained that  $S_-(\xi) \leq S_X(\xi) \leq S_0$  for any  $\xi \in [-X, X]$ .

Next, we consider  $I_X(\xi)$ . It is obvious that  $I_X(\xi) > 0$  for any  $\xi \in [-X, X]$ , by the strong maximum principle. According to Lemma 2.1, it is easy to obtain that

$$\begin{aligned} cI'_+(\xi) - d_2 \int_{-\infty}^{+\infty} J(\xi - x)\tilde{\varphi}(x)dx + \left(d_2 + \gamma - \frac{\beta\phi(\xi)}{\phi(\xi) + \varphi(\xi)}\right)I_+(\xi) \\ \geq cI'_+(\xi) - d_2(J * I_+(\xi)) - (\beta - \gamma)I_+(\xi) = 0, \end{aligned}$$

which implies that  $I_+(\xi)$  is a super-solution of (2.9). Recalling the definition of  $\tilde{\varphi}$  and applying Lemma 2.4, it is not difficult to show that  $\psi_-(\xi) \leq I_X(\xi)$  for any  $\xi \in [-X, X]$ . In summary, we have  $\psi_-(\xi) \leq I_X(\xi) \leq I_+(\xi)$  for any  $\xi \in [-X, X]$ . So, we have shown that  $\mathcal{G}$  maps  $\Gamma_X$  into  $\Gamma_X$ .

Finally, we show that the operator  $\mathcal{G}$  is completely continuous. It follows from ODE theory that system (2.8)–(2.10) admits a unique solution  $(S_X(\cdot), I_X(\cdot))$ . Moreover,  $S_X(\cdot) \in C^1([-X, X])$  and  $I_X(\cdot) \in C^1([-X, X])$ . By direct computation, we obtain that

$$\begin{aligned} (2.11) \quad S_X(\xi) &= S_-(-X) \exp \left\{ -\frac{1}{c} \int_{-X}^{\xi} \left( d_1 + \frac{\beta\varphi(\tau)}{\phi(\tau) + \varphi(\tau)} \right) d\tau \right\} \\ &\quad + \frac{d_1}{c} \int_{-X}^{\xi} \exp \left\{ -\frac{1}{c} \int_{\eta}^{\xi} \left( d_1 + \frac{\beta\varphi(\tau)}{\phi(\tau) + \varphi(\tau)} \right) d\tau \right\} g_{\phi}^1(\eta) d\eta \end{aligned}$$

and

$$(2.12) \quad I_X(\xi) = I_-(-X) \exp \left\{ -\frac{1}{c} \int_{-X}^{\xi} \left( d_2 + \gamma - \frac{\beta \phi(\tau)}{\phi(\tau) + \varphi(\tau)} \right) d\tau \right\} \\ + \frac{d_2}{c} \int_{-X}^{\xi} \exp \left\{ -\frac{1}{c} \int_{\eta}^{\xi} \left( d_2 + \gamma - \frac{\beta \phi(\tau)}{\phi(\tau) + \varphi(\tau)} \right) d\tau \right\} g_{\varphi}^2(\eta) d\eta,$$

where

$$g_{\phi}^1(\eta) = \int_{-\infty}^{-X} J(\eta - y) S_-(y) dy \\ + \int_{-X}^X J(\eta - y) \phi(y) dy \\ + \int_X^{+\infty} J(\eta - y) \phi(X) dy, \\ g_{\varphi}^2(\eta) = \int_{-\infty}^{-X} J(\eta - y) \psi_-(y) dy \\ + \int_{-X}^X J(\eta - y) \varphi(y) dy \\ + \int_X^{+\infty} J(\eta - y) \varphi(X) dy.$$

Choose  $(\phi_i, \varphi_i) \in \Gamma_X$ ,  $i = 1, 2$ , and suppose that  $S_{X,i}(\xi) = \mathcal{G}_1[\phi_i, \varphi_i](\xi)$  and  $I_{X,i}(\xi) = \mathcal{G}_2[\phi_i, \varphi_i](\xi)$ . Note that

$$|g_{\phi_1}^1(\eta) - g_{\phi_2}^1(\eta)| \leq \left| \int_{-X}^X J(\eta - y) [\phi_1(y) - \phi_2(y)] dy \right| \\ + \left| \int_X^{+\infty} J(\eta - y) [\phi_1(X) - \phi_2(X)] dy \right| \\ \leq 2 \max_{y \in [-X, X]} |\phi_1(y) - \phi_2(y)|$$

and

$$|g_{\varphi_1}^2(\eta) - g_{\varphi_2}^2(\eta)| \leq \left| \int_{-X}^X J(\eta - y) [\varphi_1(y) - \varphi_2(y)] dy \right| \\ + \left| \int_X^{+\infty} J(\eta - y) [\varphi_1(X) - \varphi_2(X)] dy \right| \\ \leq 2 \max_{y \in [-X, X]} |\varphi_1(y) - \varphi_2(y)|.$$

Then, by the definition of the operator  $\mathcal{G}$  and equations (2.11) and (2.12), we know that  $\mathcal{G}$  is continuous. Moreover, since  $S_X$  and  $I_X$  are all of class  $C^1([-X, X])$  and, according to equations (2.8) and (2.9), for any  $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_X$ , we know  $S'_X$  and  $I'_X$  are all bounded for any  $\xi \in [-X, X]$ . Thus, it is obtained that the operator  $\mathcal{G}$  is compact. Finally, we have obtained that  $\mathcal{G}$  is completely continuous.  $\square$

Furthermore, it is easy to verify that  $\Gamma_X$  is closed and convex. Thus, Schauder's fixed theorem implies that there exists  $(S_X(\cdot), I_X(\cdot)) \in \Gamma_X$  such that

$$(S_X(\xi), I_X(\xi)) = \mathcal{G}[S_X, I_X](\xi)$$

for any  $\xi \in (-X, X)$ . Define

$$C^{1,1}([-X, X]) = \{u \in C^1([-X, X]) \mid u \text{ and } u' \text{ are Lipschitz continuous}\}$$

with the norm

$$\|u\|_{C^{1,1}([-X, X])} = \max_{x \in [-X, X]} |u| + \max_{x \in [-X, X]} |u'| + \sup_{\substack{x, y \in [-X, X] \\ x \neq y}} \frac{|u'(x) - u'(y)|}{|x - y|}.$$

Then, we list some estimates about  $S_X$  and  $I_X$  as follows.

