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STABILITY FOR A CLASS OF FRACTIONAL PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we deal with a class of fractional integro-differential equations involving impulsive effects and nonlocal conditions, whose principal part is of diffusion-wave type. Our aim is to establish some existence and stability results for integral solutions to the problem at hand by use of the fixed point approach.

1. Introduction. In this paper, we study the existence and stability of solutions to the following problem in a Banach space X:

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)\,ds + f(t, u(t), u_t), \quad t>0, \quad t \neq t_k,\ k \in \Lambda, \tag{1.1}$$

$$u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad k \in \Lambda, \tag{1.2}$$

$$u(\overline{s}) + g(u)(\overline{s}) = \varphi(\overline{s}), \quad \overline{s} \in [-h, 0], \tag{1.3}$$

where A is a closed, linear and unbounded operator, and f, g and I_k are the functions which will be specified in Section 3. Here $\alpha \in (1,2)$ and $\Lambda \subset \mathbf{N}$ is an index set. By $u(t_k^+)$ and $u(t_k^-)$ we mean the right and left limit of u at t_k, respectively; u_t stands for the history of the state function up to the time t, i.e., $u_t(\overline{s}) = u(t+\overline{s}), \overline{s} \in [-h,0]$. It is obvious that the principal part

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)\,ds$$

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can be viewed as

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)\, ds + u(0). \tag{1.4}$$

The last equation in the case A is the Laplace operator was studied in [**13, 14**]. As discussed in these papers, (1.4) is intermediate between the diffusion ($\alpha = 1$) and the wave ($\alpha = 2$) equation. In addition, it is known that the following prototype of (1.1),

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \Delta_x u(x,s)\, ds + F(x,t,u(x,t)),$$

describes the anomalous diffusion processes and wave propagations in viscoelastic materials (see, e.g., [**18, 29–31**]).

The generalized Cauchy problem involving nonlocal and/or impulsive conditions has been an active subject for many investigations in recent years. It is known that nonlocal conditions give a better description for real models than classical initial ones, e.g., the condition

$$u(s) + \sum_{i=1}^{M} c_i u(\tau_i + s) = \varphi(s),$$

allows taking additional measurements instead of solely initial datum. The first result and physical meaning for nonlocal problems go back to the work of Byszewski [**6**]. It then has aroused an increasing interest in various nonlocal problems involving integer order differential equations and inclusions. For some remarkable solvability results, we quote here the works in [**8, 17, 20, 23, 26–28, 39**]. On the other hand, impulsive conditions have been used to describe the dynamical systems having abrupt changes. A comprehensive investigation for impulsive differential equations can be found in [**25**]. Evidently, generalized Cauchy problems with nonlocal conditions and impulsive effects play an important role in describing many real world problems. Due to the application of fractional derivatives in modeling and the development of fractional calculus (see, e.g., [**3, 24, 32, 34**]), the integer order differential systems have been generalized to many models involving fractional differential equations. In this direction, we refer to [**15, 19, 37, 38**] for some typical existence results.

Recently, some authors have drawn attention to the Cauchy problems driven by fractional integro-differential equations as in (1.1). The results on asymptotically periodic solutions were obtained in [**2, 10, 11**]. Considering a fractional integro-differential equation in neutral form, the authors in [**7**] showed the existence of asymptotically almost automorphic solutions. Let us take note that, in these works, the models under investigation have neither nonlocal nor impulsive condition. We are also concerned with the existence results in [**36**], in which the model considered involved a nonlocal condition. It should be noted that, in the above-mentioned works, no attempt has been made to consider stability problems. This is the main motivation for our study in the present paper.

Regarding stability for differential equations, the Lyapunov functional method is an effective tool for problems in finite-dimensional spaces (see [**12, 16**]). However, it is not easily done for fractional integro-differential equations in Banach spaces. In this work, we will employ the fixed point approach initiated by Burton and Furumochi for ordinary/functional differential equations [**4, 5**]. The main idea of this method is to construct a *stable subset*, in which the solution operator has a unique fixed point. By this approach, we will prove that the zero solution for (1.1)–(1.3) is BI-asymptotically stable, that is, $u(t) \to 0$ as $t \to +\infty$ for all bounded initial data φ. In addition, we show that, if $\#\Lambda < \infty$, a better estimate will be obtained, namely $\|u(t)\| = O(t^{-\alpha})$ as $t \to +\infty$. It should be noted that, if a solution to (1.1)–(1.3) is BI-asymptotically stable, then it is also S-asymptotically ω-periodic, so the result obtained in this paper is stronger than those in [**2, 10, 11**].

The remainder of our work is as follows. In the next section, we recall some notions and facts related to fractional resolvent operators, measures of noncompactness and the fixed point theory for condensing maps. Section 3 is devoted to the existence result under a general setting via measures of noncompactness, which extends the one in [**36**] for the non-impulsive case. Section 4 shows the stability result under Lipschitz conditions imposed on nonlinearities. The last section gives an example, which demonstrates the results obtained in a model of fractional integro-partial differential equations.

2. Preliminaries.

2.1. Fractional calculus. We first recall some notions and results related to fractional resolvent operators.

Definition 2.1. Let A be a closed and linear operator with domain $D(A)$ on a Banach space X. We say that A is the generator of an α-resolvent if there exist $\omega \in \mathbf{R}$ and a strongly continuous function $S_\alpha : \mathbf{R}^+ \to L(X)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X.$$

It is known that, in the case $\alpha = 1$, $S_\alpha(\cdot) = S_1(\cdot)$ is a C_0-semigroup while, if $\alpha = 2$, we have a cosine family $S_2(\cdot)$. By the subordination principle (see [**3**]), if A generates a β-resolvent with $\beta > \alpha$, then it also generates an α-resolvent. In particular, if A is the generator of a cosine family, there exists an α-resolvent generated by A with $\alpha \in (1, 2)$.

Another case ensuring the existence of α-resolvent was discussed in [**9**]. Specifically, let A be a closed and densely defined operator. Assume that A is a sectorial of type (ω, θ), that is, there exist $\omega \in \mathbf{R}, \theta \in (0, (\pi/2))$, $M > 0$ such that its resolvent set lies in $\mathbf{C}\backslash\Sigma_{\omega,\theta}$ and

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \Sigma_{\omega,\theta};$$

here

$$\Sigma_{\omega,\theta} = \{\omega + \lambda : \lambda \in \mathbf{C}, |\arg(-\lambda)| < \theta\}.$$

In the case $0 \le \theta < \pi(1 - \alpha/2)$, $S_\alpha(\cdot)$ exists and has the following formula:

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}\, d\lambda, \quad t \ge 0,$$

where γ is a suitable path lying outside $\Sigma_{\omega,\theta}$. Furthermore, we have the following assertion for the behavior of $S_\alpha(\cdot)$.

