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A SPECTRAL COLLOCATION METHOD FOR EIGENVALUE PROBLEMS OF COMPACT INTEGRAL OPERATORS

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ABSTRACT. We propose and analyze a new spectral collocation method to solve eigenvalue problems of compact integral operators, particularly, piecewise smooth operator kernels and weakly singular operator kernels of the form $1/|t-s|^{\mu}$, $0 < \mu < 1$. We prove that the convergence rate of eigenvalue approximation depends upon the smoothness of the corresponding eigenfunctions for piecewise smooth kernels. On the other hand, we can numerically obtain a higher rate of convergence for the above weakly singular kernel for some μ's even if the eigenfunction is not smooth. Numerical experiments confirm our theoretical results.

1. Introduction. We consider numerical approximation of the eigenvalue problem for a compact integral operator T on a Banach space. Recent years have witnessed a revitalization of this field, and various methods are applied to solve the problem. The Galerkin, Petrov-Galerkin, collocation, Nyström and degenerate kernel methods have been extensively studied for the approximation of eigenvalues and eigenvectors of integral operators. The results are well documented in the literature. Here, we mention a few related to our current work. As early as 1967, Atkinson proved a general theorem showing the convergence of numerical eigenvalues and eigenvectors to those of compact integral operators [**2**]. In 1975, he further obtained a convergence rate for the approximation [**3**], based upon which Osborn established a general spectral approximation theory for compact operators, when a sequence of $\{T_n\}$ approximates T in a collectively compact manner. The analysis of [**3, 17**] covers many methods and provides a basis for the convergence analysis of our method. In [**13**], Dellwo and Friedman proposed

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a new approach by solving a polynomial eigenvalue problem of a higher degree, based upon which Alam et al. [**1**] obtained an accelerated spectral approximation for eigenelements. Kulkarni [**16**] introduced another method by involving a new approximation operator T_n and obtained a high-order convergence rate. In addition, a multiscale method was discussed in [**11**]. Comprehensive studies for eigenvalue problem can be found in [**5, 9, 21**]. For the numerical solution of integral equations or integro-differential equations, interested readers are referred to [**4, 7**].

In this article, we approximate eigenfunctions by some appropriate orthogonal polynomial expansions. In a different manner from previous methods in the literature we find the exact integration when calculating the convolution of the singular kernel with the orthogonal polynomials. The key ingredients here are some special identities. By doing so, we: 1) avoid large numerical quadrature errors accumulated with the singular kernels and thereby obtain higher accuracy for eigenvalue approximations, and 2) avoid product integration methods and therefore reduce the computational cost. Furthermore, if the kernel is positive definite and piecewisely smooth, a refined result can be obtained.

To fix the idea, we consider problems of the form

$$\int_0^1 k(t,s)u(s)\,ds = \lambda u(t), \quad t \in [0,1], \tag{1.1}$$

where $k(t,s) = |t-s|^{-\mu}$ for $0 < \mu < 1$, $k(t,s)$ is piecewisely smooth or smooth. We will develop algorithms for all three types of problems separately.

This paper is organized as follows. In Section 2, some preliminary knowledge is given. In Section 3, algorithms for all types of equations are listed. Section 4 is devoted to convergence analysis of algorithms. Finally, we illustrate our theories with numerical examples in Section 5. Throughout the paper, C stands for a generic constant that is independent of collocation points p but which may depend upon the index μ and the number of pieces a piecewise kernel has.

2. Preliminaries. Let $T : X \to X$ be a compact linear operator on a Banach space X and $\sigma(T)$ and $\rho(T)$ the spectrum and resolvent of T, respectively. Let λ be a nonzero eigenvalue of T with multiplicity m, and let Γ be a circle centered at λ which lies in $\rho(T)$ and which encloses no other points in $\sigma(T)$. Then the spectral projection associated with T and λ is defined by

$$E = -\frac{1}{2\pi i}\int_{\Gamma}(T - zI)^{-1}dz$$

and $\max_{z\in\Gamma}\|(T - zI)^{-1}\| \leq C$.

Let $\{T_n\}$ be a sequence of operators in $\mathcal{B}(X)$ that converges to T in a collective way, i.e., the set $\{T_n x : \|x\| \leq 1,\ n = 1, 2, \dots\}$ is sequentially compact. For n large enough, $\Gamma \in \rho(T_n)$ and the associated projection,

$$E_n = -\frac{1}{2\pi i}\int_{\Gamma}(T_n - zI)^{-1}dz$$

exists and $\max_{z\in\Gamma}\|(T_n - zI)^{-1}\| \leq C$. Clearly, $\dim(E) = \dim(E_n) = m$ and $T_n E_n = E_n T_n$. Furthermore, the spectrum of T_n inside Γ contains m approximations of λ, i.e., $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,m}$, counted according to their algebraic multiplicities [**9, 17**]. Let

$$\widehat{\lambda}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \cdots + \lambda_{n,m}}{m}.$$

Then we have the following theorem.

Theorem 2.1 [**17**]. *For all n sufficiently large,*

$$|\lambda - \widehat{\lambda}_n| \leq C\|(T - T_n)|_{R(E)}\|,$$

where $R(E)$ is the range of the projection E.

This is a rather general result. We may refine the result if the kernel is positive definite. Let

$$(2.1)\quad a(u, v) = \int_0^1\int_0^1 k(t, s)u(s)v(t)\,ds\,dt, \quad b(u, v) = \int_0^1 u(t)v(t)\,dt,$$

where v is a test function in the L^2 space V. If the bilinear operator $a(u, v)$ is coercive, then we can list eigenvalues of T by

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0,$$

with zero the only possible cluster point.