**Theorem 2.6.** *There exists a constant  $\tilde{C} > 0$  such that*

$$\begin{aligned} \|S_X\|_{C^{1,1}([-X, X])} &< \tilde{C}, \\ \|I_X\|_{C^{1,1}([-X, X])} &< \tilde{C} \end{aligned}$$

and

$$\int_{-X}^X I_X(\xi) d\xi < \tilde{C}$$

for any  $X > \max\{(1/\alpha) \ln \sigma, (1/\eta) \ln [M(\lambda_1 + \eta)/\lambda_1], R\}$ , where  $R$  is the radius of  $\text{supp } J$ .

*Proof.* By the above discussion, we know that  $(S_X, I_X)$  satisfies

$$(2.13) \quad cS'_X(\xi) = d_1 \int_{-\infty}^{+\infty} J(\xi - y) \tilde{S}_X(y) dy - d_1 S_X(\xi) - \frac{\beta S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)}$$

and

$$(2.14) \quad \begin{aligned} cI'_X(\xi) &= d_2 \int_{-\infty}^{+\infty} J(\xi - y) \tilde{I}_X(y) dy - d_2 I_X(\xi) \\ &\quad + \frac{\beta S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} - \gamma I_X(\xi), \end{aligned}$$

for any  $\xi \in (-X, X)$ , where

$$\begin{aligned} \tilde{S}_X(\xi) &= \begin{cases} S_X(X), & \xi > X, \\ S_X(\xi), & |\xi| \leq X, \\ S_-(\xi), & \xi < -X, \end{cases} \\ \tilde{I}_X(\xi) &= \begin{cases} I_X(X), & \xi > X, \\ I_X(\xi), & |\xi| \leq X, \\ \psi_-(\xi), & \xi < -X. \end{cases} \end{aligned}$$

First, integrating (2.13) from  $-X$  to  $X$ , we have

$$\begin{aligned} \int_{-X}^X \frac{\beta S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} d\xi &= d_1 \int_{-X}^X \int_{-\infty}^{+\infty} J(y) (\tilde{S}_X(\xi - y) - S_X(\xi)) dy d\xi \\ &\quad - c[S_X(X) - S_X(-X)]. \end{aligned}$$

An easy calculation gives

$$\begin{aligned} &\int_{-X}^X \int_{-\infty}^{+\infty} J(y) (\tilde{S}_X(\xi - y) - S_X(\xi)) dy d\xi \\ &= \int_{-\infty}^{+\infty} J(y) \int_{-X-y}^{X-y} (\tilde{S}_X(z) - S_X(z+y)) dz dy \\ &= \left\{ \int_{-\infty}^0 J(y) \int_{-X-y}^X (S_X(z) - S_X(z+y)) dz dy \right\} \\ &\quad + \left\{ \int_0^{+\infty} J(y) \int_{-X}^{X-y} (S_X(z) - S_X(z+y)) dz dy \right\} \\ &\quad + \left\{ \int_{-\infty}^0 J(y) \int_X^{X-y} (S_X(X) - S_X(z+y)) dz dy \right. \\ &\quad \left. + \int_0^{+\infty} J(y) \int_{-X-y}^{-X} (S_-(z) - S_X(z+y)) dz dy \right\} \\ &= \Gamma_{s_1} + \Gamma_{s_2} + \Gamma_{s_3}. \end{aligned}$$

Noting that  $0 \leq S_X(\xi) \leq S_0$ , we have

$$\begin{aligned}\Gamma_{s_1} &= \int_{-\infty}^0 J(y) \int_{-X-y}^X (-y) \int_0^1 S'_X(z + ty) dt dz dy \\ &= \int_{-\infty}^0 J(y)(-y) \int_0^1 (S_X(X + ty) - S_X(-X - y + ty)) dt dy \\ &\leq S_0 \int_{-\infty}^0 J(y)(-y) dy, \\ \Gamma_{s_2} &= \int_0^{+\infty} J(y) \int_{-X}^{X-y} (-y) \int_0^1 S'_X(z + ty) dt dz dy \\ &= \int_0^{+\infty} J(y)(-y) \int_0^1 (S_X(X - y + ty) - S_X(-X + ty)) dt dy \\ &\leq S_0 \int_0^{+\infty} J(y)y dy\end{aligned}$$

and

$$\begin{aligned}\Gamma_{s_3} &\leq \int_{-\infty}^0 J(y) \int_X^{X-y} S_X(X) dz dy \\ &\quad + \int_0^{+\infty} J(y) \int_{-X-y}^{-X} S_-(z) dz dy \\ &\leq S_0 \int_{-\infty}^0 J(y)(-y) dy + S_0 \int_0^{+\infty} J(y)y dy.\end{aligned}$$

Thus, according to  $J(x) = J(-x)$ , we can obtain that

$$(2.15) \quad \int_{-X}^X \frac{\beta S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} d\xi \leq 4d_1 S_0 \int_0^{+\infty} J(y)y dy + cS_0.$$

Below, we divide two cases to obtain that some constant  $C_1$  exists independent of  $X$  such that  $\int_{-X}^X I_X(\xi) d\xi < C_1$ .