Theorem 2.1. *Let $A : D(A) \subset X \to X$ be a sectorial operator of type (ω, θ) with $0 \le \theta < \pi(1 - \alpha/2)$. Then there exists $C > 0$ independent of t such that*

$$||S_\alpha(t)|| \le \begin{cases} C(1 + \omega\, t^\alpha) e^{\omega^{1/\alpha} t} & \omega \ge 0, \\ \dfrac{C}{1 + |\omega| t^\alpha} & \omega < 0, \end{cases}$$

for $t \ge 0$.

We now look for a suitable concept of integral solutions to (1.1)–(1.3) in the form of a variation-of-constants formula. Denote by $\mathcal{L}$ the Laplace transform for X-valued functions acting on $\mathbf{R}^+$. Putting $\eta(t) = f(t, u(t), u_t)$ and applying the Laplace transform to (1.1)–(1.3), we have:

$$\lambda \mathcal{L}[u](\lambda) - u(0) - \sum_{k \in \Lambda} e^{-\lambda t_k} I_k = \frac{1}{\lambda^{\alpha - 1}} A \mathcal{L}[u](\lambda) + \mathcal{L}[\eta](\lambda).$$

Then

$$(\lambda^\alpha I - A) \mathcal{L}[u](\lambda) = \lambda^{\alpha-1} u(0) + \sum_{k \in \Lambda} e^{-\lambda t_k} \lambda^{\alpha - 1} I_k + \lambda^{\alpha-1} \mathcal{L}[\eta](\lambda).$$

So

$$\begin{aligned} \mathcal{L}[u](\lambda) &= \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} [\varphi(0) - g(u)(0)] \\ &\quad + \sum_{k \in \Lambda} e^{-\lambda t_k} \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} I_k \\ &\quad + \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} \mathcal{L}[\eta](\lambda), \end{aligned}$$

for all λ such that $\operatorname{Re} \lambda > 0$, $\lambda^\alpha \in \rho(A)$ (the resolvent set of A). Let $S_\alpha(\cdot)$ be an α-resolvent generated by A. Then

$$\begin{aligned} \mathcal{L}[u](\lambda) &= \mathcal{L}[S_\alpha](\lambda) [\varphi(0) - g(u)(0)] \\ &\quad + \sum_{k \in \Lambda} e^{-\lambda t_k} \mathcal{L}[S_\alpha](\lambda) I_k \\ &\quad + \mathcal{L}[S_\alpha](\lambda) \mathcal{L}[\eta](\lambda). \end{aligned} \tag{2.1}$$

Using the second translation and convolution theorems of the Laplace transform for the inversion of (2.1), one gets

$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k\le t} S_\alpha(t-t_k)I_k(u(t_k)) \tag{2.2}$$
$$+ \int_0^t S_\alpha(t-s)f(s,u(s),u_s)\,ds, \quad t \ge 0.$$

Given $T > 0$, we denote by $\mathcal{PC}([-h,T];X)$ the space of functions $u : [-h,T] \to X$ such that u is continuous on $[-h,T]\backslash\{t_k : k \in \Lambda\}$ and, for each $t_k, k \in \Lambda$, there exist

$$u(t_k^-) = \lim_{t\to t_k^-} u(t); \qquad u(t_k^+) = \lim_{t\to t_k^+} u(t)$$

and $u(t_k) = u(t_k^-)$. Then $\mathcal{PC}([-h,T];X)$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{PC}} := \sup_{t\in[-h,T]} \|u(t)\|.$$

Motivated by (2.2), we adopt the following definition of integral solutions for (1.1)–(1.3).

Definition 2.2. A function $u \in \mathcal{PC}([-h,T];X)$ is said to be an integral solution of problem (1.1)–(1.3) on the interval $[-h,T]$ if and only if $u(t) = g(u)(t) + \varphi(t)$ for $t \in [-h,0]$, and

$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k))$$
$$+ \int_0^t S_\alpha(t-s)f(s,u(s),u_s)\,ds,$$

for any $t \in [0,T]$.

Let $\mathcal{F} : \mathcal{PC}([-h,T];X) \to \mathcal{PC}([-h,T];X)$, where

$$\mathcal{F}(u)(t) = \begin{cases} \varphi(t) + g(u)(t) & t \in [-h,0], \\ S_\alpha(t)[\varphi(0) - g(u)(0)] & \\ \quad + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k)) & \\ \quad + \int_0^t S_\alpha(t-s)f(s,u(s),u_s)\,ds & t \in [0,T]. \end{cases}$$

Then u is an integral solution of (1.1)–(1.3) if it is a fixed point of *the solution operator* $\mathcal{F}$.

2.2. Fixed point theory for condensing operators. Denote by $\mathcal{B}(X)$ the collection of nonempty bounded subsets of X. We will use the following definition of measure of noncompactness.

Definition 2.3. A function $\beta : \mathcal{B}(X) \to \mathbf{R}^+$ is called *a measure of noncompactness* (MNC) in X if

$$\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega) \quad \text{for every } \Omega \in \mathcal{B}(X),$$

where $\overline{\mathrm{co}}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called

i) monotone if $\Omega_0, \Omega_1 \in \mathcal{B}(X)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$;

ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in X, \Omega \in \mathcal{B}(X)$;

iii) invariant with respect to union with the compact set if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset X$ and $\Omega \in \mathcal{B}(X)$;

iv) algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \le \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{B}(X)$;

v) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is the *Hausdorff* MNC $\chi(\cdot)$, which is defined as follows

$$\chi(\Omega) = \inf\{\varepsilon : \ \Omega \quad \text{has a finite } \varepsilon\text{-net}\}.$$

It should be mentioned that the Hausdorff MNC also has the following additional properties:

- semi-homogeneity: $\chi(t\Omega) \le |t|\chi(\Omega)$ for any $\Omega \in \mathcal{B}(X)$ and $t \in \mathbf{R}$;
- in a separable Banach space X,

$$\chi(\Omega) = \lim_{m\to\infty} \sup_{x\in\Omega} d(x, X_m),$$

where $\{X_m\}$ is a sequence of finite-dimensional subspaces of X such that $X_m \subset X_{m+1}$,$m = 1, 2, \ldots$ and

$$\overline{\bigcup_{m=1}^{\infty} X_m} = X.$$

Based on the Hausdorff MNC χ in X, one can define the *sequential* MNC χ_0 as follows:

$$\chi_0(\Omega) = \sup\{\chi(D) : D \in \Delta(\Omega)\}, \tag{2.3}$$

where $\Delta(\Omega)$ is the collection of all at-most-countable subsets of Ω (see [**1**]). We know that

$$\frac{1}{2}\chi(\Omega) \le \chi_0(\Omega) \le \chi(\Omega), \tag{2.4}$$

for all bounded sets $\Omega \subset X$. Then the following property is evident.

