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BLOW-UP BEHAVIOR OF HAMMERSTEIN-TYPE VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we consider the blow-up behavior of Hammerstein-type Volterra integral equations. Based on several fundamental assumptions, some necessary and sufficient conditions under which the solution blows up in finite time are given. Some examples illustrate that there may always exist a global solution for a power-law function and that the blow-up behavior only depends upon the value of the kernel in a neighborhood of zero. As an application, we give some results on the blow-up behavior of Volterra integro-differential equations of Hammerstein-type.

1. Introduction. In this paper, we investigate the blow-up behaviors of solutions of Hammerstein-type Volterra equations

$$u(t)=\phi(t)+\int_0^t k(t-s)G(s,u(s))\,\mathrm{d}s, \tag{1.1}$$

where $\phi:[0,\infty)\to[0,\infty)$ and $G:[0,\infty)\times\mathbf{R}\to[0,\infty)$ are continuous functions, the kernel $k:(0,\infty)\to[0,\infty)$ is a locally integrable function and u is an unknown (continuous) solution. We note that results on local existence and uniqueness of solutions for (1.1) may be found in [**16**].

In recent years, the literature includes extensive blow-up results of ordinary differential equations, partial differential equations and

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equations with nonlinear memory (see [**1, 5, 7, 9, 10, 16, 26**] and the references therein). Volterra integral equations arise from nonlinear partial differential equation models of explosive behavior in a reactive-diffusive medium in [**28, 30**]. The blow-up behavior of some special cases of (1.1) such as

$$u(t) = \phi(t) + \int_0^t k(t-s)g(u(s))\,\mathrm{d}s, \tag{1.2}$$

in [**8**] and

$$u(t) = \phi(t) + \int_0^t k(t-s)r(s)g(u(s))\,\mathrm{d}s, \tag{1.3}$$

in [**18**], has been investigated, where $g(u)$ is a power-law function or an exponential function and $r(s)$ is an increasing function.

The dynamical behavior of solution $u(t)$ of (1.1) will be of the following types

(i) the maximum existence interval is $[0,\infty)$ and the solution may be bounded or unbounded,

(ii) the maximum existence interval is $[0,T_b)$ and the solution is unbounded.

Definition 1.1. We say that the solution of (1.1) blows up in finite time if a finite $T_b > 0$ exists such that $\lim_{t\to T_b-} |u(t)| = \infty$.

In Section 2, some necessary and sufficient conditions of blow-up behavior are investigated. In Section 3, some examples are given to illustrate how the known functions in (1.1) influence both global and blow-up solutions. At last, the blow-up results are applied to Volterra integro-differential equations of Hammerstein-type.

2. Blow-up behaviors.

2.1. Fundamental assumptions and properties. In this section, according to practical models in [**8, 25, 29**], we give some fundamental assumptions. Assume that $G(s,u)$ is continuous with respect to (s,u) and uniformly continuous of u with respect to s belonging to any finite interval and satisfies:

(G1) $G(s,0) \equiv 0$ and $G(s_2,u_2) > G(s_1,u_1)$ for two positive vectors $(s_1,u_1),(s_2,u_2)$ with $(s_2,u_2) \geqslant (s_1,u_1)$ in component and $u_2 \neq u_1$,

(G2) $\lim_{u\to\infty}[G(0,u)]/u = \infty$.

Also assume that

(P) the function $\phi(t)$ is a positive, non-decreasing, continuous function,

(K) the kernel $k(z)$ is a locally integrable function and $K(t) = \int_0^t k(z)\,\mathrm{d}z > 0$ is a non-decreasing function.

Remark 2.1. Conditions (P) and (K) come from physical models (see [**8, 25, 27–30**]).

Remark 2.2. If the function $\phi(t)$ satisfies Condition (P) and a positive function $G(s,u)$ satisfies conditions (G1) and (G2) only but $G(s,0) \not\equiv 0$, then $G_1(s,u) := G(s,u) - G(s,0)$ satisfies conditions (G1) and (G2), and $\phi_1(t) := \phi(t) + \int_0^t k(t-s)G(s,0)\,\mathrm{d}s$ satisfies Condition (P).

Let us consider the case, i.e., $G(s,u) = r(s)g(u)$. Condition (G1) means that $r(s)$ is non-decreasing and $g(u)$ is increasing, which has been required in [**13, 18, 24, 27, 28, 30**]. Condition (G2) means that $\lim_{u\to\infty}[g(u)]/u = \infty$, which, in fact, is a necessary condition under which the solution of (1.1) blows up in finite time (see [**2**] and Example 4.3 in [**29**] in detail).

Under these fundamental assumptions, by a comparison theorem, we will show that the unique solution of (1.1) is positive and non-decreasing.

Lemma 2.3. *Assume that conditions* (P), (K) *and* (G1) *hold. Then* $u(t) > \phi(0)$ *for all* $t > 0$.

Lemma 2.4. *Assume that conditions* (P), (K) *and* (G1) *hold. If* $v(t)$ *is a continuous nonnegative solution of*

$$v(t) \leqslant \phi(t) + \int_0^t k(t-s)G(s,v(s))\,\mathrm{d}s \quad \text{for } t \in [0,T],$$

then $u(t) \geqslant v(t)$ *for* $t \in [0,T]$.