Let us consider a numerical approximation of the first eigenpair (λ, u). Let (λ_p, v_p) be their Galerkin approximation, and let u_p be the Legendre expansion of u. We have

$$\lambda = \frac{a(u,u)}{b(u,u)} = \sup_{v\in V} \frac{a(v,v)}{b(v,v)}, \qquad \lambda_p = \frac{a(v_p,v_p)}{b(v_p,v_p)} = \max_{v\in\mathcal{P}_p} \frac{a(v,v)}{b(v,v)}. \tag{2.2}$$

Here $\mathcal{P}_p$ is the polynomial space with degree no more than p. Denote $\widetilde{\lambda}_p = a(u_p,u_p)/b(u_p,u_p)$; then we have the following lemma.

Lemma 2.2. *Let λ, λ_p and $\widetilde{\lambda}_p$ be defined as above and $a(u,v)$ coercive. Then*

$$0 \le \lambda - \lambda_p \le \lambda - \widetilde{\lambda}_p = \lambda \frac{\|u-u_p\|_b^2}{\|u\|_b^2} - \frac{\|u-u_p\|_a^2}{\|u\|_b^2}. \tag{2.3}$$

Proof. From [**5**, page 701, Lemma 9.1], we have

$$0 \le \nu_p - \nu \le \widetilde{\nu}_p - \nu \le \frac{\|u-u_p\|_b^2}{\|u\|_a^2} - \nu \frac{\|u-u_p\|_a^2}{\|u\|_a^2}, \tag{2.4}$$

where $\nu = 1/\lambda$, $\nu_p = 1/\lambda_p$ and $\widetilde{\nu}_p = 1/\widetilde{\lambda}_p$. Hence,

$$0 \le \frac{\lambda - \lambda_p}{\lambda} \le \lambda_p \frac{\|u-u_p\|_b^2}{\|u\|_a^2} - \frac{\lambda_p}{\lambda} \frac{\|u-u_p\|_a^2}{\|u\|_a^2}. \tag{2.5}$$

Using the fact that

$$a(u_p,u_p) = \lambda_p b(u_p,u_p),$$

we derive (2.3) from (2.5). $\square$

Next, we introduce some identities, which will be essential in this paper. Towards this end, we define the class of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Under the normalization $P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}$, one has the expression, namely,

$$P_k^{(\alpha,\beta)}(x) = \frac{1}{2^k} \sum_{l=0}^{k} \binom{k+\alpha}{k-l} \binom{k+\beta}{l} (x-1)^l (x+1)^{k-l}. \tag{2.6}$$

Jacobi polynomials satisfy the three-term recursive relations:

$$
\begin{aligned}
&P_0^{(\alpha,\beta)}(x) = 1, \qquad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[(\alpha-\beta) + (\alpha+\beta+2)x], \\
&a_{1,k}P_{k+1}^{(\alpha,\beta)}(x) = a_{2,k}P_k^{(\alpha,\beta)}(x) - a_{3,k}P_{k-1}^{(\alpha,\beta)}(x),
\end{aligned} \tag{2.7}
$$

where

$$
\begin{aligned}
a_{1,k} &= 2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta), \\
a_{2,k} &= (2k+\alpha+\beta+1)(\alpha^2-\beta^2) + x\Gamma(2k+\alpha+\beta+3)/\Gamma(2k+\alpha+\beta), \\
a_{3,k} &= 2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2).
\end{aligned} \tag{2.8}
$$

Especially if $\alpha = 0$ and $\beta = 0$, Jacobi polynomials become Legendre polynomials.

Lemma 2.3 [**19**]. *Let a, b be positive constants and $L_n(x)$ the Legendre polynomials with degree n on $[-1,1]$. Then*

$$
\int_a^b (s-a)^{\alpha-1} L_n\left(\frac{s}{b}\right) ds = \frac{n!}{(\alpha)_{n+1}}(b-a)^\alpha P_n^{(\alpha,-\alpha)}\left(\frac{a}{b}\right), \quad -b < a < b,\ \alpha > 0, \tag{2.9}
$$

$$
\int_{-a}^b (b-s)^{\beta-1} L_n\left(\frac{s}{a}\right) ds = \frac{n!}{(\beta)_{n+1}}(b+a)^\beta P_n^{(-\beta,\beta)}\left(\frac{b}{a}\right), \quad -a < b < a,\ \beta > 0, \tag{2.10}
$$

where $(k)_{n+1} = k(k+1)\cdots(k+n)$.

Specifically, if we choose $a = 1$, $b = x$, $\beta = 1-\mu$ in (2.10), then we obtain

$$
\int_{-1}^x \frac{L_n(t)}{(x-t)^\mu} dt = \frac{n!}{(1-\mu)_{n+1}}(1+x)^{1-\mu} P_n^{(\mu-1,1-\mu)}(x), \tag{2.11}
$$

and $a = x$, $b = 1$, $\alpha = 1-\mu$ in (2.9), we arrive at

$$
\int_x^1 \frac{L_n(t)}{(t-x)^\mu} dt = \frac{n!}{(1-\mu)_{n+1}}(1-x)^{1-\mu} P_n^{(1-\mu,\mu-1)}(x). \tag{2.12}
$$

Remark 1. We use identities (2.11) and (2.12) in our algorithm for weakly singular kernels after we expand eigenvectors by Legendre polynomials.