Proposition 2.2. *Let χ be the Hausdorff* MNC *in X and $\Omega \subset X$ a bounded set. Then, for every $\epsilon > 0$, there exists a sequence $\{x_n\} \subset \Omega$ such that*

$$\chi(\Omega) \le 2\chi(\{x_n\}) + \epsilon.$$

We need the following assertion, whose proof can be found in [**22**].

Proposition 2.3. *If $\{w_n\} \subset L^1(0,T;X)$ such that*

$$\|w_n(t)\|_X \le \nu(t), \quad \textit{for a.e. } t \in [0,T],$$

for some $\nu \in L^1(0,T)$, then we have

$$\chi\Big(\Big\{ \int_0^t w_n(s)\,ds \Big\}\Big) \le 2\int_0^t \chi(\{w_n(s)\})\,ds$$

for $t \in [0,T]$.

Let $\chi_{\mathcal{PC}}$ be the Hausdorff MNC in $\mathcal{PC}([-h,T];X)$. We recall the following facts (see [**21**]), which will be used later. For each bounded set $D \subset \mathcal{PC}([-h,T];X)$, one has

• $\chi(D(t)) \le \chi_{\mathcal{PC}}(D)$, for all $t \in [-h,T]$, where $D(t) := \{x(t) : x \in D\}$.

• If D is an equicontinuous set on each interval $(t_k, t_{k+1}] \subset [-h,T]$, then

$$\chi_{\mathcal{PC}}(D) = \sup_{t\in[-h,T]} \chi(D(t)).$$

Definition 2.4. Let E be a Banach space. A continuous map $\mathcal{F} : Z \subseteq E \to E$ is said to be condensing with respect to an MNC β (β-condensing) if, for any bounded set $\Omega \subset Z$, the relation

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in E. The application of the topological degree theory for condensing maps (see, e.g., [**2, 22**]) yields the following fixed point principle.

Theorem 2.4. [**22**, Corollary 3.3.1]. *Let $\mathcal{M}$ be a bounded convex closed subset of E, and let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a β-condensing map. Then* $\mathrm{Fix}\,(\mathcal{F}) := \{x \in E : x = \mathcal{F}(x)\}$ *is a non-empty compact set.*

3. Existence results. Let $\mathcal{C}_h = C([-h, 0]; X)$ and χ_h be the Hausdorff MNC in $\mathcal{C}_h$. Concerning problem (1.1)–(1.3), we give the following assumptions.

(A) The operator A is sectorial of type (ω, θ) with $0 \leq \theta < \pi(1-\alpha/2)$ so that the α-resolvent $S_\alpha(\cdot)$ generated by A is norm continuous for $t > 0$.

(F) The nonlinear function $f : \mathbf{R}^+ \times X \times \mathcal{C}_h \to X$ satisfies:

(i) $f(\cdot, v, w)$ is measurable for each $(v, w) \in X \times \mathcal{C}_h$, $f(t, \cdot, \cdot)$ is continuous for almost every $t \in [0, T]$ and

$$\|f(t, v, w)\|_X \leq k(t)\Psi_f(\|v\|_X + \|w\|_{\mathcal{C}_h}),$$

for all $(v, w) \in X \times \mathcal{C}_h$, where $k \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$, Ψ_f is a real-valued, continuous and nondecreasing function;

(ii) there exists a function $m : \mathbf{R}^2_+ \to \mathbf{R}^+$ such that $m(t, \cdot) \in L^1(0, t)$, $t > 0$, and for all bounded subsets $V \subset X$, $W \subset \mathcal{C}_h$,

$$\chi(S_\alpha(t-s)f(s, V, W)) \leq m(t, s)[\chi(V) + \chi_h(W)],$$

for almost every $t, s \in [0, T]$, $s \leq t$.

(I) The impulsive function $I_k : X \to X$, $k \in \Lambda$, satisfies:

(i) I_k is continuous, and there exists $l_k \geq 0$ verifying that

$$\|I_k(x)\| \leq l_k \Psi_I(\|x\|),$$

where Ψ_I is a real-valued, continuous and nondecreasing function;

(2) there is a number $\mu_k \geq 0$ such that

$$\chi(I_k(V) \leq \mu_k \chi(V),$$

for all bounded sets $V \subset X$.

(G) The nonlocal function $g : \mathcal{PC}([-h,T];X) \to \mathcal{C}_h$ obeys the following conditions:

(i) g is continuous and

$$\|g(u)\|_{\mathcal{C}_h} \leq \Psi_g(||u||_{\mathcal{PC}}),$$

for all $u \in \mathcal{PC}$, where Ψ_g is a continuous and nondecreasing function on $\mathbf{R}^+$;

(ii) there exists $\eta \geq 0$ such that, for any bounded set $D \subset \mathcal{PC}([-h,T];X)$,

$$\chi_h(g(D)) \leq \eta \chi_{\mathcal{PC}}(D).$$

Remark 3.1. Let us give some comments on assumptions (F) (ii), (G) (ii) and (I) (ii).

(1) If $f(t,\cdot,\cdot)$ satisfies the Lipschitz condition, i.e.,

$$\|f(t,v_1,w_1) - f(t,v_2,w_2)\|_X \leq k_f(t)(\|v_1 - v_2\|_X + \|w_1 - w_2\|_{\mathcal{C}_h}),$$

for some $k_f \in L^p_{\text{loc}}(\mathbf{R}^+)$, then (F)(ii) holds for $k(t,s) = \|S_\alpha(t-s)\| k_f(s)$. On the other hand, if $S_\alpha(t)$, $t > 0$, is compact or $f(t,\cdot,\cdot)$ is completely continuous (for each fixed t) then $(F)(ii)$ is obviously fulfilled with $k = 0$.