**Case I.** We consider the case  $I'_X(X) < 0$ . Using the fact that  $I'_X(-X) \geq \psi'_-(-X) > 0$ , there exists some  $\xi_M \in (-X, X)$  such that  $I_X(\xi_M) = \max_{\xi \in [-X, X]} I_X(\xi)$ . Hence, it follows from (2.14) that

$$\begin{aligned}\gamma I_X(\xi_M) &= d_2 \left\{ \int_{-\infty}^{+\infty} J(\xi_M - y) \tilde{I}_X(y) dy - I_X(\xi_M) \right\} \\ &\quad + \frac{\beta S_X(\xi_M) I_X(\xi_M)}{S_X(\xi_M) + I_X(\xi_M)}.\end{aligned}$$

Since

$$\begin{aligned}
& \int_{-\infty}^{+\infty} J(\xi_M - y) \tilde{I}_X(y) dy - I_X(\xi_M) \\
&= \int_{-\infty}^{-X} J(\xi_M - y) I_-(y) dy \\
&\quad + \int_{-X}^X J(\xi_M - y) I_X(y) dy \\
&\quad + \int_X^{+\infty} J(\xi_M - y) I_X(X) dy - I_X(\xi_M) \\
&\leq \|J\|_{L^\infty(\mathbf{R})} \int_{-\infty}^{-X} \psi_-(y) dy
\end{aligned}$$

and

$$\frac{\beta S_X(\xi_M) I_X(\xi_M)}{S_X(\xi_M) + I_X(\xi_M)} \leq \beta S_0,$$

we have

$$I_X(\xi_M) \leq \frac{d_2 \|J\|_{L^\infty(\mathbf{R})}}{\gamma} \int_{-\infty}^{-X} \psi_-(y) dy + \frac{\beta S_0}{\gamma}.$$

Then, integrating (2.14) from  $-X$  to  $X$ , we have

$$\begin{aligned}
(2.16) \quad & \gamma \int_{-X}^X I_X(\xi) d\xi = d_2 \int_{-X}^X \left\{ \int_{-\infty}^{+\infty} J(y) \tilde{I}_X(\xi - y) dy - I_X(\xi) \right\} d\xi \\
&+ \int_{-X}^X \frac{\beta S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} d\xi \\
&- c[I_X(X) - I_X(-X)].
\end{aligned}$$

It is noticed that

$$\begin{aligned}
& \int_{-X}^X \left\{ \int_{-\infty}^{+\infty} J(y) \tilde{I}_X(\xi - y) dy - I_X(\xi) \right\} d\xi \\
&= \int_{-\infty}^{+\infty} J(y) \int_{-X-y}^{X-y} (\tilde{I}_X(z) - I_X(z+y)) dz dy \\
&= \left\{ \int_{-\infty}^0 J(y) \int_{-X-y}^X (I_X(z) - I_X(z+y)) dz dy \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \int_0^{+\infty} J(y) \int_{-X}^{X-y} (I_X(z) - I_X(z+y)) dz dy \right\} \\
& + \left\{ \int_{-\infty}^0 J(y) \int_X^{X-y} (I_X(X) - I_X(z+y)) dz dy \right. \\
& \left. + \int_0^{+\infty} J(y) \int_{-X-y}^{-X} (\psi_-(z) - I_X(z+y)) dz dy \right\} \\
& = \Gamma_{I_1} + \Gamma_{I_2} + \Gamma_{I_3}.
\end{aligned}$$

Since  $I_X(\xi) \leq I_X(\xi_M)$  for any  $\xi \in [-X, X]$ , by the same computation as above, we obtain that

$$\Gamma_{I_1} + \Gamma_{I_2} \leq 2 \left\{ \frac{d_2 \|J\|_{L^\infty(\mathbf{R})}}{\gamma} \int_{-\infty}^{-X} \psi_-(y) dy + \frac{\beta S_0}{\gamma} \right\} \int_0^{+\infty} J(y) y dy.$$

Moreover, we have

$$\begin{aligned}
\Gamma_{I_3} & \leq \left( \frac{d_2 \|J\|_{L^\infty(\mathbf{R})}}{\gamma} \int_{-\infty}^0 J(y)(-y) dy + 1 \right) \int_{-\infty}^{-X} \psi_-(y) dy \\
& \quad + \frac{\beta S_0}{\gamma} \int_{-\infty}^0 J(y)(-y) dy.
\end{aligned}$$

Thus, it is obtained from (2.15) and (2.16) that

$$\begin{aligned}
\gamma \int_{-X}^X I_X(\xi) d\xi & \leq \frac{1}{\gamma} \left( \beta S_0 + d_2 \|J\|_{L^\infty(\mathbf{R})} \int_{-\infty}^{-X} \psi_-(y) dy \right) \\
& \quad \times \left( c + 3d_2 \int_0^{+\infty} J(y)y dy \right) \\
& \quad + 4d_1 S_0 \int_0^{+\infty} J(y)y dy \\
& \quad + cS_0 + d_2 \int_{-\infty}^{-X} \psi_-(z) dz.
\end{aligned}$$

Here, since  $X < \xi_0$ , it is easy to see that  $\int_{-\infty}^{-X} \psi_-(y) dy < C$  for some constant  $C$  independent of  $X$ .