Lemma 2.4 [**18**]. *Let* $\alpha > -1$, $\beta > -1$ *and* $0 < \nu < 1$. *Then, for* $-1 < x < 1$,

$$\begin{aligned} &\int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta} P_m^{(\alpha,\beta)}(t)\,dt}{|x-t|^{\nu}} \\ &\quad = \frac{\cos(\pi\nu/2)\Phi_1(x) + \cos\pi((\nu/2)-\beta)\Phi_2(x)}{\Gamma(\nu)\cos(\pi\nu/2)}, \quad m = 0,1,2,\dots, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \Phi_1(x) = {}& \frac{\Gamma(m+\alpha+1)\Gamma(m+\nu)\Gamma(\beta-\nu+1)(-1)^m}{2^{-\alpha-\beta+\nu-1}\Gamma(m+\alpha+\beta-\nu+2)m!} \\ & \times {}_2F_1\left(m+\nu, \nu-m-\alpha-\beta-1; -\beta+\nu; \frac{1+x}{2}\right), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \Phi_2(x) = {}& \frac{\Gamma(m+\beta+1)\Gamma(\nu-\beta-1)(-1)^{m+1}}{2^{-\alpha}(1+x)^{\nu-\beta-1}m!} \\ & \times {}_2F_1\left(m+\beta+1, -m-\alpha; \beta-\nu+2; \frac{1+x}{2}\right). \end{aligned} \tag{2.15}$$

Here, ${}_2F_1(a,b;c;z)$ *is known as Gauss's hypergeometric functions.*

For the sake of convergence analysis, we need to introduce the error estimate of Gauss quadrature.

Lemma 2.5 [**12**]. *Let* $f \in C^{2n}$, x_i *and* w_i *be the Gauss points and their corresponding quadrature weights on the interval* $[a,b]$. *Then*

$$\int_a^b f(x)\,dx - \sum_{i=0}^{n} w_i f(x_i) = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \quad \xi \in (a,b). \tag{2.16}$$

3. Algorithms. In this section, we develop algorithms for eigenproblem with all three kinds of kernels mentioned before. Models that we consider in this article are:

(1) Weakly singular kernels

$$\lambda y(t) = \int_0^1 \frac{y(s)}{|t-s|^\mu} ds, \quad 0<\mu<1,\ t \in [0,1]; \tag{3.1}$$

(2) Piecewise smooth kernels

$$\begin{gathered} \lambda y(t) = \int_0^1 k(t,s) y(s)\, ds, \quad t\in[0,1], \\ \text{where } k(t,s) = \begin{cases} t - s/2 & \text{if } 0\le t\le s\le 1, \\ s/2 & \text{if } 0\le s<t\le 1; \end{cases} \end{gathered} \tag{3.2}$$

(3) Smooth kernels

$$\lambda y(t) = \int_0^1 e^{st} y(s)\, ds, \quad t\in[0,1]. \tag{3.3}$$

3.1. The first algorithm for (3.1). It is clear that (3.1) is equivalent to

$$\lambda y(t) = \int_0^t \frac{y(s)}{(t-s)^\mu} ds + \int_t^1 \frac{y(s)}{(s-t)^\mu} ds. \tag{3.4}$$

We make a change of variable $t=(1+x)/2$ and obtain

$$\int_0^{(1+x)/2} \left(\frac{1+x}{2} - s\right)^{-\mu} y(s)\, ds + \int_{(1+x)/2}^1 \left(s - \frac{1+x}{2}\right)^{-\mu} y(s)\, ds = \lambda u(x), \tag{3.5}$$

where $x\in[-1,1]$ and $u(x) = y((1+x)/2)$. Next, we make another change of variable, $s=(1+\tau)/2$ and reach

$$\begin{gathered} \left(\frac{1}{2}\right)^{1-\mu} \int_{-1}^x (x-\tau)^{-\mu} u(\tau)\, d\tau + \left(\frac{1}{2}\right)^{1-\mu} \int_x^1 (\tau - x)^{-\mu} u(\tau)\, d\tau = \lambda u(x), \\ x\in[-1,1]. \end{gathered} \tag{3.6}$$

Let $u_p(x) = \sum_{j=0}^{p} c_j L_j(x)$ be the approximation of $u(x)$. Obviously, the c_j's satisfy the equation

$$\left(\frac{1}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} \frac{L_j(\tau)}{(x_i-\tau)^{\mu}}\, d\tau + \left(\frac{1}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{x_i}^{1} \frac{L_j(\tau)}{(\tau-x_i)^{\mu}} d\tau = \lambda_p \sum_{j=0}^{p} c_j L_j(x_i). \tag{3.7}$$

Substituting (2.11) and (2.12) into (3.7), we obtain

$$\begin{aligned} \sum_{j=0}^{p} c_j \Bigg[\left(\frac{1}{2}\right)^{1-\mu} & \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\ & + \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \Bigg] \\ & = \lambda_p \sum_{j=0}^{p} c_j L_j(x_i), \quad i = 0, \ldots, p. \end{aligned} \tag{3.8}$$

If we write

$$\begin{aligned} a_{ij} &= \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\ &\quad + \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \\ b_{ij} &= L_j(x_i), \end{aligned}$$

then we have $AC_p = \lambda_p BC_p$, where $A = (a_{ij})$, $B = b_{ij}, C_p = (c_0, c_1, \ldots, c_p)^T$.