(2) Regarding (G) (ii), if g is Lipschitzian, that is,

$$\|g(u) - g(v)\|_{\mathcal{C}_h} \leq \eta \|u - v\|_{\mathcal{PC}},$$

then (G) (ii) takes place. This condition is also satisfied with $\eta = 0$ if g is completely continuous.

(3) Similarly for (I) (ii), if I_k is Lipschitzian, that is,

$$\|I_k(x) - I_k(y)\| \le \mu_k \|x - y\|, \quad \text{for all } x, y \in X,$$

then (I) (ii) takes place. Obviously, (I) (ii) is also fulfilled with $\mu_k = 0$ if I_k is completely continuous.

It should be mentioned that, since f, I and g may not be Lipschitzian, the existence of integral solutions of (1.1)–(1.3) cannot be obtained by the Banach contraction principle. In this paper, we deploy the fixed point theory for condensing maps by establishing the so called MNC-estimate (i.e., estimate via MNCs) to prove the condensivity of $\mathcal{F}$.

We need the following result, which was proved in [**35**, Lemma 1].

Lemma 3.1. *Let $\Phi(t,s)$ be a family of bounded linear operators on X for $t, s \in [0,T]$, $s \le t$. Assume that Φ satisfies the following conditions*:

(Φ1) *there exists a function $\rho \in L^q(0,T)$, $q \ge 1$, such that $\|\Phi(t,s)\| \le \rho(t-s)$ for all $t, s \in [0,T]$, $s \le t$;*

(Φ2) *$\|\Phi(t,s) - \Phi(r,s)\| \le \epsilon$ for $0 \le s \le r - \epsilon, r < t = r + h \le T$ with $\epsilon = \epsilon(h) \to 0$ as $h \to 0$.*

Then the operator $\mathbf{S} : L^{q'}(0,T;X) \to C([0,T];X)$ defined by

$$(\mathbf{S}g)(t) := \int_0^t \Phi(t,s) g(s)\, ds$$

sends any bounded set to an equicontinuous one, where q' is the conjugate of q ($q' = +\infty$ if $q = 1$).

Now we prove the condensivity of the solution operator.

Lemma 3.2. *Let the hypotheses (A), (F), (G) and (I) hold. Then the solution operator $\mathcal{F}$ satisfies*

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \le \left[\left(\eta + \sum_{t_k \in [0,T]} \mu_k \right) S_\alpha^T + 8 \sup_{t \in [0,T]} \int_0^t m(t,s)\, ds \right] \chi_{\mathcal{PC}}(D),$$

for all bounded sets $D \subset \mathcal{PC}([-h,T];X)$, here $S_\alpha^T = \sup_{t \in [0,T]} \|S_\alpha(t)\|$.

Proof. Let $D \subset \mathcal{PC}([-h,T];X)$ be a bounded set. Then we have

$$\mathcal{F}(D) = \mathcal{F}_1(D) + \mathcal{F}_2(D) + \mathcal{F}_3(D),$$

where

$$\mathcal{F}_1(u)(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u)(0)] & t \in [0,T], \\ \varphi(t) - g(u)(t) & t \in [-h,0]; \end{cases}$$
$$\mathcal{F}_2(u)(t) = \begin{cases} \sum\limits_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k)) & t \in [0,T], \\ 0 & t \in [-h,0]; \end{cases}$$
$$\mathcal{F}_3(u)(t) = \begin{cases} \int_0^t S_\alpha(t-s)f(s,u(s),u_s)\,ds & t \in [0,T], \\ 0 & t \in [-h,0]. \end{cases}$$

From the algebraically semi-additive property of $\chi_{\mathcal{PC}}$, we have

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \le \chi_{\mathcal{PC}}(\mathcal{F}_1(D)) + \chi_{\mathcal{PC}}(\mathcal{F}_2(D)) + \chi_{\mathcal{PC}}(\mathcal{F}_3(D)).$$

For $z_1, z_2 \in \mathcal{F}_1(D)$, there exist $u_1, u_2 \in D$ such that

$$z_1(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u_1)(0)] & t \in [0,T], \\ \varphi(t) - g(u_1)(t) & t \in [-h,0] \end{cases}$$
$$z_2(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u_2)(0)] & t \in [0,T], \\ \varphi(t) - g(u_2)(t) & t \in [-h,0]. \end{cases}$$

Then

$$\|z_1(t) - z_2(t)\| \le \begin{cases} \|S_\alpha(t)\|\|g(u_1) - g(u_2)\|_{\mathcal{C}_h} & t \in [0,T], \\ \|g(u_1) - g(u_2)\|_{\mathcal{C}_h} & t \in [-h,0]. \end{cases}$$

Therefore,

$$||z_1 - z_2||_{\mathcal{PC}} \le S_\alpha^T \|g(u_1) - g(u_2)\|_{\mathcal{C}_h},$$

thanks to the fact that $S_\alpha^T \ge 1$. This implies

$$\chi_{\mathcal{PC}}(\mathcal{F}_1(D)) \le S_\alpha^T \chi_h(g(D)).$$

Employing (G) (ii), we have

$$\chi_{\mathcal{PC}}(\mathcal{F}_1(D)) \leq \eta S_\alpha^T \chi_{\mathcal{PC}}(D). \tag{3.1}$$

Now, letting $z_1, z_2 \in \mathcal{F}_2(D)$, one can find $u_1, u_2 \in D$ such that

$$\|z_1(t) - z_2(t)\| = \sum_{0<t_k<t} S_\alpha(t-t_k)[I_k(u_1(t_k)) - I_k(u_2(t_k))].$$

Hence,

$$\|z_1 - z_2\|_{\mathcal{PC}} \leq S_\alpha^T \sum_{t_k \in [0,T]} \|I_k(u_1(t_k)) - I_k(u_2(t_k))\|.$$

This inequality deduces that

$$\begin{aligned} \chi_{\mathcal{PC}}(\mathcal{F}_2(D)) &\leq S_\alpha^T \sum_{t_k \in [0,T]} \chi(I_k(D(t_k))) \\ &\leq S_\alpha^T \sum_{t_k \in [0,T]} \mu_k \chi(D(t_k)) \\ &\leq \Big(S_\alpha^T \sum_{t_k \in [0,T]} \mu_k \Big) \chi_{\mathcal{PC}}(D), \end{aligned} \tag{3.2}$$

thanks to (I) (ii).