Proof. For any $c > 0$ we define

$$v_c(t) := \begin{cases} v(0) & t \in [0, c], \\ v(t-c) & t \in (c, T]. \end{cases}$$

Similarly, we define $\phi_c(t)$. Hence,

$$v_c(t) \leqslant \phi_c(t) + \int_0^t k(t-s)G(s, v_c(s))\,\mathrm{d}s.$$

It follows from Lemma 2.3 that $u(t) > v_c(t)$ for $t \in (0, c]$. We suppose that $u(t) = v_c(t)$ and $0 \leqslant v_c(s) < u(s)$ for $s \in (0, t)$. Then conditions (K) and (G1) imply that

$$\begin{aligned} 0 &= u(t) - v_c(t) \\ &= \phi(t) - \phi_c(t) \\ &\quad + \int_0^t k(t-s)(G(s, u(s)) - G(s, v_c(s)))\,\mathrm{d}s \\ &\geqslant \int_0^t k(t-s)(G(s, u(s)) - G(s, v_c(s)))\,\mathrm{d}s > 0, \end{aligned}$$

which is a contradiction. As a result, the proof is complete. □

Corollary 2.5. *Assume that conditions* (P), (K) *and* (G1) *hold and* $u^\varepsilon(t)$ *is a non-continuable solution of*

$$u^\varepsilon(t) = \phi(t) + \int_0^t k^\varepsilon(t-s)G(s, u^\varepsilon(s))\,\mathrm{d}s \quad \text{for } t \in [0, T),$$

where $k^\varepsilon(z) \geqslant k(z)$ *for all* $z \in [0, \infty)$ *satisfies Condition* (K). *Then* $u(t) \leqslant u^\varepsilon(t)$ *for* $t \in [0, T)$.

Theorem 2.6. *Assume that conditions* (P), (K) *and* (G1) *hold. Then* $u(t) \geqslant \phi(0)$ *is also non-decreasing.*

Proof. Let $\overline{u}(t) := \max_{s \in [0,t]} u(s)$. Then $\overline{u}(t)$ is a continuous, non-decreasing function satisfying

$$u(t) \leqslant \overline{u}(t) \quad \text{for all } t > 0.$$

Therefore, it follows from conditions (K) and (G1) that, for all $0 < s \leqslant t$,

$$u(s) \leqslant \phi(t) + \int_0^t k(t-r)G(r, \overline{u}(r))\,\mathrm{d}r,$$

which implies that

$$\overline{u}(t) \leqslant \phi(t) + \int_0^t k(t-r)G(r, \overline{u}(r))\,\mathrm{d}r.$$

Hence, using Lemma 2.4 we conclude that $\overline{u}(t) \leqslant u(t)$, and the proof is complete. $\square$

2.2. Sufficient conditions. In this subsection, we will give some sufficient conditions under which the solution of (1.1) blows up in finite time.

Lemma 2.7. *Assume that conditions* (K), (G1) *and* (G2) *hold. Define* $F(t,u) := \int_0^t k(t-s)G(s,u)\,\mathrm{d}s - u$. *Then, for any given* $t \in (0,\infty)$, *the minimum value* $F_{\min}(t) := \min_{u\in[0,\infty)} F(t,u)$ *exists and* $\lim_{t\to 0} F_{\min}(t) = -\infty$. *Moreover,* $F_{\min}(t) \leqslant 0$ *is a non-decreasing function.*

Proof. For any given $t \in (0,\infty)$, in view of conditions (K) and (G1),

$$\frac{F(t,u)}{u} + 1 = \int_0^t k(t-s)\frac{G(s,u)}{u}\,\mathrm{d}s \geqslant \frac{G(0,u)}{u}K(t).$$

Hence, Condition (G2) yields $\lim_{u\to\infty} F(t,u) = \infty$, which implies that the minimum value of $F(t,u)$ exists, i.e., $F_{\min}(t)$ is well-defined. Since $\lim_{t\to 0+} \int_0^t k(t-s)G(s,u)\,\mathrm{d}s = 0$ for any given $u > 0$, a sufficiently small $t > 0$ exists such that $F(t,u) < -(1/2)u$, which implies that $\lim_{t\to 0} F_{\min}(t) = -\infty$.

By conditions (K) and (G1), $F_{\min}(t) \leqslant F(t,0) = 0$ and, for any given $t_2 > t_1 > 0$,

$$\begin{aligned} F(t_2,u) &= \int_0^{t_2} k(s)G(t_2-s,u)\,\mathrm{d}s - u \\ &\geqslant \int_0^{t_1} k(s)G(t_1-s,u)\,\mathrm{d}s - u = F(t_1,u). \end{aligned}$$

Hence, $F_{\min}(t_1) \leqslant F(t_2, u)$ for all $u \in [0, \infty)$, which implies that $F_{\min}(t)$ is a non-decreasing function. The proof is complete. $\square$

Example 2.8. Consider the case of $G(s,u) = r(s)u^p$ with $p > 1$. Then $F(t,u) = I(t)u^p - u$, and its minimum value

$$F_{\min}(t) = \left(\frac{1}{pI(t)}\right)^{1/(p-1)} \frac{1-p}{p} < 0$$

obtains at the minimum value point $(1/(pI(t)))^{1/(p-1)}$, where $I(t) = \int_0^t k(t-s)r(s)\,\mathrm{d}s$. For example, $k(z) \equiv 1$ and $G(s,u) = su^p$ with $p > 1$, $F(t,u) = (t^2/2)u^p - u$ and

$$F_{\min}(t) = \left(\frac{2}{pt^2}\right)^{1/(p-1)} \frac{1-p}{p} < 0$$

obtains at the minimum value point $(2/(pt^2))^{1/(p-1)}$.

Theorem 2.9. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold, and that the solution* $u(t)$ *of* (1.1) *exists globally in the interval* $[0, \infty)$*. Then* $\lim_{t\to\infty} u(t) = \infty$*, provided that a* $t^* \in (0, \infty)$ *exists such that*

$$\phi(t^*) + F_{\min}(t^*) > 0. \tag{2.1}$$

Proof. Otherwise, by Theorem 2.6, $\lim_{t\to\infty} u(t) = u_\infty \in (\phi(t^*), \infty)$ exists. Thus, for any given $0 < \varepsilon < u_\infty$ with $\phi(t^*) - \varepsilon + F_{\min}(t^*) > 0$, a $T_\varepsilon > t^*$ exists such that $u_\infty - \varepsilon < u(t) < u_\infty$ for all $t > T_\varepsilon$. Thus, it follows from conditions (K) and (G1) that, for all $t > T_\varepsilon + t^*$,

$$\begin{aligned} u_\infty > u(t) &= \phi(t) + \int_0^t k(t-s)G(s,u(s))\,\mathrm{d}s \\ &\geqslant \phi(t) + \int_0^{T_\varepsilon} k(t-s)G(s,u(s))\,\mathrm{d}s \\ &\quad + \int_{T_\varepsilon}^t k(t-s)G(s,u_\infty-\varepsilon)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\geqslant \phi(t) + \int_0^{t-T_\varepsilon} k(s)G(t-s, u_\infty - \varepsilon)\,\mathrm{d}s \\
&\geqslant \phi(t) + \int_0^{t-T_\varepsilon} k(s)G(t-T_\varepsilon - s, u_\infty - \varepsilon)\,\mathrm{d}s.
\end{aligned}
$$