**Case II.** We now consider the case that  $I'_X(X) \geq 0$ . Note that if  $I_X(\xi)$  attains its maximum at an interior point  $\xi_M \in (-X, X)$ , we have

done as in Case I. Thus, we assume  $I_X(X) = \max_{\xi \in [-X, X]} I_X(\xi)$ . Since  $I_X(\xi) \in C^1([-X, X])$ , some point sequence  $\xi_n \in (-X, X)$  exists such that

$$\lim_{n \rightarrow +\infty} I_X(\xi_n) = I_X(X)$$

and

$$\begin{aligned} cI'_X(\xi_n) &= d_2 \left( \int_{-\infty}^{+\infty} J(\xi_n - y) \tilde{I}_X(y) dy - I_X(\xi_n) \right) \\ &\quad + \frac{\beta S_X(\xi_n) I_X(\xi_n)}{S_X(\xi_n) + I_X(\xi_n)} - \gamma I_X(\xi_n) \end{aligned}$$

holds. Note that, for any  $\varepsilon > 0$ , there exists some  $N_0 > 0$  such that for each  $n > N_0$  we have  $|I_X(X) - I_X(\xi_n)| < \varepsilon$  and  $I'_X(\xi_n) \geq 0$ . Thus, for every  $n > N_0$ , there is

$$\begin{aligned} \gamma I_X(\xi_n) &\leq d_2 \left( \int_{-\infty}^{+\infty} J(\xi_n - y) \tilde{I}_X(y) dy - I_X(\xi_n) \right) + \beta S_0 \\ &\leq d_2 \|J\|_{L^\infty(\mathbf{R})} \int_{-\infty}^{-X} \psi_-(y) dy + d_2 \varepsilon + \beta S_0. \end{aligned}$$

Now, we can pass to the limit on the above expression to get a uniform bound of  $I_X(X)$ . Then, discussing as above, we also can obtain that some constant  $C_1$  exists independent of  $X$  such that  $\int_{-X}^X I_X(\xi) d\xi < C_1$ .

It is noticed that  $S_X(\xi) \leq S_0$  for any  $\xi \in [-X, X]$ . By simple computation, we have

$$\begin{aligned} |S'_X(\xi)| &\leq \frac{d_1}{c} \int_{-\infty}^{+\infty} J(\xi - y) \tilde{S}_X(y) dy \\ &\quad + \frac{d_1}{c} S_X(\xi) + \frac{\beta}{c} \frac{S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} \\ &\leq \frac{2d_1 + \beta}{c} S_0. \end{aligned}$$

According to the discussion as above, we know that there exists some constant  $C_*$  independent of  $X$  such that  $I_X(\xi) < C_*$  for any  $\xi \in [-X, X]$ . Then, it follows from (2.14) that

$$|I'_X(\xi)| \leq \frac{d_2 \|J\|_{L^\infty}}{c} \int_{-\infty}^{-X} \psi_-(y) dy + \frac{(2d_2 + \gamma)C_*}{c} + \frac{\beta}{c} S_0.$$

Consequently, it follows from the above two inequalities that some constant  $C_0$  exists independent of  $X$  such that  $|S'_X(\xi)| < C_0$  and  $|I'_X(\xi)| < C_0$ . Thus, for any  $\xi, \eta \in [-X, X]$ , some constant  $L_1 > 0$  exists such that

$$|S_X(\xi) - S_X(\eta)| < L_1|\xi - \eta| \quad \text{and} \quad |I_X(\xi) - I_X(\eta)| < L_1|\xi - \eta|.$$

On the other hand, it follows from (2.13) that

$$\begin{aligned} |S'_X(\xi) - S'_X(\eta)| &\leq \frac{d_1}{c} \left| \int_{-\infty}^{+\infty} J(\xi - y) \tilde{S}_X(y) dy \right. \\ &\quad \left. - \int_{-\infty}^{+\infty} J(\eta - y) \tilde{S}_X(y) dy \right| \\ &\quad + \frac{d_1}{c} |S_X(\xi) - S_X(\eta)| \\ &\quad + \frac{\beta}{c} \left| \frac{S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} - \frac{S_X(\eta) I_X(\eta)}{S_X(\eta) + I_X(\eta)} \right| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It is noticed that

$$\begin{aligned} &\left| \int_{-X}^X (J(\xi - y) - J(\eta - y)) S_X(y) dy \right| \\ &= \left| \int_{\xi-X}^{\xi+X} J(z) S_X(\xi - z) dz - \int_{\eta-X}^{\eta+X} J(z) S_X(\eta - z) dz \right| \\ &= \left| \int_{\eta+X}^{\xi+X} J(z) S_X(\xi - z) dz - \int_{\eta-X}^{\xi-X} J(z) S_X(\eta - z) dz \right. \\ &\quad \left. + \int_{\xi-X}^{\eta+X} J(z) (S_X(\xi - z) - S_X(\eta - z)) dz \right| \\ &\leq (2\|J\|_{L^\infty} S_0 + L_1)|\xi - \eta|. \end{aligned}$$

According to assumption (J), we know  $J$  is Lipschitz continuous. Thus, we have

$$I_1 = \frac{d_1}{c} \left| \int_{-\infty}^{+\infty} J(\xi - y) \tilde{S}_X(y) dy - \int_{-\infty}^{+\infty} J(\eta - y) \tilde{S}_X(y) dy \right|$$

$$\begin{aligned}
&= \frac{d_1}{c} \left| \int_{-\infty}^{-X} (J(\xi - y) - J(\eta - y)) S_-(y) dy \right. \\
&\quad + \int_{-X}^X (J(\xi - y) - J(\eta - y)) S_X(y) dy \\
&\quad \left. + \int_X^{+\infty} (J(\xi - y) - J(\eta - y)) S_X(X) dy \right| \\
&\leq \frac{d_1}{c} \left\{ S_0 \left| \int_{\xi+X}^{\eta+X} J(z) dz \right| + \left| \int_{\eta-X}^{\xi-X} J(y) dy \right| S_0 \right. \\
&\quad + \sigma S_0 \int_{-\infty}^{-X} |J(\xi - y) - J(\eta - y)| e^{\alpha y} dy \\
&\quad \left. + \left| \int_{-X}^X (J(\xi - y) - J(\eta - y)) S_X(y) dy \right| \right\} \\
&\leq \frac{d_1}{c} (\sigma L S_0 + 4S_0 \|J\|_{L^\infty} + L_1) |\xi - \eta|
\end{aligned}$$

for the Lipschitz constant  $L$  of  $J$ . In addition, a simple computation yields

$$\begin{aligned}
I_3 &= \frac{\beta}{c} \left| \frac{S_X(\xi) I_X(\xi)}{S_X(\xi) + I_X(\xi)} - \frac{S_X(\eta) I_X(\eta)}{S_X(\eta) + I_X(\eta)} \right| \\
&\leq \frac{\beta}{c} [|I_X(\xi) - I_X(\eta)| + |S_X(\xi) - S_X(\eta)|].
\end{aligned}$$