Regarding $\mathcal{F}_3(D)$, for $\epsilon > 0$, one can choose a sequence $\{u_n\} \subset D$ such that

$$\chi_{\mathcal{PC}}(\mathcal{F}_3(D)) \leq 2\chi_{\mathcal{PC}}(\mathcal{F}_3(\{u_n\})) + \epsilon. \tag{3.3}$$

It follows from assumption (A) that $\Phi(t,s) = S_\alpha(t-s)$ verifies $(\Phi1)-(\Phi2)$ in Lemma 3.1. Then we get that $\mathcal{F}_3(\{u_n\})$ is an equicontinuous set in $C([0,T];X)$. This leads to

$$\begin{aligned} \chi_{\mathcal{PC}}(\mathcal{F}_3(\{u_n\})) &= \sup_{t\in[0,T]} \chi(\mathcal{F}_3(\{u_n\})(t)) \\ &\leq 2 \sup_{t\in[0,T]} \int_0^t \chi\big(S_\alpha(t-s) f(s, u_n(s), (u_n)_s)\big)\, ds \end{aligned}$$

$$\begin{aligned}&\le 2\sup_{t\in[0,T]}\int_0^t m(t,s)[\chi(\{u_n(s)\})\\&\quad+\sup_{\tau\in[-h,0]}\chi(\{u_n(s+\tau)\})]\,ds\\&\le 4\chi_{\mathcal{PC}}(\{u_n\})\sup_{t\in[0,T]}\int_0^t m(t,s)\,ds\\&\le 4\chi_{\mathcal{PC}}(D)\sup_{t\in[0,T]}\int_0^t m(t,s)\,ds;\end{aligned}$$

here we have used Proposition 2.3. In view of (3.3), one has

$$\chi_{\mathcal{PC}}(\mathcal{F}_3(D)) \le 8\chi_{\mathcal{PC}}(D)\sup_{t\in[0,T]}\int_0^t m(t,s)\,ds, \tag{3.4}$$

since $\epsilon > 0$ can be chosen arbitrarily.

Combining (3.1), (3.2) and (3.4), we arrive at

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \le \left[\eta S_\alpha^T + S_\alpha^T\sum_{t_k\in[0,T]}\mu_k + 8\sup_{t\in[0,T]}\int_0^t m(t,s)\,ds\right]\chi_{\mathcal{PC}}(D).$$

The proof is complete. □

Theorem 3.3. *Assume that the hypotheses of Lemma* 3.2 *hold. Then the problem* (1.1)–(1.3) *has at least one integral solution in* $\mathcal{PC}([-h,T];X)$, *provided that*

$$\left(\eta + \sum_{t_k\in[0,T]}\mu_k\right)S_\alpha^T + 8\sup_{t\in[0,T]}\int_0^t m(t,s)\,ds < 1, \tag{3.5}$$

and

$$\begin{aligned}\liminf_{r\to\infty}\frac{1}{r}\Bigg[\Bigg(\Psi_g(r) + \Psi_I(r)\sum_{t_k\in[0,T]} l_k\Bigg)S_\alpha^T\\ + \Psi_f(2r)\sup_{t\in[0,T]}\int_0^t \|S_\alpha(t-s)\|k(s)\,ds\Bigg] < 1,\end{aligned} \tag{3.6}$$

where S_α^T *is given in Lemma* 3.2.

Proof. By (3.5), we obtain the $\chi_{\mathcal{PC}}$-condensing property for $\mathcal{F}$ thanks to Lemma 3.2. Indeed, let $D \subset \mathcal{PC}([-h,T];X)$ be a bounded set satisfying $\chi_{\mathcal{PC}}(D) \le \chi_{\mathcal{PC}}(\mathcal{F}(D))$. Then, by Lemma 3.2, we have

$$\chi_{\mathcal{PC}}(D) \le \chi_{\mathcal{PC}}(\mathcal{F}(D)) \le \ell\, \chi_{\mathcal{PC}}(D),$$

where

$$\ell = \Big(\eta + \sum_{t_k \in [0,T]} \mu_k\Big) S_\alpha^T + 8 \sup_{t\in[0,T]} \int_0^t m(t,s)\, ds < 1.$$

This implies that $\chi_{\mathcal{PC}}(D) = 0$. By the regularity of $\chi_{\mathcal{PC}}$, one gets that D is relatively compact.

In order to apply Theorem 2.4, it remains to show that $\mathcal{F}(B_R) \subset B_R$ for some $R > 0$, where B_R is the closed ball in $\mathcal{PC}([-h,T];X)$ centered at 0 with radius R.

Assume to the contrary that a sequence $\{v_n\} \subset \mathcal{PC}([-h,T];X)$ exists such that $\|v_n\|_{\mathcal{PC}} \le n$ but $\|\mathcal{F}(v_n)\|_{\mathcal{PC}} > n$. From the formulation of $\mathcal{F}$, we have

$$\begin{aligned}
\|\mathcal{F}(v_n)(t)\|_X &\le \sup_{t\in[0,T]} \|S_\alpha(t)\| \Big(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(\|v_n\|_{\mathcal{PC}}) \\
&\quad + \sum_{t_k\in[0,T]} \|I_k(v_n(t_k))\| \Big) \\
&\quad + \int_0^t \|S_\alpha(t-s)\| \|f(s, v_n(s), (v_n)_s)\|\, ds \\
&\le S_\alpha^T \Big(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \sum_{t_k\in[0,T]} l_k \Psi_I(\|v_n(t_k)\|) \Big) \\
&\quad + \int_0^t \|S_\alpha(t-s)\|\, k(s) \Psi_f(\|v_n(s)\| + ||(v_n)_s||_{\mathcal{C}_h})\, ds \\
&\le S_\alpha^T \Big(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \Psi_I(n) \sum_{t_k\in[0,T]} l_k \Big) \\
&\quad + \Psi_f(2n) \int_0^t \|S_\alpha(t-s)\|\, k(s)\, ds.
\end{aligned}$$

Therefore,

$$n < \|\mathcal{F}(v_n)\|_{\mathcal{PC}} \le S_\alpha^T \Bigg(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \Psi_I(n) \sum_{t_k \in [0,T]} l_k \Bigg) + \Psi_f(2n) \sup_{t\in[0,T]} \int_0^t \|S_\alpha(t-s)\| \, k(s)\, ds.$$