Therefore, Lemma 2.7 implies that

$$\phi(t) - \varepsilon + F(t - T_\varepsilon, u_\infty - \varepsilon) < 0.$$

This contradicts $\phi(t^*) - \varepsilon + F_{\min}(t^*) > 0$, and the proof is complete. □

Corollary 2.10. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold, and that the solution $u(t)$ of* (1.1) *exists globally in the interval* $[0, \infty)$. *Then* $\lim_{t\to\infty} u(t) = \infty$ *if* $\lim_{t\to\infty} \phi(t) = \infty$ *or*

$$\lim_{t\to\infty} \int_0^t k(t-s)G(s,u)\,\mathrm{d}s = \infty \quad \textit{for all } u \in (0,\infty). \tag{2.2}$$

Proof. In fact, from Theorem 2.9, we only need to show that $\lim_{t\to\infty} F_{\min}(t) = 0$ under Condition (2.2). For $t > 0$, let $u_*(t) := \sup\{u : F(t,u) \leqslant 0\}$. Then it follows from Condition (G2) that $u_*(t) \in [0, \infty)$.

We claim that $\liminf_{t\in(0,\infty)} u_*(t) = 0$ under (2.2). Otherwise, there exist a constant $\delta > 0$ and a sequence $t_n \to \infty$ as $n \to \infty$ such that $\delta \leqslant u_*(t_n) \leqslant 2\delta$. On the other hand, in view of Condition (G1), ones obtain

$$1 = \frac{\int_0^{t_n} k(t_n - s)G(s, u_*(t_n))\,\mathrm{d}s}{u_*(t_n)} \geqslant \frac{\int_0^{t_n} k(t_n - s)G(s,\delta)\,\mathrm{d}s}{2\delta}.$$

This contradicts (2.2) which, together with $F_{\min}(t) \geqslant -u_*(t)$, implies that the proof is complete. □

In the following, we will estimate the growing-up time of $u(t)$.

Lemma 2.11. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold, and that the solution $u(t)$ of* (1.1) *exists globally in the interval* $[0, \infty)$.

If a t^* exists such that (2.1)*, then, for any given $R > 1$, a sequence t_n exists such that $u(t_n) = R^n$ for all $n \geqslant \max\{1, [\log(\phi(0))/(\log R)]\}$, $\lim_{n\to\infty} t_n = \infty$ and $t_{n+1} - t_n$ tends to zero as $n \to \infty$.*

Proof. It follows from Theorems 2.6 and 2.9 that a sequence t_n exists such that $u(t_n) = R^n$ and $\lim_{n\to\infty} t_n = \infty$. Moreover, conditions (P), (K) and (G1) imply that

$$\begin{aligned} u(t_{n+1}) &= \phi(t_{n+1}) + \int_0^{t_{n+1}} k(t_{n+1} - s)G(s, u(s))\,\mathrm{d}s \\ &= \phi(t_{n+1}) + \int_0^{t_n} k(t_{n+1} - s)G(s, u(s))\,\mathrm{d}s \\ &\quad + \int_{t_n}^{t_{n+1}} k(t_{n+1} - s)G(s, u(s))\,\mathrm{d}s \\ &\geqslant G(t_n, u(t_n))K(t_{n+1} - t_n) \\ &\geqslant G(0, u(t_n))K(t_{n+1} - t_n). \end{aligned}$$

In view of Condition (G2),

$$\lim_{n\to\infty} \frac{R^{n+1}}{G(0, R^n)} = 0,$$

which implies that $t_{n+1} - t_n \to 0$ as $n \to \infty$. □

Theorem 2.12. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold. If*

(i) *$k(z) = z^{\beta-1}k_1(z)$ where $\beta > 0$ and $k_1(z)$ is a nonnegative function satisfying that a $\delta > 0$ exists such that $k^1_{\inf} := \inf_{z\in[0,\delta]} k_1(z) > 0$,*

a $t^* > 0$ exists such that (2.1) *and*

$$\int_U^\infty \left(\frac{u}{G(t^*, u)}\right)^{1/\beta} \frac{\mathrm{d}u}{u} < \infty \quad \textit{for all } U > 0, \tag{2.3}$$

then the solution of (1.1) *blows up in finite time.*

Proof. Otherwise, by Lemma 2, there exist sequences t_n such that $u(t_n) = r^{n\beta}$ for some $r > 1$ and $h_n := t_{n+1} - t_n$ tends to zeros as $n \to \infty$. Therefore, an $N > 0$ exists such that $h_n \leqslant \delta$ and $t_n \geqslant t^*$ for all $n \geqslant N$ which, together with conditions (P), (K) and (G1), implies that for $n \geqslant N$

$$\begin{aligned} u(t_{n+1}) &= \phi(t_{n+1}) + \int_0^{t_{n+1}} k(t_{n+1}-s)G(s,u(s))\,\mathrm{d}s \\ &\geqslant G(t_n,u(t_n))K(t_{n+1}-t_n) \\ &\geqslant G(t^*,u(t_n))K(t_{n+1}-t_n). \end{aligned}$$