Then, there exists some constant  $L_2 > 0$  such that

$$|S'_X(\xi) - S'_X(\eta)| < L_2 |\xi - \eta|$$

for any  $\xi, \eta \in [-X, X]$ . In addition, it follows from

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} J(y) \tilde{I}_X(\xi - y) dy - \int_{-\infty}^{+\infty} J(y) \tilde{I}_X(\eta - y) dy \right| \\
&\leq \int_{-\infty}^{-X} |J(\xi - y) - J(\eta - y)| \psi_-(y) dy \\
&\quad + \int_{-X}^X |J(\xi - y) - J(\eta - y)| I_X(y) dy \\
&\quad + C_* \left| \int_{\eta-X}^{\xi-X} J(z) dz \right|
\end{aligned}$$

$$\begin{aligned} &\leq \left( L \int_{-\infty}^{-X} \psi_-(y) dy \right. \\ &\quad \left. + L \int_{-X}^X I_X(y) dy + C_* \|J\|_{L^\infty(\mathbf{R})} \right) |\xi - \eta| \end{aligned}$$

and the estimate of  $I_3$  that

$$|I'_X(\xi) - I'_X(\eta)| < L_2 |\xi - \eta|$$

for any  $\xi, \eta \in [-X, X]$ . So far, we have shown that some constant  $\tilde{C} > 0$  exists independent of  $X$  such that

$$\|S_X\|_{C^{1,1}([-X, X])} < \tilde{C}, \quad \|I_X\|_{C^{1,1}([-X, X])} < \tilde{C}.$$

Thus, we have completed the whole proof.  $\square$

Now, choose a sequence  $\{X_n\}_1^\infty$  satisfying

$$X_n > \max \left\{ \frac{1}{\alpha} \ln \sigma, \frac{1}{\eta} \ln \frac{M(\lambda_1 + \eta)}{\lambda_1}, R \right\}$$

for each  $n$  and  $\lim_{n \rightarrow +\infty} X_n = +\infty$ . We know some  $(S_{X_n}, I_{X_n}) \in \Gamma_{X_n}$  exists satisfying Theorem 2.6. Thus, there exists a subsequence  $\{X_{n_k}\}$  by the diagonal extraction argument such that  $X_{n_k} \rightarrow +\infty$  as  $k \rightarrow +\infty$  and

$$S_{X_{n_k}} \rightarrow \tilde{S} \quad \text{and} \quad I_{X_{n_k}} \rightarrow \tilde{I} \text{ in } C_{\text{loc}}^1(\mathbf{R}) \text{ as } k \rightarrow +\infty.$$

Since  $J$  is compactly supported on  $\mathbf{R}$ , using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_{-\infty}^{+\infty} J(\xi - y) \tilde{S}_{X_{n_k}}(y) dy &= \int_{-\infty}^{+\infty} J(\xi - y) \tilde{S}(y) dy \\ &= J * \tilde{S}(\xi), \end{aligned}$$

and

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_{-\infty}^{+\infty} J(\xi - y) \tilde{I}_{X_{n_k}}(y) dy &= \int_{-\infty}^{+\infty} J(\xi - y) \tilde{I}(y) dy \\ &= J * \tilde{I}(\xi), \end{aligned}$$

for any  $\xi \in \mathbf{R}$ . Moreover, we have that  $(\tilde{S}, \tilde{I})$  satisfies (2.1) and

$$\begin{aligned} S_-(\xi) &< \tilde{S}(\xi) < S_0, \\ \psi_-(\xi) &< \tilde{I}(\xi) < I_+(\xi), \\ \int_{-\infty}^{+\infty} \tilde{I}(\xi) d\xi &< +\infty. \end{aligned}$$

By the definition of  $S_-(\xi)$ , it is easy to obtain that  $\tilde{S}(-\infty) = S_0$ . In addition, since  $\int_{-\infty}^{+\infty} \tilde{I}(\xi) d\xi < +\infty$  and  $\tilde{I}'(\xi)$  is uniformly bounded according to the proof of Theorem 2.6, we have  $\tilde{I}(\pm\infty) = 0$ . Moreover, it is obvious that

$$0 \leq \liminf_{\xi \rightarrow +\infty} \tilde{S}(\xi) \leq \limsup_{\xi \rightarrow +\infty} \tilde{S}(\xi) \leq S_0.$$

Then, we have the following result.