Then

$$1 < \frac{1}{n}\|\mathcal{F}(v_n)\|_{\mathcal{PC}} \le \frac{1}{n}\Bigg[S_\alpha^T \Bigg(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \Psi_I(n) \sum_{t_k \in [0,T]} l_k \Bigg) + \Psi_f(2n) \sup_{t\in[0,T]} \int_0^t \|S_\alpha(t-s)\| \, k(s)\, ds \Bigg].$$

Passing the last inequality into limits, one gets a contradiction. The proof is now complete. □

4. Stability results. In order to study the stability results for problem (1.1)–(1.3), we consider the function space

$$\mathcal{PC}_0 = \{u \in \mathcal{PC}([-h,+\infty);X) : \lim_{t\to\infty} u(t) = 0\}$$

with the norm

$$\|u\|_\infty = \sup_{t\ge 0} \|u(t)\|,$$

where $\mathcal{PC}([-h,\infty);X)$ is defined similarly to $\mathcal{PC}([-h,T];X)$ as $T = +\infty$.

Then $\mathcal{PC}_0$ is a Banach space. In this section, we replace assumptions (A), (F), (G) and (I) by the following:

(A′) The operator A is sectorial of type (ω,θ) such that $\omega < 0$ and $0 \le \theta < \pi(1-\alpha/2)$.

(F′) $f(\cdot, v, w)$ is measurable for each $v \in X, w \in \mathcal{C}_h$, $f(t,\cdot,\cdot)$ is continuous for almost every $t \in \mathbf{R}^+$, $f(t,0,0) = 0$, and there exists $k \in L^1(\mathbf{R}^+)$, such that

$$\|f(t,v_1,w_1) - f(t,v_2,w_2)\| \le k(t)(\|v_1 - v_2\| + \|w_1 - w_2\|_{\mathcal{C}_h}),\ t \in \mathbf{R}^+,$$

for all $v_1, v_2 \in X$, $w_1, w_2 \in \mathcal{C}_h$.

(G′) g is a continuous function satisfying that $g(0) = 0$ and there is a nonnegative number η such that

$$\|g(w_1) - g(w_2)\|_{\mathcal{C}_h} \le \eta \|w_1 - w_2\|_{\mathcal{PC}},$$

for all $w_1, w_2 \in \mathcal{PC}([-h, T]; X)$, with all $T > 0$.

(I′) I_k, $k \in \Lambda$, is continuous, $I_k(0) = 0$, and there exist a sequence $\{\mu_k\}$, $k \in \Lambda$, such that $\sum_{k\in\Lambda} \mu_k < \infty$, and

$$\|I_k(x) - I_k(y)\| \le \mu_k \|x - y\|, \quad \text{for all } x, y \in X.$$

Theorem 4.1. *Let* (A′), (F′), (G′) *and* (I′) *hold. Then problem* (1.1)–(1.3) *has a unique solution* $u \in \mathcal{PC}_0$, *provided that*

$$\left(\eta + \sum_{k\in\Lambda} \mu_k\right) S_\alpha^\infty + 2 \sup_{t\ge 0} \int_0^t \|S_\alpha(t-s)\| k(s)\, ds < 1, \tag{4.1}$$

where $S_\alpha^\infty = \sup_{t\ge 0} \|S_\alpha(t)\|$.

Proof. In the context of this theorem, we make use of the contraction mapping principle. We will show that the solution operator $\mathcal{F}$ maps $\mathcal{PC}_0$ into itself, and it is a contraction map. Here we recall that

$$\mathcal{F}(u)(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum\limits_{0<t_k<t} S_\alpha(t-t_k) I_k(u(t_k)) \\ \quad + \int_0^t S_\alpha(t-s) f(s, u(s), u_s)\, ds, t > 0, \\ \varphi(t) - g(u)(t), \quad t \in [-h, 0]. \end{cases}$$

Let $u \in \mathcal{PC}_0$ be such that $R = \|u\|_\infty > 0$. We first prove that $\mathcal{F}(u) \in \mathcal{PC}_0$, i.e., $\mathcal{F}(u)(t) \to 0$, as $t \to +\infty$.

Let $\epsilon > 0$ be given. Then there exists $T_1 > 0$ such that

$$\|u(t)\| \le \epsilon, \quad \text{for all } t > T_1, \tag{4.2}$$

$$\|u_t\|_{\mathcal{C}_h} = \sup_{\tau\in[-h,0]} \|u(t+\tau)\| \le \epsilon, \quad \text{for all } t > T_1 + h. \tag{4.3}$$

On the other hand, from the assumption that $\sum_{k\in\Lambda}\mu_k < +\infty$, there exists $N_0 \in \mathbf{N}$ such that

$$\sum_{k>N_0} \mu_k \le \epsilon.$$

Then, for $t > 0$,

$$\begin{aligned}
\|\mathcal{F}(u)(t)\| &\le \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \|g(u)\|_{\mathcal{C}_h}) \\
&\quad + \sum_{k\le N_0} \|S_\alpha(t-t_k)\|\,\|I_k(u(t_k))\| \\
&\quad + \sum_{k>N_0} \|S_\alpha(t-t_k)\|\,\|I_k(u(t_k))\| \\
&\quad + \int_0^t \|S_\alpha(t-s)\|\,\|f(s,u(s),u_s)\|\,ds \\
&\le \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \eta R) \\
&\quad + R\sum_{k\le N_0} \|S_\alpha(t-t_k)\|\,\mu_k + RS_\alpha^\infty \sum_{k>N_0} \mu_k \\
&\quad + \int_0^t \|S_\alpha(t-s)\|\,k(s)\,(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\,ds \\
&= E_1(t) + E_2(t) + E_3(t),
\end{aligned}$$

where

$$\begin{aligned}
E_1(t) &= \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \eta R), \\
E_2(t) &= R\sum_{k\le N_0} \|S_\alpha(t-t_k)\|\,\mu_k + RS_\alpha^\infty \sum_{k>N_0} \mu_k, \\
E_3(t) &= \int_0^t \|S_\alpha(t-s)\|\,k(s)\,(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\,ds.
\end{aligned}$$