3.2. The second algorithm for (3.1). From [**20**, Theorem 1], we assume that the first true eigenvector is of the form

$$y(t) = \widehat{d}_1 t^{1-\mu} + \widehat{d}_2 (1-t)^{1-\mu} + \text{a smoother function } \phi(t). \tag{3.9}$$

Hence, we approximate the eigenvector by $u_p(t) = d_1 t^{1-\mu} + d_2(1-t)^{1-\mu} + \sum_{j=0}^{p} c_j P_j(t)$, where $P_j(t)$ is the shifted Legendre polynomial on $[0, 1]$,

$j = 0, 1, \dots, p$. Substituting it into (3.1) and taking the same change of variables as the previous algorithm, we obtain

$$
\begin{aligned}
(3.10)\quad & \left(\frac{1}{2}\right)^{2-2\mu}\left(d_1 \int_{-1}^{1} \frac{(1+\tau)^{1-\mu}}{|x-\tau|^{\mu}}\, d\tau + d_2 \int_{-1}^{1} \frac{(1-\tau)^{1-\mu}}{|x-\tau|^{\mu}}\, d\tau\right) \\
& + \left(\frac{1}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \left(\int_{-1}^{x} (x-\tau)^{-\mu} L_j(\tau)\, d\tau + \int_{x}^{1} (\tau - x)^{-\mu} L_j(\tau)\, d\tau\right) \\
& = d_1 \left(\frac{1+x}{2}\right)^{1-\mu} + d_2 \left(\frac{1-x}{2}\right)^{1-\mu} + \sum_{j=0}^{p} c_j L_j(x), \quad x \in [-1, 1].
\end{aligned}
$$

From Lemmas 2.3 and 2.4 and (3.10), we obtain

(3.11)

$$
\begin{aligned}
& \left(\frac{1}{2}\right)^{2-2\mu} \frac{\Gamma(2-2\mu)}{2^{2\mu-2}\Gamma(3-2\mu)}\, {}_2F_1\left(\mu, 2\mu-2; 2\mu-1; \frac{1+x_i}{2}\right) d_1 \\
& \quad + \left(\frac{1}{2}\right)^{2-2\mu} \frac{1}{\Gamma(\mu)} \left(\frac{\Gamma(2-\mu)\Gamma(\mu)\Gamma(1-\mu)}{2^{2\mu-2}\Gamma(3-2\mu)}\, {}_2F_1\left(\mu, 2\mu-2; \mu; \frac{1+x_i}{2}\right) \right. \\
& \qquad \left. - \frac{\Gamma(\mu-1)}{2^{\mu-1}(1+x_i)^{\mu-1}}\, {}_2F_1\left(1, \mu-1; 2-\mu; \frac{1+x_i}{2}\right) \right) d_2 \\
& \quad + \sum_{j=0}^{p} c_j \left[\left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1, 1-\mu)}(x_i) \right. \\
& \qquad \left. + \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu, \mu-1)}(x_i) \right] \\
& = \lambda_p \left(\sum_{j=0}^{p} c_j L_j(x_i) + d_1 \left(\frac{1+x_i}{2}\right)^{1-\mu} + d_2 \left(\frac{1-x_i}{2}\right)^{1-\mu} \right), \\
& \qquad\qquad i = 0, \dots, p+2.
\end{aligned}
$$

Note that the first hypergeometric function is not well defined when $\mu = 1/2$. However, the integration of the two singular terms with the kernel are simpler, in which case, the linear system is

(3.12)

$$\begin{aligned}&\Bigg(\frac{\pi(1+x_i)}{4}+\sqrt{\frac{1-x_i}{2}}+\frac{1+x_i}{2}\\&\qquad\qquad\times\log\left(1+\sqrt{\frac{1-x_i}{2}}\right)-\frac{1+x_i}{4}\log\left(\frac{1+x_i}{2}\right)\Bigg)d_1\\&+\left(\sqrt{\frac{1+x_i}{2}}-\frac{x_i-1}{2}\tanh^{-1}\left(\sqrt{\frac{1+x_i}{2}}\right)-\frac{\pi(x_i-1)}{4}\right)d_2\\&\quad+\sum_{j=0}^{p}c_j\Bigg[\left(\frac{1}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i)\\&\qquad\qquad+\left(\frac{1}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1-x_i)^{1-\mu}P_j^{(1-\mu,\mu-1)}(x_i)\Bigg]\\&=\lambda_p\left(d_1\sqrt{\frac{1+x_i}{2}}+d_2\sqrt{\frac{1-x_i}{2}}+\sum_{j=0}^{p}c_jL_j(x_i)\right),\quad i=0,\ldots,p+2.\end{aligned}$$

3.3. Algorithm for (3.2). We make the change of variable as before and let $u(x)=y((1+x)/2)$. We obtain

$$\lambda u(x)=\int_{-1}^{1}\frac{1+\tau}{8}u(\tau)\,d\tau+\frac{1}{4}\int_{x}^{1}(x-\tau)u(\tau)\,d\tau. \tag{3.13}$$

Let $u_p(x)=\sum_{j=0}^{p}c_jL_j(x)$ be the approximation of $u(x)$. Then the c_j's satisfy

(3.14)

$$\lambda_p\sum_{j=0}^{p}c_jL_j(x_i)=\sum_{j=0}^{p}c_j\int_{-1}^{1}\frac{1+\tau}{8}L_j(\tau)\,d\tau+\sum_{j=0}^{p}c_j\int_{x_i}^{1}(x_i-\tau)L_j(\tau)\,d\tau,$$
$$i=0,\ldots,p,$$

i.e.,

$$\begin{aligned}\lambda_p \sum_{j=0}^{p} c_j L_j(x_i) &= \left(\frac{c_0}{4} + \frac{c_1}{12}\right) \\ &\quad + \frac{1-x_i}{8} \sum_{j=0}^{p} c_j \sum_{k=0}^{p} w_k (x_i - x_k) L_j\left(\frac{1+x_i}{2} + \frac{1-x_i}{2} x_k\right), \\ &\qquad i = 0, \dots, p.\end{aligned} \tag{3.15}$$