**Theorem 2.7.** *Suppose  $R_0 = \beta/\gamma > 1$ . If  $c > c_*$ , then a nonnegative traveling wave solution  $(\tilde{S}(x + ct), \tilde{I}(x + ct))$  of (2.1) exists satisfying (2.2).*

*Proof.* By the above discussion, we need only prove that there exists some  $S_\infty$  such that  $\lim_{\xi \rightarrow +\infty} \tilde{S}(\xi) = S_\infty$ . Firstly, we claim that  $\liminf_{\xi \rightarrow +\infty} \tilde{S}(\xi) < S_0$ . Otherwise, we have  $\lim_{\xi \rightarrow +\infty} \tilde{S}(\xi) = S_0$ . Then, integrating the first equation of (2.1) from  $-x$  to  $x$  yields

$$\begin{aligned} c[\tilde{S}(x) - \tilde{S}(-x)] &= d_1 \int_{-\infty}^{+\infty} J(y)(-y) \int_0^1 (\tilde{S}(x - ty) - \tilde{S}(-x - ty)) dt dy \\ &\quad - \int_{-x}^x \frac{\beta \tilde{S}(\xi) \tilde{I}(\xi)}{\tilde{S}(\xi) + \tilde{I}(\xi)} d\xi. \end{aligned}$$

Letting  $x \rightarrow +\infty$ , we obtain that

$$\int_{-\infty}^{+\infty} \frac{\beta \tilde{S}(\xi) \tilde{I}(\xi)}{\tilde{S}(\xi) + \tilde{I}(\xi)} d\xi = 0,$$

which contradicts  $\int_{-\infty}^{+\infty} (\beta \tilde{S}(\xi) \tilde{I}(\xi)) / (\tilde{S}(\xi) + \tilde{I}(\xi)) d\xi > 0$ .

Now, assume  $\limsup_{\xi \rightarrow +\infty} \tilde{S}(\xi) > \liminf_{\xi \rightarrow +\infty} \tilde{S}(\xi)$  to the contrary. Then, some point sequence  $\xi_n$  exists satisfying  $\xi_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  such that  $\lim_{n \rightarrow +\infty} \tilde{S}(\xi_n) = \limsup_{\xi \rightarrow +\infty} \tilde{S}(\xi) := \sigma_1$  and  $\tilde{S}'(\xi_n) = 0$ . Furthermore,  $\lim_{n \rightarrow +\infty} J * \tilde{S}(\xi_n) \leq \sigma_1$ . And  $\eta_n$  exists satisfying  $\eta_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  such that  $\lim_{n \rightarrow +\infty} \tilde{S}(\eta_n) = \liminf_{\xi \rightarrow +\infty} \tilde{S}(\xi) := \sigma_2$  and  $\tilde{S}'(\eta_n) = 0$ . Moreover,  $\liminf_{n \rightarrow \infty} J * S(\eta_n) \geq \sigma_2 > 0$ . In view of

$$0 = c\tilde{S}'(\xi_n) = d_1(J * \tilde{S}(\xi_n) - \tilde{S}(\xi_n)) - \frac{\beta\tilde{S}(\xi_n)\tilde{I}(\xi_n)}{\tilde{S}(\xi_n) + \tilde{I}(\xi_n)},$$

letting  $n \rightarrow +\infty$ , we can obtain that  $\lim_{n \rightarrow +\infty} J * \tilde{S}(\xi_n) = \sigma_1$ .

Let  $S_n(y) = \tilde{S}(\xi_n + y)$ . Next, we want to show that  $S_n(y) \rightarrow \sigma_1$  for any  $y \in \text{supp } J = \Omega$ . Take  $\delta > 0$  small enough, and let  $\Omega_\delta = \Omega \cap \{\lim_{n \rightarrow +\infty} S_n(y) < \sigma_1 - \delta\}$ . Then

$$\begin{aligned} \sigma_1 &= \lim_{n \rightarrow +\infty} \int_{\Omega} J(y)S_n(y) dy \\ &\leq \limsup_{n \rightarrow +\infty} \int_{\Omega \setminus \Omega_\delta} J(y)S_n(y) dy + \limsup_{n \rightarrow +\infty} \int_{\Omega_\delta} J(y)S_n(y) dy \\ &\leq \sigma_1 \int_{\Omega \setminus \Omega_\delta} J(y) dy + (\sigma_1 - \delta) \int_{\Omega_\delta} J(y) dy \\ &= \sigma_1 - \delta \int_{\Omega_\delta} J(y) dy, \end{aligned}$$

which implies  $m(\Omega_\delta) = 0$  for  $\delta > 0$  small enough. Here,  $m(\cdot)$  denotes the measure. Thus, we obtain that  $S_n(y) \rightarrow \sigma_1$  almost everywhere in  $\Omega$ . However,  $\{S_n\}$  is an equicontinuous family, so the convergence is everywhere in  $\Omega$ . That is,  $\lim_{n \rightarrow +\infty} S_n(y) = \sigma_1$  for any  $y \in \Omega$ .

Since  $J \in C^1$ ,  $R' > 0$  and  $\delta' > 0$  must exist with  $R' \geq \delta'$  such that

$$[R' - \delta', R' + \delta'] \cup [-R' - \delta', -R' + \delta'] \subset \Omega.$$

Let  $\xi_n^\pm = \xi_n \pm R'$ . Using similar arguments, we get

$$\tilde{S}(\xi_n^\pm + y) \rightarrow \sigma_1 \quad \text{as } n \rightarrow \infty \text{ for any } y \in \Omega.$$

In particular,  $\tilde{S}(\xi_n + y) \rightarrow \sigma_1$  as  $n \rightarrow \infty$  if  $y \in [-\delta', \delta']$ . Repeating the above procedures, we obtain

$$\tilde{S}(\xi_n + y) \rightarrow \sigma_1 \quad \text{as } n \rightarrow \infty$$

for any  $y \in [-R, R]$ , where  $R > 0$  denotes the radius of  $\Omega$ .