Thus,

$$\frac{k_{\mathrm{inf}}^1}{\beta} h_n^{\beta} \leqslant \frac{r^{(n+1)\beta}}{G(t^*,r^{n\beta})}.$$

Hence,

$$\begin{aligned} h_n &\leqslant \left(\frac{\beta}{k_{\mathrm{inf}}^1}\right)^{1/\beta} \frac{r^n - r^{n-1}}{(G(t^*,r^{n\beta}))^{1/\beta}} \frac{r}{1-r^{-1}} \\ &\leqslant \frac{r}{1-r^{-1}} \left(\frac{\beta}{k_{\mathrm{inf}}^1}\right)^{1/\beta} \int_{r^{n-1}}^{r^n} \left(\frac{1}{G(t^*,s^{\beta})}\right)^{1/\beta} \mathrm{d}s \\ &= \frac{r^2}{(r-1)\beta} \left(\frac{\beta}{k_{\mathrm{inf}}^1}\right)^{1/\beta} \int_{r^{(n-1)\beta}}^{r^{n\beta}} \left(\frac{u}{G(t^*,u)}\right)^{1/\beta} \frac{\mathrm{d}u}{u}, \end{aligned}$$

which, together with (2.3), implies that

$$\lim_{n\to\infty} t_n = t_1 + \lim_{n\to\infty} \sum_{i=1}^{n} h_i < \infty.$$

This is a contradiction, and the proof is complete. □

Corollary 2.13. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold and that* (2.2) *holds. Then the solution of* (1.1) *blows up in finite time, if*

$k(z) = z^{\beta-1}k_1(z)$ *where* $\beta > 0$ *and* $k_1(z)$ *is a nonnegative function satisfying that a* $\delta > 0$ *exists such that* $k_{\mathrm{inf}}^1 = \inf_{z\in[0,\delta]} k_1(z) > 0$,

a $t^* > 0$ *exists such that* (2.3).

Remark 2.14. It is easy to see that the following statements are valid.

(i) There exists at most one such function $k_1(z)$ satisfying (i) of Theorem 2.12, which is bounded in a neighborhood of zero.

(ii) $k_1(z) \equiv 1$ and $k^1_{\inf} = 1$ for the important kernels $\exp(-z)$ and $z^{-\alpha}$, $0 < \alpha < 1$.

(iii) There exists no function $k_1(z)$ satisfying (i) of Theorem 2.12 for some kernels, such as $\exp(-1/z)$.

2.3. Necessary conditions. In subsection 2.2, two sufficient conditions leading to the blow-up behavior of the solution of (1.1) are given in Theorem 2.12. In this subsection, we will show that (2.1) and (2.3) are also necessary.

Lemma 2.15. *Assume that conditions* (K), (G1) *and* (G2) *hold. Then, for any given $t \in (0,\infty)$, the function $u_F(t) := \inf\{U : F(t,u) > F_{\min}(t)$ for $u \in [U,\infty)\} < \infty$ is well-defined and $u_F(t) \geqslant -F_{\min}(t)$ for all $t > 0$. Moreover, $\lim_{t\to 0} u_F(t) = \infty$.*

Example 2.16. Consider the case of $G(s,u) = r(s)u^p$ with $p > 1$. Then the minimum value $F_{\min}(t)$ is obtained at the unique minimum value point $u_F(t) = (1/(pI(t)))^{1/(p-1)}$, where $I(t) = \int_0^t k(t-s)r(s)\,\mathrm{d}s$. For example, if $k(z) \equiv 1$ and $G(s,u) = su^p$ with $p > 1$, then $u_F(t) = (2/(pt^2))^{1/(p-1)}$.

Theorem 2.17. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold. Suppose that $F_{\min}(t) \leqslant 0$ for all $t \in [0,\infty)$, and $\phi(t) + F_{\min}(t) \leqslant 0$ for all $t \in [0,\infty)$. Then $u(t)$ exists globally in the interval $[0,\infty)$ and $u(t) \leqslant u_F(t)$ for all $t \in (0,\infty)$.*

Proof. In fact, in view of Lemma 2.15, we only need to show that $u(t) \leqslant u_F(t)$ for all $t \in (0,\infty)$.

Since $u(t)$ is non-decreasing by Theorem 2.6 and $\lim_{t\to 0} u_F(t) = \infty$, we suppose that $t = \inf\{s : u(s) > u_F(s)\} < \infty$, where we denote $\inf \varnothing = \infty$. Then it follows from conditions (K) and (G1) that

$$\begin{aligned} u(t) &= \phi(t) + \int_0^t k(t-s)G(s,u(s))\,\mathrm{d}s \\ &< \phi(t) + \int_0^t k(t-s)G(s,u(t))\,\mathrm{d}s \\ &\leqslant -F_{\min}(t) + \int_0^t k(t-s)G(s,u_F(t))\,\mathrm{d}s \\ &= u_F(t). \end{aligned}$$

This contradiction implies that the proof is complete. □

Example 2.18. Let $k(z) = (3/2)\exp(-z)$ and

$$G(s,u) = g(u) = \begin{cases} \frac{u}{2} & u \in [0,1/2], \\ u - \frac{1}{4} & u \in (1/2,1], \\ \frac{3}{4}u^2 & u \in (1,\infty). \end{cases}$$

Thus, $I(t) = 3(1-\exp(-t))/2$, and

$$F(t,u) = \begin{cases} \frac{I(t)}{2}u - u & u \in [0,1/2], \\ (I(t)-1)u - \frac{I(t)}{4} & u \in (1/2,1], \\ \frac{3I(t)}{4}u^2 - u & u \in (1,\infty), \end{cases}$$

$$F_{\min}(t) = \begin{cases} -\frac{1}{3I(t)} & t \in (0,\ln 9 - \ln 5], \\ \frac{3I(t)-4}{4} & t \in (\ln 9 - \ln 5, \ln 3], \\ \frac{I(t)-2}{4} & t \in (\ln 3,\infty), \end{cases}$$

$$u_F(t) = \begin{cases} \frac{2}{3I(t)} & t \in (0,\ln 9 - \ln 5], \\ 1 & t \in (\ln 9 - \ln 5, \ln 3], \\ 1/2 & t \in (\ln 3,\infty). \end{cases}$$

Therefore, the solution $u(t)$ is bounded by $1/2$ when $\phi(t) \leqslant 1/8$ for all $t \in [0,\infty)$ and the solution $u(t)$ blows up in finite time if $\phi(t^*) > 1/8$ for some $t^* \geqslant 0$.