Here, the numerical integration is exact. The scheme is of the form

$$AC_p = \lambda_p BC_p,$$

where

$$b_{ij} = L_j(x_i)$$

$$a_{ij} = \begin{cases} (1-x_i)/8 \sum_{k=0}^{p} w_k(x_i - x_k) L_j((1+x_i)/2 + (1-x_i)/2x_k) \\ \hfill \text{if } j \neq 0, 1, \\ (1-x_i)/8 \sum_{k=0}^{p} w_k(x_i - x_k) L_j((1+x_i)/2 + (1-x_i)/2x_k) + 1/4 \\ \hfill \text{if } j = 0; \\ (1-x_i)/8 \sum_{k=0}^{p} w_k(x_i - x_k) L_j((1+x_i)/2 + (1-x_i)/2x_k) + 1/12 \\ \hfill \text{if } j = 1. \end{cases}$$

3.4. Algorithm for (3.3). Substituting the Legendre expansion $y(t) = \sum_{j=0}^{p} y_j L_j(t)$ into (3.3) and collocating at n Gaussian points, we have

$$\sum_{j=0}^{p} y_j \int_0^1 e^{st_i} L_j(s)\, ds = \lambda \sum_{j=0}^{p} y_j L_j(t_i), \quad i = 1, 2, \dots, n. \tag{3.16}$$

The matrix form of (3.16) is

$$K\mathbf{y} = \lambda L\mathbf{y}, \tag{3.17}$$

where

$$K_{ij} = \int_0^1 e^{st_i} L_j(s)\, ds, \quad L_{ij} = L_j(t_i), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n)^T. \tag{3.18}$$

K_{ij} can be calculated by the n-point Gaussian quadrature

$$\int_0^1 e^{st_i} L_j(S)\, ds \approx \sum_{l=0}^{p} e^{s_l t_i} L_j(s_l) w_l, \quad s_k = t_k. \tag{3.19}$$

4. Convergence analysis. Let L_k be the standard Legendre polynomial of degree k, and let $\pi_p f \in P_p[-1,1]$ interpolate a smooth function f at $(p+1)$-Gauss points: $-1 < x_0 < \cdots < x_p < 1$. Let T_k be the first kind Chebyshev polynomial of degree k. Then the remainder of the interpolation is

$$f(x) - \pi_p f(x) = f[x_0, x_1, \ldots, x_p, x]\nu(x), \tag{4.1}$$

where $\nu(x) = (x - x_0)(x - x_1)\cdots(x - x_p)$.

Note that

$$\begin{aligned} L_n(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{j=0}^{n} \binom{n}{j} x^{2(n-j)}(-1)^j \\ &= \frac{1}{2^n n!} \sum_{j=0}^{n} \binom{n}{j} (2n-2j)(2n-2j-1)\cdots(2n-2j-n+1)x^{n-2j}(-1)^j. \end{aligned} \tag{4.2}$$

From the term with $j = 0$, we get the leading coefficient

$$\frac{1}{2^n n!}\binom{n}{0}(2n)(2n-1)\cdots(2n-n+1)(-1)^0 = \frac{(2n)!}{2^n (n!)^2} \tag{4.3}$$

By the Stirling formula,

$$\frac{(2n)!}{2^n (n!)^2} \approx \frac{(2n/e)^{2n}\sqrt{4\pi n}}{2^n[(n/e)^n\sqrt{2\pi n}]^2} = 2^n. \tag{4.4}$$

Hence,

$$f(x) - \pi_p f(x) \approx \frac{f[x_0, x_1, \dots, x_p, x]}{2^{p+1}} L_{p+1}(x). \tag{4.5}$$

If $f \in C^{p+1}[-1, 1]$, the divided difference

$$f[x_0, x_1, \dots, x_p, x] = \frac{f^{(p+1)}(\xi_x)}{(p+1)!}, \qquad \xi_x \in (-1, 1). \tag{4.6}$$

The result can be concluded as the following theory.

Theorem 4.1. (1) *If* $y(t)$ *satisfies condition* (K): $\|y^{(k)}\|_{L^\infty[0,1]} \le Ck!R^{-k}$, *then*

$$\|y - \pi_p y\|_{L^\infty[0,1]} \le \frac{C}{(4R)^{p+1}}; \tag{4.7}$$

(2) *If* $y(t)$ *satisfies condition* (M): $\|y^{(k)}\|_{L^\infty[0,1]} \le CM^k$, *we have*

$$\|y - \pi_p y\|_{L^\infty[0,1]} \le \frac{C}{\sqrt{p+1}} \left(\frac{eM}{4(p+1)} \right)^{p+1}. \tag{4.8}$$

Proof. We make the change of variables

$$t = \frac{1+x}{2}, \qquad s = \frac{1+\tau}{2}, \quad x, \tau \in [-1, 1],$$

and let $u(x) = y((1+x)/2)$. Then the result for y under condition (K) follows directly from (4.6) and the fact that $dt = (1/2)dx$.

If y satisfies condition (M), by applying the Stirling's formula,

$$\begin{aligned} \|y - \pi_p y\|_{L^\infty[0,1]} &= \|u - \pi_p u\|_{L^\infty[-1,1]} \\ &\le \frac{CM^{p+1}}{(4R)^{p+1}(p+1)!} \approx \frac{cM^{p+1}}{\sqrt{2\pi(p+1)}(4(p+1)/e)^{p+1}} \\ &= \frac{C}{\sqrt{p+1}} \left(\frac{eM}{4(p+1)} \right)^{p+1}. \end{aligned} \tag{4.9}$$

For non-smooth functions, we need some other estimates.