On the other hand, it follows from  $\tilde{I}(+\infty) = 0$  and

$$J * \tilde{S}(\eta_n) = \tilde{S}(\eta_n) + \frac{\beta \tilde{S}(\eta_n) \tilde{I}(\eta_n)}{\tilde{S}(\eta_n) + \tilde{I}(\eta_n)}$$

that  $\lim_{n \rightarrow \infty} J * \tilde{S}(\eta_n) = \sigma_2 < \sigma_1$ . Using similar arguments, we can prove that

$$\tilde{S}(\eta_n + y) \rightarrow \sigma_2 \quad \text{as } n \rightarrow \infty$$

for any  $y \in \Omega$ . Consequently, we have

$$\tilde{S}(\eta_n + y) \rightarrow \sigma_2 \quad \text{as } n \rightarrow \infty \text{ for any } y \in [-R, R].$$

Note that  $\int_{\eta_n}^{\xi_n} [\tilde{S}(\xi) \tilde{I}(\xi)] / [\tilde{S}(\xi) + \tilde{I}(\xi)] d\xi \rightarrow 0$  as  $n \rightarrow \infty$ . Integrating the two sides of the first equation of (2.1) from  $\eta_n$  to  $\xi_n$ , we have

$$\begin{aligned} 0 &< c(\sigma_1 - \sigma_2) \\ &= c \lim_{n \rightarrow \infty} [\tilde{S}(\xi_n) - \tilde{S}(\eta_n)] \\ &= d_1 \lim_{n \rightarrow \infty} \int_{\eta_n}^{\xi_n} [J * \tilde{S}(\xi) - \tilde{S}(\xi)] d\xi \\ &\quad - \lim_{n \rightarrow \infty} \int_{\eta_n}^{\xi_n} \frac{\beta \tilde{S}(\xi) \tilde{I}(\xi)}{\tilde{S}(\xi) + \tilde{I}(\xi)} d\xi \\ &= d_1 \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} J(y) y \int_0^1 [\tilde{S}(\eta_n - ty) - \tilde{S}(\xi_n - ty)] dt dy \\ &\quad - \lim_{n \rightarrow \infty} \int_{\eta_n}^{\xi_n} \frac{\beta \tilde{S}(\xi) \tilde{I}(\xi)}{\tilde{S}(\xi) + \tilde{I}(\xi)} d\xi \\ &= 0, \end{aligned}$$

which yields a contradiction. Hence, we have

$$\liminf_{\xi \rightarrow \infty} \tilde{S}(\xi) = \limsup_{\xi \rightarrow \infty} \tilde{S}(\xi) := S_\infty < S_0.$$

This completes the proof.  $\square$

*Remark 2.8.* Note that the solution  $S(\xi)$  of (1.3) is monotone in [35]. However, due to the appearance of the nonlocal dispersal operator in our problem (1.3), we don't know whether  $S(\xi)$  is monotone or not; even the kernel function is symmetric. Also, for the case  $R_0 > 1$  ( $\beta > \gamma$ ) and  $c = c_*$ , the existence of traveling waves remains open. We shall leave these problems for further research.

**3. Nonexistence of traveling waves.** In this section, we prove that, if  $R_0 > 1$  ( $\beta > \gamma$ ) and  $c < c_*$ , then there does not exist a nontrivial and nonnegative traveling wave solution pair for (2.1). In addition, we will show there do not always exist traveling wave solutions as  $R_0 \leq 1$  ( $\beta \leq \gamma$ ).

**Theorem 3.1.** *Suppose that  $R_0 = \beta/\gamma > 1$ . For any speed  $0 < c < c_*$ , no nontrivial traveling wave solutions  $(S(x+ct), I(x+ct))$  exist of system (2.1) satisfying (2.2).*

*Proof.* On the contrary, assume  $(S(x), I(x))$  is a pair of traveling wave solutions to (2.1) and (2.2). Since  $(\beta S)/(S + I) \rightarrow \beta$  as  $x \rightarrow -\infty$ , some  $x^*$  exists such that

$$\frac{\beta S}{S + I} - \gamma \rightarrow \beta - \gamma > \delta = \frac{\beta - \gamma}{2} > 0$$

for all  $x < x^*$ . Thus, the second equation of (2.1) yields

$$(3.1) \quad cI'(x) > d_2(J * I(x) - I(x)) + \frac{\beta - \gamma}{2}I(x)$$

for all  $x < x^*$ . Since  $I$  is integrable, we can define

$$Q(x) = \int_{-\infty}^x I(y) dy.$$

Integrating (3.1) from  $-\infty$  to  $x$ , we have

$$\delta Q(x) < cI(x) - d_2 \int_{-\infty}^x (J * I(y) - I(y)) dy.$$

Noting that

$$\int_{-\infty}^x J * I(y) dy = \int_{-\infty}^{+\infty} J(z) \int_{-\infty}^{x-z} I(\tau) d\tau dz = J * Q(x),$$

we have

$$(3.2) \quad \delta Q(x) < cI(x) - d_2(J * Q(x) - Q(x)).$$

Moreover, by simple computation, we can obtain that

$$\begin{aligned} (3.3) \quad & \int_{-\infty}^x (J * Q(t) - Q(t)) dt \\ &= \int_{-\infty}^x \int_{-\infty}^{+\infty} J(z)(Q(t-z) - Q(t)) dz dt \\ &= \int_{-\infty}^x \int_{-\infty}^{+\infty} (-z)J(z) \int_0^1 Q'(t-\theta z) d\theta dz dt \\ &= \int_{-\infty}^{+\infty} J(z)(-z) \int_0^1 Q(x-\theta z) d\theta dz. \end{aligned}$$

Then, an integration of (3.2) yields

$$\delta \int_{-\infty}^x Q(y) dy < cQ(x) + d_2 \int_{-\infty}^{+\infty} J(z)z \int_0^1 Q(x-\theta z) d\theta dz.$$