The following lemma is useful to show that (2.3) is also necessary.

Lemma 2.19. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold, and that a* $t^* \in (0,\infty)$ *exists such that* (2.1) *is true. If*

(i) $k(z) = z^{\beta-1}$, $\beta > 0$,

(ii) *a* $U > 0$ *exists such that*

$$\int_U^\infty \left(\frac{u}{G(t,u)}\right)^{1/\beta} \frac{\mathrm{d}u}{u} = \infty \quad \textit{for all } t \geqslant 0, \tag{2.4}$$

then the solution of (1.1) *does not blow up in finite time.*

Proof. Suppose that a number $T_b > 0$ exists such that $\lim_{t\to T_b-} u(t) = \infty$. Now, choose an increasing sequence $t_n \to T_b$ as $n \to \infty$ such that $u(t_n) = r^{n\beta}$ for

$$r = \begin{cases} (2(1+\phi(T_b)))^{1/\beta} & 0 < \beta < 1, \\ (2(2^\beta + \phi(T_b)))^{1/\beta} & \beta \geqslant 1. \end{cases}$$

Hence, $h_n = t_{n+1} - t_n \to 0$ as $n \to \infty$, and an $N > 0$ exists such that $h_n < \min\{1, t_n\}$ for all $n \geqslant N$. We claim that, for $n \geqslant N$,

$$u(t_{n+1}) \leqslant \left(2^\beta + \frac{2}{\beta}\right) h_n^\beta G(T_b, u(t_{n+1})). \tag{2.5}$$

Let $\beta \geqslant 1$. Then $(t_{n+1} - s)^{\beta-1} \leqslant 2^{\beta-1}((t_n - s)^{\beta-1} + h_n^{\beta-1})$, which together with conditions (K) and (G1) implies that, for $n \geqslant N$,

$$\begin{aligned} u(t_{n+1}) &= \phi(t_{n+1}) + \int_0^{t_n} k(t_{n+1} - s) G(s, u(s))\,\mathrm{d}s \\ &\quad + \int_{t_n}^{t_{n+1}} k(t_{n+1} - s) G(s, u(s))\,\mathrm{d}s \\ &\leqslant \phi(t_{n+1}) + 2^{\beta-1} \int_0^{t_n} k(t_n - s) G(s, u(s))\,\mathrm{d}s \\ &\quad + 2^{\beta-1} h_n^{\beta-1} \int_0^{t_n} G(s, u(s))\,\mathrm{d}s + K(h_n) G(T_b, u(t_{n+1})) \\ &\leqslant \phi(T_b) + 2^\beta u(t_n) + 2^{\beta-1} h_n^{\beta-1} \int_0^{h_n} G(s, u(s))\,\mathrm{d}s \\ &\quad + K(h_n) G(T_b, u(t_{n+1})) \\ &\leqslant \frac{1}{2} u(t_{n+1}) + \left(2^{\beta-1} + \frac{1}{\beta}\right) h_n^\beta G(T_b, u(t_{n+1})), \end{aligned}$$

which yields (2.5).

If $0 < \beta < 1$, then $(t_{n+1} - s)^{\beta-1} \leqslant (t_n - s)^{\beta-1}$, which together with conditions (K) and (G1) implies that, for $n \geqslant N$,

$$u(t_{n+1}) = \phi(t_{n+1}) + \int_0^{t_n} k(t_{n+1} - s) G(s, u(s))\,\mathrm{d}s$$

$$
\begin{aligned}
&+ \int_{t_n}^{t_{n+1}} k(t_{n+1}-s)G(s,u(s))\,\mathrm{d}s \\
&\leqslant \phi(t_{n+1}) + \int_0^{t_n} k(t_n-s)G(s,u(s))\,\mathrm{d}s \\
&\quad + K(h_n)G(T_b,u(t_{n+1})) \\
&\leqslant \phi(T_b) + u(t_n) + K(h_n)G(T_b,u(t_{n+1})) \\
&\leqslant \frac{1}{2}u(t_{n+1}) + (2^{\beta-1}+\frac{1}{\beta})h_n^{\beta}G(T_b,u(t_{n+1})),
\end{aligned}
$$

which also yields (2.5).

Therefore, one obtains that

$$
\begin{aligned}
h_n &\geqslant (C_1(\beta))^{-1/\beta}\,\frac{r^{n+1}}{(G(T_b,r^{(n+1)\beta}))^{1/\beta}} \\
&= \frac{1}{r-1}\,(C_1(\beta))^{-1/\beta}\,\frac{r^{n+2}-r^{n+1}}{(G(T_b,r^{(n+1)\beta}))^{1/\beta}} \\
&\geqslant \frac{1}{r-1}\,(C_1(\beta))^{-1/\beta}\int_{r^{n+1}}^{r^{n+2}} \frac{1}{(G(T_b,s^{\beta}))^{1/\beta}}\,\mathrm{d}s,
\end{aligned}
$$

where $C_1(\beta) := 2^{\beta} + (2/\beta)$ which, together with (2.4), implies that

$$\lim_{n\to\infty} t_n = t_1 + \lim_{n\to\infty}\sum_{i=1}^{n} h_i = \infty.$$

This is a contradiction and the proof is complete. $\square$

Theorem 2.20. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold. If*

(i) $k(z) = z^{\beta-1}k_1(z)$ *where* $\beta > 0$ *and* $k_1(z) \geqslant 0$ *is bounded in any finite interval,*