Theorem 4.2 [**8**]. (1) *For any* $f \in H^k(-1,1)$,

$$\|f - \pi_p f\|_{L^2(-1,1)} \leq Cp^{-k}|f|_{H^{k,p}(-1,1)}. \tag{4.10}$$

(2) *For any* $f \in H^k_w(-1,1)$,

$$\|f - \pi_p^c f\|_{L^2(-1,1)} \leq Cp^{-k}|f|_{H^{k,p}_w(-1,1)}, \tag{4.11}$$

where two seminorms are defined by

$$|f|_{H^{k,p}(-1,1)} = \left(\sum_{s=\min(k,p+1)}^{k} \|f^{(s)}\|^2_{L^2(-1,1)} \right)^{1/2},$$
$$|f|_{H^{k,p}_w(-1,1)} = \left(\sum_{s=\min(k,p+1)}^{k} \|f^{(s)}\|^2_{L^2_w(-1,1)} \right)^{1/2},$$

and the weight $w(x) = (1-x)^{-1/2}(1+x)^{-1/2}$ *and* π_p^c *is the interpolatory operator on Chebyshev points.*

Let $R(E)$ and $R(E_p)$ be the range of E and E_p, respectively. Define $\pi_p : R(E) \to R(E_p)$ as an interpolatory projection by $\pi_p(t) = \sum_{j=0}^{p} \xi_j L_j(t)$ and ξ_j is determined by

$$\sum_{j=0}^{p} \xi_j L_j(t_i) = x(t_i), \quad i = 0, \dots, p.$$

Then our algorithms can be written as

$$T_p u_p = \lambda_p u_p, \quad \text{where } T_p = \pi_p T. \tag{4.12}$$

Theorem 4.3. *Let* y *be the exact first eigenvector and* T *a compact operator in* (3.1), (3.2) *or* (3.3) *and* T_p *defined as above. Then*

(1) *If* $y \in H^k(0,1)$,

$$|\lambda - \widehat{\lambda}_p| \le \frac{C}{(2p)^k}; \tag{4.13}$$

(2) *Furthermore, if* y *satisfies condition* (K),

$$|\lambda - \widehat{\lambda}_p| \le \frac{C}{(4R)^{p+1}}; \tag{4.14}$$

(3) *Furthermore, if* y *satisfies condition* (M),

$$|\lambda - \widehat{\lambda}_p| \le \frac{C}{\sqrt{p+1}}\left(\frac{eM}{4(p+1)}\right)^{p+1}. \tag{4.15}$$

Proof. The result follows directly from Theorems 2.1, 4.1, 4.2 and [**16**, Theorem 2.2]. To make the paper self contained, we put the proof here.

Let $\widehat{E}_p = E_p|_{R(E)} : R(E) \to R(E_p)$. Then, for large p, $\widehat{E_p}$ is bijective and $\|\widehat{E}_p^{-1}\| \le 2$ [**17**]. Define $\widehat{T} = T_{R(E)}$, and $\widehat{T}_p := \widehat{E}_p^{-1} T_p \widehat{E}_p$. Then

$$\begin{aligned} |\lambda - \widehat{\lambda}_p| &= \frac{1}{m}|\mathrm{trace}\,(\widehat{T} - \widehat{T}_p)| \le \|\widehat{T} - \widehat{T}_p\| \\ &= \|\widehat{E}_n^{-1}(\widehat{E}_p T - \widehat{E} T_p)\| \le C\|(T - T_p)|_{R(E)}\| \\ &= C\|(I - \pi_p)T|_{R(E)}\|. \end{aligned} \tag{4.16}$$

Since Tu is smoother than u, see [**10, 15**]; then the result follows.

Theorem 4.4. *Let* λ *and* λ_p *be the exact eigenvalue and its numerical approximation of a positive definite operator* T *whose kernel is a piecewise smooth function, respectively. Then*

(1) *if* u *satisfies condition* (K),

$$|\lambda - \lambda_p| \le C\left(\frac{1}{(4R)^{2p+2}} + \frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right); \tag{4.17}$$

(2) *if* u *satisfies condition* (M),

$$|\lambda - \lambda_p| \le C\left(\frac{1}{p+1}\left(\frac{eM}{4(p+1)}\right)^{2p+2} + \frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right); \tag{4.18}$$

(3) *if* $u \in H^k[0,1]$,

$$|\lambda - \lambda_p| \le C\left(\frac{1}{(2p)^{2k}} + \frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right), \tag{4.19}$$

Proof. By our algorithms, we have

$$\int_0^1 k(t_i, s)u_p(s)\, ds = \lambda_p u_p(t_i), \tag{4.20}$$

where t_i are $(p+1)$-Gauss points on [0,1].

Multiplying both sides by $L_j(t_i)w_i$ and summing up from 0 to p, we obtain

$$\sum_{j=0}^{p}\int_0^1 k(t_i, s)u_p(s)L_j(t_i)w_i\, ds = \lambda_p \sum_{j=0}^{p} u_p(t_i)L_j(t_i)w_i. \tag{4.21}$$

Here, w_i are weights of the Gauss quadrature.

If we write $\widetilde{A} = (\int_0^1\int_0^1 k(t,s)L_j(s)L_i(t)\, dsdt)_{ij}$, $\widetilde{B} = (\int_0^1 L_j(t)L_i(t)\, dt)_{ij}$ and recall that $u_p(x) = \sum_{i=0}^{p} \widetilde{u}_i L_i(x)$, we obtain

$$\widetilde{A}\widetilde{u} = \widetilde{\lambda}_p \widetilde{B}\widetilde{u},$$

where $\widetilde{u} = [\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_p]^T$.

However, for most cases, we can only apply numerical quadrature to find elements of $\widetilde{A}$ and $\widetilde{B}$. If the kernel is piecewise smooth, we apply the Gauss quadrature piece by piece. Therefore, the system that we actually solve is

$$Au = \lambda_p Bu. \tag{4.22}$$

Now we are ready to analyze errors of eigenvalue approximations.