It is noticed that  $zQ(x-\theta z)$  is non-increasing with respect to  $\theta \in [0, 1]$ . Thus, we have  $zQ(x-\theta z) \leq zQ(x)$ . Consequently, the following holds:

$$\delta \int_{-\infty}^x Q(y) dy \leq cQ(x).$$

Note that

$$\begin{aligned} \delta \int_{-\infty}^x Q(y) dy &= \delta \int_0^{+\infty} Q(x-y) dy \\ &\geq \delta \int_0^\tau Q(x-y) dy \\ &> \delta\tau Q(x-\tau) \end{aligned}$$

for  $\tau > 0$ . Thus, there exists  $\tau > 0$  such that  $Q(x - \tau) < 1/2Q(x)$ . Denote  $\mu_0 = (\ln 2)/\tau > 0$ . Letting  $L(x) = Q(x)e^{-\mu_0 x}$ , it is easy to verify that  $L(x - \tau) < L(x)$  for all  $x < x^*$ , which implies that  $L(x)$  is bounded as  $x \rightarrow -\infty$ . It is noticed that

$$cI'(x) \leq d_2(J * I(x) - I(x)) + \beta I(x).$$

Then, we conclude that  $I(x)e^{-\mu_0 x}$  and  $I'(x)e^{-\mu_0 x}$  are all bounded as  $x \rightarrow -\infty$ . Since  $I(\xi)$  is bounded, we have

$$\sup_{x \in \mathbf{R}} \{I(x)e^{-\mu_0 x}\} < \infty, \quad \sup_{x \in \mathbf{R}} \{I'(x)e^{-\mu_0 x}\} < \infty.$$

Moreover, since  $I/(I + S) \leq 1$  and  $S(x) + I(x) \rightarrow S_0$  as  $x \rightarrow -\infty$ , it is known that  $(e^{-\mu_0 x}I(x))/(S(x) + I(x))$  is also uniformly bounded on the real line. Below, we introduce a two-sided Laplace transform on the second equation of (2.1):

$$(3.4) \quad f(\mu, c) \int_{-\infty}^{+\infty} I(x)e^{-\mu x} dx = \int_{-\infty}^{+\infty} e^{-\mu x} \frac{\beta I^2(x)}{S(x) + I(x)} dx,$$

where  $\mu \in (0, \mu_0)$  and  $f(\mu, c)$  is defined in Section 2. In fact, since  $e^{-\mu_0 x}I(x)/(S(x) + I(x))$  is uniformly bounded on the real line, the two-side Laplace integrals can be analytically continued to the whole right half plane. Otherwise, the integral on the right has a singularity at  $\mu = \mu^*$ , and it is analytic for all  $\mu < \mu^*$  (see [41]). But, the integral on the right is actually analytic for all  $\mu < \mu_0 + \mu^*$ , which is a contradiction. Moreover, by the definition of  $f(\mu, c)$ , we know  $f(\mu, c) > 0$  for all  $\mu > 0$ ; thus,  $\int_{-\infty}^{+\infty} I(x)e^{-\mu x} dx$  is analytic in the right half plane. However, according to Lemma 2.1, we know  $f(\mu, c) \rightarrow +\infty$  as  $\mu \rightarrow \hat{\lambda}$ . This leads to a contradiction from (3.4), see [36, 38] for earlier ideas in different settings. So, we conclude that no traveling wave solutions exist which satisfy the boundary conditions.  $\square$

**Theorem 3.2.** *Suppose that  $R_0 = \beta/\gamma \leq 1$ . Then, for any speed  $c > 0$ , no traveling wave solutions  $(S(x+ct), I(x+ct))$  of (2.1) satisfying (2.2) exist.*

*Proof.* For the case  $R_0 < 1$ , since  $I(x)$  satisfies

$$cI'(x) = d_2(J * I(x) - I(x)) + \frac{\beta S(x)I(x)}{S(x) + I(x)} - \gamma I(x),$$

integrating this equation from  $-\infty$  to  $+\infty$ , by simple computation, we have

$$\gamma \int_{-\infty}^{+\infty} I(x) dx \leq d_2 \int_{-\infty}^{+\infty} (J * I(x) - I(x)) dx + \beta \int_{-\infty}^{+\infty} I(x) dx.$$

That is,

$$\int_{-\infty}^{+\infty} I(x) dx \leq \frac{d_2}{\gamma + d_2 - \beta} \int_{-\infty}^{+\infty} J * I(x) dx < \int_{-\infty}^{+\infty} I(x) dx,$$

which is a contradiction.

For the case  $R_0 = 1$ , that is  $\beta = \gamma$ , we have

$$cI'(x) = d_2(J * I(x) - I(x)) + \beta \left( \frac{S(x)}{S(x) + I(x)} - 1 \right) I(x).$$

Since  $I(\pm\infty) = 0$  and  $\int_{\mathbf{R}} (J * I(x) - I(x)) dx = 0$ , the following can be obtained:

$$\beta \int_{-\infty}^{+\infty} \left( \frac{S(x)}{S(x) + I(x)} - 1 \right) I(x) dx = 0.$$

Moreover, since  $S(x)/[S(x) + I(x)] \leq 1$  and  $I(x)$  is nonnegative, we get

$$\left( \frac{S(x)}{S(x) + I(x)} - 1 \right) I(x) = 0$$

according to the continuity of  $S$  and  $I$ . It is noted that  $\text{Int}(\text{supp } I)$  is not empty. Then, for any  $x \in \text{supp } I$ , we have  $S(x)/[S(x) + I(x)] - 1 = 0$ , which implies  $I(x) = 0$ . This is a contradiction.  $\square$

*Remark 3.3.* Theorems 2.7, 3.1 and 3.2 combined provide a threshold condition for the existence of traveling wave solutions in terms of the basic reproduction number  $R_0$  and the minimal wave speed  $c_*$ . We do anticipate that  $c_*$  is the asymptotic speed of propagation for our model(2.1) following the work described in [24], but we do not verify it. We leave it for further study.

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