(ii) *a* $U > 0$ *exists such that* (2.4),

then the solution of (1.1) *does not blow up in finite time.*

Proof. Suppose that the solution of (1.1) blows up at finite time T_b. Then, in view of Corollary 2.5, the solution $\overline{u}(t)$ of

$$\overline{u}(t) = \phi(t) + \lambda\int_0^t (t-s)^{\beta-1}G(T_b,\overline{u}(s))\,\mathrm{d}s,$$

also blows up in finite time, where $\lambda = \sup_{z\in[0,T_b]} k_1(z) + T_b$. This contradicts Lemma 2.19 and the proof is complete. □

Corollary 2.21. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold. If* $k(z) = z^{\beta-1}k_1(z)$ *where* $\beta > 0$ *and* $k_1(z)$ *satisfies conditions in Theorems* 2.12 *and* 2.20*, then the solution of* (1.1) *blows up in finite time if and only if a* $t^* \in (0,\infty)$ *exists such that* (2.1) *and* (2.3)*.*

Corollary 2.22. *Assume that conditions* (P), (K), (G1) *and* (G2) *hold, and that* $\lim_{t\to\infty}\phi(t) = \infty$ *or* (2.2) *holds. If* $k(z) = z^{\beta-1}k_1(z)$*, where* $\beta > 0$ *and* $k_1(z)$ *satisfies conditions in Theorems* 2.12 *and* 2.20*, then the solution of* (1.1) *blows up in finite time if and only if a* $t^* > 0$ *exists such that* (2.3)*.*

2.4. Several remarks. For a more general Volterra integral equation with Hammerstein-type

$$u(t) = \phi(t) + \int_{t_0}^{t} k(t-s)G(s,u(s))\,\mathrm{d}s, \tag{2.6}$$

where t_0 is a finite number or $-\infty$, similarly to the above discussion, the blow-up conditions are given in the following remarks.

Remark 2.23. Assume that conditions (P), (K), (G1) and (G2) hold. Then the global solution satisfies $\lim_{t\to\infty} u(t) = \infty$ if and only if a $t^* \in (t_0,\infty)$ exists such that

$$\phi(t^*) + \min_{u\in[0,\infty)}\left(\int_{t_0}^{t^*} k(t-s)G(s,u)\,\mathrm{d}s - u\right) > 0. \tag{2.7}$$

Moreover, if $\lim_{t\to\infty}\phi(t) = \infty$ or

$$\lim_{t\to\infty}\int_{t_0}^{t} k(t-s)G(s,u)\,\mathrm{d}s = \infty \quad \text{for all } u \in (0,\infty), \tag{2.8}$$

then the global solution satisfies $\lim_{t\to\infty} u(t) = \infty$.

Remark 2.24. Assume that conditions (P), (K), (G1) and (G2) hold. If $k(z) = z^{\beta-1}k_1(z)$ where $\beta > 0$ and $k_1(z)$ satisfies conditions in

Theorems 2.12 and 2.20, then the solution of (2.6) blows up in finite time if and only if a $t^* \in (t_0, \infty)$ exists such that (2.3) and (2.7) hold.

Remark 2.25. Assume that conditions (P), (K), (G1) and (G2) hold, and that $\lim_{t\to\infty} \phi(t) = \infty$ or (2.8) holds. If $k(z) = z^{\beta-1}k_1(z)$, where $\beta > 0$ and $k_1(z)$ satisfies conditions in Theorems 2.12 and 2.20, then the solution of (1.1) blows up in finite time if and only if a $t^* > t_0$ exists such that (2.3).

Remark 2.26. Equation (2.3) depends only upon the value of the kernel in a neighborhood of zero but is independent of the value of $G(s, u)$ in a neighborhood of zero.

Authors in [**4, 18, 27–30**] investigate the blow-up behavior of

$$v(t) = \int_{t_0}^{t} k(t-s)r(s)g(v(s)+\phi(s))\,\mathrm{d}s. \tag{2.9}$$

The authors in [**29**] give some conditions such that the solution blows up in finite time when $g''(v) > 0$ and $k(z)$ is decreasing. When $k(z) = z^{\beta-1}$ with $\beta > 0$, the author in [**18**] proves a generalization result of [**28**] under the following conditions, i.e.,

(A1) $\phi(s)$ and $r(s)$ are nondecreasing;

(A2) $g(v)$ is increasing;

(A3) $k(z) > 0$ and $K(t) := \int_0^t k(z)\,\mathrm{d}z < \infty$ for $t > 0$.

Note that the blow-up behavior of $v(t)$ is the same as that of the corresponding solution $u(t) = v(t) + \phi(t)$ of (1.1) with $G(s, u) = r(s)g(u)$. Therefore, the blow-up conditions of (2.9) are obtained from Remarks 2.24 and 2.25.

Remark 2.27. Assume that conditions (A1)–(A3) hold. Then (2.1) if and only if

$$\phi(t^*) + \min_{v\in[0,\infty)} (g(v)I(t^*) - v) > 0,$$

where $I(t) := \int_{t_0}^t k(t-s)r(s)\,\mathrm{d}s$ is defined in [**28**]. Hence, there is a strong influence of functions $\phi(t)$, $k(z)$, $r(s)$ and $g(v)$ when $\phi(t)$ and $I(t)$

are both bounded, while only the value of the kernel in a neighborhood of zero and the value of $g(v)$ far away from zero influence (2.3).

Remark 2.28. Assume that conditions (A1)–(A3) hold, and that $k(z) = z^{\beta-1}k_1(z)$ where $\beta > 0$ and $k_1(z)$ satisfies conditions in Theorems 2.12 and 2.20. Then solutions of (2.9) blow up in finite time if and only if (2.1) and

$$\int_V^\infty \left(\frac{v}{g(v)}\right)^{1/\beta} \frac{\mathrm{d}v}{v} < \infty \quad \text{for some } V > 0. \tag{2.10}$$

Therefore, all of solutions of (3.1), (3.6) and (3.10) in [**30**] blow up in finite time, while the solution of (3.15) in [**30**] does not blow up in finite time since (2.1) is not satisfied.