First, we analyze the case when the kernel is a linear piecewise polynomial. Noting the fact that $(p+1)$-Gauss quadrature is exact for all polynomials of degree less than or equal to $2p+1$, we have

$$A = \widetilde{A}, \quad \text{and} \quad B = \widetilde{B}. \tag{4.23}$$

Here, the integration is piecewise, so is the numerical integration.

Denote the arithmetic mean of the approximation of λ by λ_p again, if it is a multiple eigenvalue. We derive from Lemma 2.2 that

$$\begin{aligned} |\lambda - \lambda_p| &= |\lambda - \tilde{\lambda}_p| \\ &\le C \begin{cases} 1/4^{2p+2} & \text{if } u \text{ satisfies condition (K);} \\ 1/(p+1)(eM/(4(p+1)))^{2p+2} & \text{if } u \text{ satisfies condition (M);} \\ 1/(2p)^{2k} & \text{if } u \in H^k[0,1]. \end{cases} \end{aligned} \tag{4.24}$$

If the kernel is piecewise smooth or smooth, from the analysis of the previous case, we only need to estimate $A - \widetilde{A}$ since $B = \widetilde{B}$. If we write the remainder of Gaussian quadrature as ε, then

$$A - \widetilde{A} \le C\varepsilon \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{4.25}$$

Here, we define $E < F$ if and only if $|(E)_{ij}| < |(F)_{ij}|$.

By the error estimate of the Gauss quadrature and Stirling's formula, we have

$$\begin{aligned} \varepsilon &\le C\left(\frac{[p!]^4}{(2p+1)[(2p)!]^3}\right) \\ &\approx C\left(\frac{[\sqrt{2\pi p}(p/e)^p]^4}{(2p+1)[\sqrt{2\pi(2p)}(2p/e)^{2p}]^3}\right) \\ &\le C\left(\frac{e^{2p}}{p^{2p+1/2}2^{6p}}\right). \end{aligned} \tag{4.26}$$

Hence,

$$\|A - \widetilde{A}\|_n \le C\left(\frac{pe^{2p}}{p^{2p+1/2}2^{6p}}\right), \quad n = 1, \infty. \tag{4.27}$$

Clearly, (4.22) is equivalent to

$$B^{-1}Au = \lambda_p u. \tag{4.28}$$

Thus,

$$\begin{aligned} \|\widetilde{B}^{-1}\widetilde{A} - B^{-1}A\|_n &\le \|B^{-1}\|_n \|\widetilde{A} - A\|_n \\ &\le C\bigg(\frac{p^2 e^{2p}}{p^{2p+1/2}2^{6p}}\bigg), \quad n = 1, \infty, \end{aligned} \tag{4.29}$$

by noting that $B = \widetilde{B} = \operatorname{diag}(1, 1/3, \ldots, 1/(2p+1))$. Therefore, by a perturbation theory, see [**9**, page 30], we have

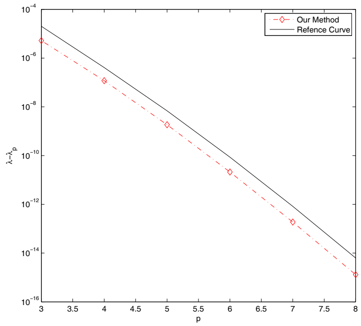
$$|\lambda_p - \widetilde{\lambda}_p| \le C\bigg(\frac{e^{2p}}{p^{2p-3/2}2^{6p}}\bigg). \tag{4.30}$$

Denote the arithmetic mean of the approximation of λ by λ_p again, if it is a multiple eigenvalue. We derive from Lemma 2.2 and (4.30) that

$$\begin{aligned} |\lambda - \lambda_p| &\le |\lambda - \widetilde{\lambda}_p| + |\widetilde{\lambda}_p - \lambda_p| \\ &\le C \begin{cases} ((1/4^{2p+2}) + (e^{2p}/(p^{2p-1/2}2^{6p}))) \\ \qquad \text{if } u \text{ satisfies condition (K);} \\ (1/(p+1)(eM/(4(p+1)))^{2p+2} + (e^{2p}/(p^{2p-1/2}2^{6p}))) \\ \qquad \text{if } u \text{ satisfies condition (M);} \\ ((1/(2p)^{2k}) + (e^{2p}/(p^{2p-1/2}2^{6p}))) \\ \qquad \text{if } u \in H^k[0,1]. \end{cases} \end{aligned}$$

Remark 2. Theorem 4.4 shows that, although numerical integration contributes to the error of eigenvalue approximation, it is trivial compared with truncation error for our method. Hence, in our numerical experiments, we ignore it for reference curves.

5. Numerical examples. In this section, we will find numerical approximations to solutions of some examples to demonstrate our theory.



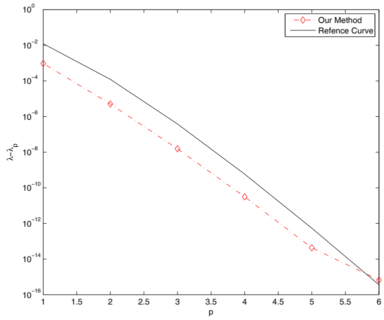


FIGURE 1. (Left) Linear piecewise kernel. (Right) Kernel: e^{st}.