Observing from Theorem 2.1 that there is $T_2 > 0$ verifying

$$\|S_\alpha(t)\| \le \epsilon, \quad \text{for all } t > T_2,$$

so

$$E_1(t) \le (1+\eta)R\epsilon, \quad \text{for all } t > T_2. \tag{4.4}$$

In addition,

$$E_2(t) \leq \Big(\sum_{k\leq N_0} \mu_k + S_\alpha^\infty\Big) R\epsilon, \quad \text{for all } t > T_2 + t_{N_0}. \tag{4.5}$$

Concerning $E_3(t)$, for $t > T_1 + h$ one has

$$\begin{aligned} E_3(t) &= \Big(\int_0^{T_1+h} + \int_{T_1+h}^t\Big) \|S_\alpha(t-s)\|\, k(s)\, (\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds \\ &\leq 2R \int_0^{T_1+h} ||S_\alpha(t-s)||\, k(s)\, ds \\ &\quad + 2\epsilon \int_{T_1+h}^t \|S_\alpha(t-s)\|\, k(s)\, ds \end{aligned}$$

thanks to (4.2)–(4.3). Therefore,

$$E_3(t) \leq 2R\epsilon \int_0^{T_1+h} k(s)\, ds + 2\epsilon \int_{T_1+h}^t \|S_\alpha(t-s)\|\, k(s)\, ds,$$

for all $t > T_2 + T_1 + h$. Then

$$E_3(t) \leq (2R\|k\|_{L^1(\mathbf{R}^+)} + 1)\epsilon, \tag{4.6}$$

for all $t > T_2 + T_1 + h$. Here we use the fact that

$$\int_{T_1+h}^t \|S_\alpha(t-s)\|\, k(s)\, ds < 1,$$

due to (4.1). Combining (4.4), (4.5) and (4.6), gives

$$\|\mathcal{F}(u)(t)\| \leq C\epsilon$$

for all $t > \max\{T_2 + T_1 + h, T_2 + t_{N_0}\}$, where

$$C = (1+\eta)R + \Big(\sum_{k\in\Lambda} \mu_k + S_\alpha^\infty\Big) R + 2R\|k\|_{L^1(\mathbf{R}^+)} + 1.$$

This derives the claim that $\mathcal{F}(\mathcal{PC}_0) \subset \mathcal{PC}_0$.

It remains to show that $\mathcal{F}$ is contractive. Let $u, v \in \mathcal{PC}_0$. Then, by using (F′), (G′) and (I′), we have

$$\begin{aligned}
\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\| &\le \|S_\alpha(t)\|\,\|g(u)-g(v)\|_{\mathcal{C}_h}\\
&\quad+\sum_{0<t_k<t}\|S_\alpha(t-t_k)\|\,\|I_k(u(t_k))-I_k(v(t_k))\|\\
&\quad+\int_0^t\|S_\alpha(t-s)\|\|f(s,u(s),u_s)-f(s,v(s),v_s)\|ds\\
&\le S_\alpha^\infty\,\eta\|u-v\|_\infty+\Big(S_\alpha^\infty\sum_{0<t_k<t}\mu_k\Big)\|u-v\|_\infty\\
&\quad+\Big(2\int_0^t\|S_\alpha(t-s)\|\,k(s)\,ds\Big)\|u-v\|_\infty.
\end{aligned}$$

So

$$\|\mathcal{F}(u)-\mathcal{F}(v)\|_\infty \le \ell\|u-v\|_\infty,$$

where

$$\ell=\Big(\eta+\sum_{k\in\Lambda}\mu_k\Big)S_\alpha^\infty+2\sup_{t\ge 0}\int_0^t\|S_\alpha(t-s)\|k(s)\,ds<1.$$

We get the desired conclusion. □

We now consider the case $\#\Lambda<\infty$, e.g., the number of impulsive moments is finite. We will show that $\|u(t)\|=O(t^{-\alpha})$ as $t\to\infty$.

Theorem 4.2. *Let the hypotheses of Theorem* 4.1 *hold. Assume further that* $\#\Lambda<\infty$*. Then problem* (1.1)–(1.3) *has a unique integral solution* $u\in\mathcal{PC}_0$ *satisfying* $\|u(t)\|=O(t^{-\alpha})$ *as* $t\to\infty$*.*

Proof. Let $\mathcal{PC}_\alpha=\{u\in\mathcal{PC}([-h,\infty);X):\ \|u(t)\|=O(t^{-\alpha})\text{ as }t\to\infty\}$. It is easily seen that $\mathcal{PC}_\alpha$ is a closed subspace of $\mathcal{PC}_0$. We have to prove that the solution operator $\mathcal{F}$ is a contraction on $\mathcal{PC}_\alpha$. As a matter of fact, it suffices to verify that $\mathcal{F}(\mathcal{PC}_\alpha)\subset\mathcal{PC}_\alpha$.

We proceed similarly as in the proof of Theorem 4.1. Let $u \in \mathcal{PC}_\alpha$ be such that $\|u\|_\infty = R$. Then

$$t^\alpha \|u(t)\| \le M, \tag{4.7}$$

$$t^\alpha \|u_t\|_{\mathcal{C}_h} = t^\alpha \sup_{\tau\in[-h,0]} \|u(t+\tau)\| \le M, \quad \text{for all } t > h, \tag{4.8}$$

for some $M > 0$.

Denote by $E_1(t)$ and $E_3(t)$ the notations as in the proof of Theorem 4.1. Let $\Lambda = \{1, 2, \dots, N_0\}$ and $E_2(t) = R\sum_{k\le N_0} \|S_\alpha(t-t_k)\|\,\mu_k$. Taking into account that

$$\|S_\alpha(t)\| \le \frac{C}{1+|\omega|t^\alpha},$$

we have

$$\begin{aligned} t^\alpha E_1(t) &= t^\alpha \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \eta R) \\ &\le \frac{Ct^\alpha}{1+|\omega|t^\alpha}(\|\varphi\|_{\mathcal{C}_h} + \eta R) = O(1), \quad \text{as } t \to \infty. \end{aligned}$$

Similarly, for $t > t_{N_0}$,

$$\begin{aligned} t^\alpha E_2(t) &= Rt^\alpha \sum_{k\le N_0} \|S_\alpha(t-t_k)\|\,\mu_k \\ &\le \sum_{k\le N_0} \frac{RCt^\alpha}{1+|\omega|(t-t_k)^\alpha}\mu_k \\ &\le \frac{RCt^\alpha}{1+|\omega|(t-t_{N_0})^\alpha} \sum_{k\le N_0} \mu_k = O(1), \quad \text{as } t \to \infty. \end{aligned}$$