It is easily seen that (2.1) is true if $k(z) = z^{\beta-1}$ with $\beta > 0$, which implies that solutions of (2.9) blow up in finite time if and only if (2.10). This is the same as Theorem 02.4 in [**18**].

Remark 2.29. The following equations

$$v(t) = \int_0^t \frac{g(v(s) + \phi(s))}{\sqrt{\pi(t-s)}} \,\mathrm{d}s \tag{2.11}$$

and

$$v(t) = \gamma \int_0^t \frac{(1+s)^q (v(s) + \phi(s))^p}{\sqrt{\pi(t-s)}} \,\mathrm{d}s \tag{2.12}$$

where γ, p, q are nonnegative parameters, are considered in [**30**]. In fact, from Remark 2.28, the blow-up conditions of (2.12) are $\gamma > 0$, $q \geqslant 0$ and $p > 1$ and the blow-up condition of (2.11) is

$$\int_V^\infty \frac{v}{g(v)^2} \,\mathrm{d}v < \infty \quad \text{for some } V > 0,$$

which is sharper than (2.11) of Theorem 2.3 in [**30**] (see Example 3.2 in subsection 3.2.1).

Remark 2.30. The authors in [**31**] also investigate the blow-up condition of (2.9) with $G(s,v) = r(s)(g(v) + \phi(s))$. It is easy to see that the solution $v(t)$ satisfies

$$v(t) = \phi_1(t) + \int_{t_0}^{t} k(t-s)r(s)g(v(s))\,\mathrm{d}s,$$

where $\phi_1(t) := \int_{t_0}^{t} k(t-s)r(s)\phi(s)\,\mathrm{d}s$. Hence, the blow-up conditions are also able to be obtained from Remarks 2.24 and 2.25.

In [**17, 19, 20–23**], the existence of the nontrivial solution of a simple Volterra integral equation

$$v(t) = \int_{0}^{t} k(t-s)g(v(s))\,\mathrm{d}s, \tag{2.13}$$

is discussed, where $g(0) = 0$, $g(v)$ is a continuous, increasing function and $k(z)$ is a local integrable function such that $K(t) = \int_0^t k(z)\,\mathrm{d}z$ is increasing. The blow-up behavior of the nontrivial solution is investigated in [**17, 24**]. Some sufficient and necessary conditions of the existence of blow-up of the nontrivial solution are given in [**12**] by using the following equality

$$t = \int_{0}^{g(t)} K(v^{-1}(t) - v^{-1}(g^{-1}(s)))\,\mathrm{d}s,$$

which is established in [**21**], and it is shown in [**13**] that

$$\int_0^t K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)} < \infty \quad \text{for any } t \in (0,1)$$

is a necessary and sufficient condition for blow-up of solutions to (2.13) with a power-law function $g(v)$, where v^{-1}, K^{-1} and g^{-1} are the inverse functions of v, K and g. In [**14**], the authors discuss a simple lower estimate of the blow-up time. Recently, the blow-up condition was expressed in terms of the convergence of some integrals in [**11**], and some examples with power and exponential nonlinear functions are presented.

Remark 2.31. We assume that a unique, nontrivial solution of (2.13) exists. Then(2.1) is obtained from $\lim_{t\to\infty} K(t) = \infty$, which is required in [**11–14**]. Hence, the nontrivial solution of (2.13) blows up in finite time if and only if (2.10).

Remark 2.32. If the kernel $k(z)$ also satisfies conditions in Corollary 2.21, then the nontrivial solution blows up in finite time if and only if (2.10) holds. Hence, all of the solutions of Examples 4.1, 4.2 in [**12**], Example 6.2 in [**13**], Examples 7.1, 7.3, 7.5 in [**11**] blow up in finite time, and the solution of Example 4.3 in [**12**] does not blow up in finite time.

3. Examples and applications.

3.1. Examples. We will give some examples to illustrate how the functions $\phi(t)$, $k(z)$ and $r(s)$ influence conditions (2.1) and (2.3), where $G(s,u) = r(s)g(u)$.

3.1.1. Critical exponents. For nonlinear parabolic equations

$$\begin{aligned} &u_t = \triangle u + u^p, && x \in \Omega \subseteq \mathbf{R}^N,\ t > 0,\\ &u(x,0) = u_0(x) \geqslant 0, && x \in \Omega \subseteq \mathbf{R}^N, \end{aligned}$$

it is shown in [**6**] that, if $\Omega = \mathbf{R}^N$, then

(i) all nontrivial nonnegative solutions blow up in finite time for $1 < p < 1 + (2/N)$;

(ii) global solutions exist when the initial data are sufficiently small $u_0(x) \geqslant 0$ for $p > 1 + (2/N)$.

The number $p^* = 1+(2/N)$ is called the *critical exponent*, which should belong to case (i). While $p^* = 1$ when the set $\Omega \subseteq \mathbf{R}^N$ is bounded, that is to say, finite-time blow-up does not persist (see [**9, 15**]). Including [**3, 9**], there are a lot of papers concerning the critical exponent for partial differential equations.

In applications there are many problems modeled by a Volterra integral equation with a power-law nonlinear function $g(u) = u^p$ (see [**28**]),

$$u(t) = c + \int_0^t k(t-s)r(s)u^p(s)\,\mathrm{d}s, \tag{3.1}$$

where $p > 1$, $k : (0, \infty) \to [0, \infty)$ is a locally integrable function satisfying Condition (K) and $r(s)$ is an increasing function. It is easy to see that conditions (P), (G1), (G2) and (2.3) hold. Therefore, it is seen from Theorems 2.9 and 2.17 that the solution of (3.1) blows up in finite time if and only if a t^* exists such that

$$c + \min_{u \in [0,\infty)} (u^p I(t^*) - u) > 0.$$

Hence, the solution of (3.1) blows up in finite time if $\lim_{t\to\infty} I(t) = \infty$. To show that global solutions of (3.1) always exist when $I(t)$ is bounded for $t \in [0, \infty)$, the following lemmas are useful and the proofs are trivial.