TABLE 1. Example 5.1: $\lambda - \lambda_p$.

p	3	4	5	6
error	5.2298e-06	1.1826e-07	1.8490e-09	2.1241e-11
p	7	8		
error	1.8759e-13	1.3045e-15		

TABLE 2. Example 5.2: λ_p (The first algorithm).

p	10	20	30	40
	1.805741190980	1.805772959954	1.805776000579	1.805776693513
p	50	55	60	65
	1.805776926254	1.805776984244	1.805777024037	1.805777051585

Example 5.1 [**1**]. We consider a problem with form (3.2). Then each $\lambda_j = 1/((2j-1)^2\pi^2)$, $j = 1, 2, \dots,$ is an eigenvalue of T of algebraic multiplicity $m = 2$. Let $\widehat{\lambda}$ denote the arithmetic mean of the two eigenvalues of T_p to the largest two eigenvalues $\lambda = 1/\pi^2$. Numerical errors are presented in Table 1 and the left part of Figure 1, from which we see that the error decays super-geometrically. Here *Reference Curve* is the graph of

$$f(p) = \frac{1}{100(p+1)}\left(\frac{e\pi}{4(p+1)}\right)^{2p+2}.$$

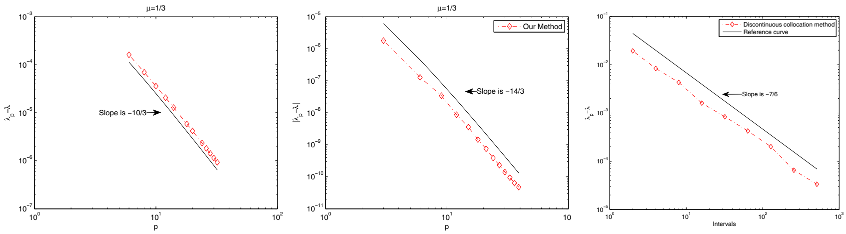


FIGURE 2. Kernel: $|t-s|^{1/3}$. The first algorithm (Left), the second algorithm (Middle) and the discontinuous linear elements method (Right).

TABLE 3. Example 5.2: λ_p (The second algorithm).

p	10	20	30	40
	1.805777162409	1.805777143959	1.80577714315	1.805777143861
p	50	55	60	65
	1.805777143840	1.805777143837	1.805777143834	1.805777143833

Example 5.2. Now let us consider an eigenproblem of form (3.1) with $\mu = 1/3$. From [**20**], eigenfunctions belong to $H^{(7/6)-\varepsilon}(0,1)$, where ε is a sufficiently small positive number and we expect to obtain a convergence rate of $\mathcal{O}(p^{-7/3})$ based on Theorem 4.3 for the first algorithm. Here, we apply both our spectral collocation methods and the three-point Gaussian collocation on equally spaced intervals with discontinuous piecewise quadratic elements method. Unfortunately, we do not know the exact eigenvalues for such types of kernels. However, we list some of our numerical approximations in Table 2, and we use the numerical approximation of the second algorithm for $p = 70$ as our "exact" value to obtain Figure 2. It is easy to see that we can only obtain a 7 digit accuracy for the first algorithm; we obtain an 11 digit of accuracy and a convergence rate of $\mathcal{O}(p^{-14/3})$ for the second algorithm, see Table 3. However, the convergence rate for the three-point Gaussian collocation with discontinuous piecewise quadratic element is only $\mathcal{O}(h^{7/6})$. This fact also confirms results in [**6**], which says that the convergence rate for p-version methods doubles the convergence rate for the h-version method if the true solution is singular.

Example 5.3. We consider an eigenproblem of form (3.1) with $\mu = 1/2$. In this case, eigenfunctions belong to $H^1(0,1)$. Again, we use both algorithms to solve it and consider the numerical approximation of the second algorithm for $p = 70$ as the "exact" first eigenvalue. Numerical results are shown in Table 4, Table 5 and Figure 3.

Example 5.4. Consider the eigenvalue problem of the form (3.3). We apply the algorithm in Section 3. Since the kernel is smooth, the first eigenvalue converges very fast, see Table 6 and the right part of Figure 1. In this case, Reference Curve is the graph of $f(p) = 1/(10(p+1))(e/(2(p+1)))^{2p+2}$.

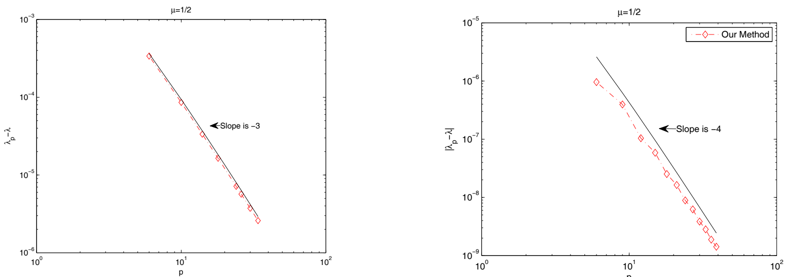


FIGURE 3. (Left) Kernel: $|t-s|^{1/2}$ (The first algorithm). (Right) Kernel: $|t-s|^{1/2}$ (The second algorithm).

TABLE 4. Example 5.3: λ_p (The first algorithm).

p	10	20	30	40
	2.682832453413	2.682906259562	2.682914644141	2.682916773709
p	50	55	60	65
	2.682917548869	2.682917752298	2.682917896142	2.682917998252

TABLE 5. Example 5.3: λ_p (The second algorithm).

p	10	20	30	40
	2.682918574502	2.682918399599	2.682918386165	2.682918383536
p	50	55	60	65
	2.682918382751	2.682918382579	2.682918382452	2.682918382377

TABLE 6. Example 5.4: $\lambda - \lambda_p$.

p	1	2	3	4
error	9.4969e-04	5.0595e-06	1.5456e-08	3.1190e-11
p	5	6		
error	4.3077e-14	6.6613e-16		

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