As for $E_3(t)$, for $t > 0$, one has

$$\begin{aligned} t^\alpha E_3(t) &= t^\alpha \left(\int_0^{t/2} + \int_{t/2}^t \right) \|S_\alpha(t-s)\|\, k(s)\, (\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds \\ &= t^\alpha E_{3a}(t) + t^\alpha E_{3b}(t); \end{aligned}$$

$$
\begin{aligned}
t^\alpha E_{3a}(t) &= t^\alpha \int_0^{t/2} \|S_\alpha(t-s)\| \|k(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds \\
&\le \frac{2RCt^\alpha}{1+|\omega|(t/2)^\alpha} \int_0^{t/2} k(s)\, ds \\
&\le \frac{2RCt^\alpha}{1+|\omega|(t/2)^\alpha} \|k\|_{L^1(\mathbf{R}^+)} = O(1), \quad \text{as } t \to \infty, \\
t^\alpha E_{3b}(t) &= t^\alpha \int_{t/2}^{t} \|S_\alpha(t-s)\| k(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds \\
&= \int_{t/2}^{t} \|S_\alpha(t-s)\| \left(\frac{t}{s}\right)^\alpha k(s)(s^\alpha\|u(s)\| + s^\alpha \|u_s\|_{\mathcal{C}_h})\, ds \\
&\le 2MC \int_{t/2}^{t} \frac{(t/s)^\alpha}{1+|\omega|(t-s)^\alpha} k(s)\, ds \\
&\le 2^{\alpha+1}MC \int_{t/2}^{t} k(s)\, ds \\
&\le 2^{\alpha+1}MC\|k\|_{L^1(\mathbf{R}^+)}.
\end{aligned}
$$

The above estimates yield

$$
\begin{aligned}
t^\alpha \|\mathcal{F}(u)(t)\| &\le t^\alpha(E_1(t) + E_2(t) + E_3(t)) \\
&= O(1), \quad \text{as } t \to \infty.
\end{aligned}
$$

The proof is complete. □

5. Example. We give an application to the abstract results. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$. We consider the following system:

(5.1)

$$
\frac{\partial u}{\partial t}(x,t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} L_x u(x,s)\, ds + k(t)\widetilde{f}(x, u(x,t), u(x,t-h)),
$$

$$
\alpha \in (1,2), \quad t \in \mathbf{R}^+ \backslash \{t_1, t_2, \dots, t_N\},\ x \in \Omega,
$$

(5.2) $$u(x,t) = 0, \quad x \in \partial\Omega,\ t \in \mathbf{R}^+,$$

(5.3) $$u(x, t_k^+) = u(x,t_k) + \widetilde{I}_k(x, u(x,t_k)), \quad x \in \Omega,\ 1 \le k \le N,$$

(5.4) $$u(x,s) + \sum_{i=1}^{M} c_i u(x, \tau_i + s) = \varphi(s), \quad s \in [-h, 0],\ x \in \Omega,$$

where the operator

$$L_x = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

has the property

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \theta|\xi|^2, \quad \text{for all } \xi \in \mathbf{R}^n$$

with $\theta > 0$. Let $X = L^2(\Omega)$, $A = L_x$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then system (5.1)–(5.4) is in the form of the abstract model (1.1)–(1.3) with

$$\begin{aligned} f(t,v,w)(x) &= k(t)\widetilde{f}(x,v(x),w(x,-h)), \quad v \in X,\ w \in C([-h,0];X), \\ I_k(v)(x) &= \widetilde{I}_k(x,v(x)), \quad v \in X, \\ g(u)(s)(x) &= \sum_{i=1}^{M} c_i u(x,\tau_i+s), \quad u \in \mathcal{PC}([-h,+\infty);X). \end{aligned}$$

It is known that (see [**33**]) A is a sectorial operator and it generates an analytical semigroup in X. Moreover, one can check that A is sectorial of type $(\lambda_1, 0)$ where $\lambda_1 < 0$ is the first eigenvalue of A.

Assume that $k \in L^1(\mathbf{R})$ and $\widetilde{f} : \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that

$$|\widetilde{f}(x,y_1,z_1) - \widetilde{f}(x,y_2,z_2)| \leq \kappa(x)(|y_1-y_2| + |z_1-z_2|), \quad \kappa \in X,$$

for all $x \in \Omega$, $y_1, y_2, z_1, z_2 \in \mathbf{R}$. Then we have

$$\begin{aligned} &\|f(t,v_1,w_1) - f(t,v_2,w_2)\| \\ &\qquad \leq k(t)\|\kappa\|(\|v_1-v_2\| + \|w_1(\cdot,-h) - w_2(\cdot,-h)\|) \\ &\qquad \leq k(t)\|\kappa\|(\|v_1-v_2\| + \|w_1-w_2\|_{\mathcal{C}_h}), \end{aligned}$$

for all $v_1, v_2 \in X$, $w_1, w_2 \in \mathcal{C}_h$.

Let $\widetilde{I}_k : \Omega \times \mathbf{R} \to \mathbf{R}$ be such that

$$|\widetilde{I}_k(x,y_1) - \widetilde{I}_k(x,y_2)| \leq \ell_k(x)|y_1-y_2|, \quad \ell_k \in X,$$

for all $x \in \Omega$, $y_1, y_2 \in \mathbf{R}$. Then

$$\|I_k(v_1) - I_k(v_2)\| \leq \|\ell_k\|\,\|v_1-v_2\|, \quad \text{for all } v_1, v_2 \in X.$$

Regarding the nonlocal function g, it is obvious that

$$\|g(u_1) - g(u_2)\|_{\mathcal{C}_h} \leq \left(\sum_{i=1}^{M} c_i\right)\|u_1 - u_2\|_{\mathcal{PC}},$$
$$\text{for all } u_1, u_2 \in \mathcal{PC}([-h,T];X), \quad \text{for all } T > 0.$$

Under the above settings and applying Theorem 4.2, one can state that problem (5.1)–(5.4) has a unique integral solution in $\mathcal{PC}_\alpha$, provided that

$$\left(\sum_{i=1}^{M} c_i + \sum_{i=1}^{N} \|\ell_i\|\right) S_\alpha^\infty + 2\|\kappa\| \sup_{t\geq 0} \int_0^t \|S_\alpha(t-s)\| k(s)\, ds < 1.$$

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