Lemma 3.1. *Let $p > 1$, $\lim_{t\to\infty} I(t) = I_\infty$ exist, $F(u) := I_\infty u^p - u$ and $F_{\min}(p) := \min_{u\in[0,\infty)} F(u)$. Then*

$$F_{\min}(p) = \frac{1}{p}\left(\frac{1}{pI_\infty}\right)^{1/(p-1)} (1 - p)$$

and the solution of (3.1) blows up in finite time if and only if $c + F_{\min}(p) > 0$.

Lemma 3.2. *Let $p > 1$ and $\lim_{t\to\infty} I(t) = I_\infty$ exist. Then*

(i) *if $I_\infty \geqslant 1$, $F_{\min}(p)$ is a decreasing function of p on $(1, \infty)$ satisfying*

$$\lim_{p\to\infty} F_{\min}(p) = -1 \quad \text{and} \quad \lim_{p\to 1+} F_{\min}(p) = 0;$$

(ii) *if $0 < I_\infty < 1$, $F_{\min}(p)$ is increasing for $p \in (1, (1/I_\infty))$ and decreasing for $p \in ((1/I_\infty), \infty)$, and*

$$\min_{p\in[1,\infty)} F_{\min}(p) = I_\infty - 1, \lim_{p\to\infty} F_{\min}(p) = -1 \text{ and } \lim_{p\to 1+} F_{\min}(p) = -\infty.$$

We are now ready to prove the following results.

Theorem 3.3. *Assume that $\lim_{t\to\infty} I(t) = I_\infty < \infty$.*

(i) *If $I_\infty \geqslant 1$, then*

(a) *when* $c \geqslant 1$, *the solution of* (3.1) *blows up in finite time for all* $p > 1$;

(b) *when* $0 < c < 1$, *a number* p_1 *exists defined by* $F_{\min}(p) = -c$ *such that the solution of* (3.1) *blows up in finite time if and only if* $1 < p < p_1$.

(ii) *If* $I_\infty < 1$, *then*

(a) *when* $c \geqslant 1$, *a number* p_1 *exists defined by* $F_{\min}(p) = -c$ *such that the solution of* (3.1) *blows up in finite time if and only if* $p > p_1$;

(b) *when* $1 - I_\infty < c < 1$, *there exist two numbers* $p_1 < p_2$ *defined by* $F_{\min}(p) = -c$ *such that the solution of* (3.1) *blows up in finite time if and only if* $p_1 < p < p_2$;

(c) *when* $0 < c \leqslant 1 - I_\infty$, *the solution of* (3.1) *is bounded for all* $p > 1$.

Theorem 3.4. *In* (3.1), *let* $k(z) = \exp(\alpha_1 z)$ *and* $r(s) = \exp(\alpha_2 s)$ *with* $\alpha_1 \in \mathbf{R}$ *and* $\alpha_2 \geqslant 0$.

(i) *Let* $\alpha_2 = 0$ *and* $-1 \leqslant \alpha_1 < 0$. *Then* (i) *in Theorem* 3.3 *holds.*

(ii) *Let* $\alpha_2 = 0$ *and* $\alpha_1 < -1$. *Then* (ii) *in Theorem* 3.3 *holds.*

(iii) *Let* $\alpha_2 > 0$ *or* $\alpha_2 = 0$ *and* $\alpha_1 \geqslant 0$. *Then the solution of* (3.1) *blows up in finite time for all* $p > 1$ *and* $c > 0$.

Theorem 3.5. *In* (3.1), *let* $k(z) = (1+z)^\alpha$ *with* $\alpha \in \mathbf{R}$ *and* $r(s) \equiv 1$.

(i) *Let* $-2 \leqslant \alpha < -1$. *Then* (i) *in Theorem* 3.3 *holds.*

(ii) *Let* $\alpha < -2$. *Then* (ii) *in Theorem* 3.3 *holds.*

(iii) *Let* $\alpha \geqslant -1$. *Then the solution of* (3.1) *blows up in finite time for all* $p > 1$ *and* $c > 0$.

3.1.2. The influence of $k(z)$ and $g(u)$. Consider the following Volterra integral equation

$$u(t) = \phi(t) + \int_0^t k(t-s)g(u(s))\,\mathrm{d}s, \tag{3.2}$$

where $\phi(t)$ and $k(z)$ satisfy Corollary 2.21 and $G(s,u) = g(u)$ satisfies only conditions (G1) and (G2) but $g(0) \neq 0$. In order to see the influence of the kernel $k(z)$ and the nonlinear function $g(u)$ to the blow-up behavior, we assume that $\lim_{t\to\infty}\phi(t)=\infty$. Hence, (2.1) is true and some examples will illustrate that (2.3) only depends upon the value of the kernel in a neighborhood of zero but is independent of the value of $g(u)$ in a neighborhood of zero.

Example 3.6. In (3.2), let $g(u)=g_\lambda(u)=\exp(u)-\lambda$ with $\lambda\leqslant 1$ and $k(z)=z^{\beta-1}$. Then

$$g_{1,\lambda}(u)=g_\lambda(u)-(1-\lambda)$$

and

$$\phi_1(t)=\phi(t)+(1-\lambda)\frac{1}{\beta}t^\beta$$

satisfy Corollary 2.21 and $g_{1,\lambda}(u)\sim\exp(u)$ as $u\to\infty$. Hence, the solution blows up in finite time for all $\lambda\leqslant 1$.

In (3.2), let $g(u)=g_\lambda(u)=(u+\lambda)(\ln(u+\lambda))^p$ with $\lambda\geqslant 1$ and $k(z)=z^{\beta-1}$. Then

$$g_{1,\lambda}(u)=g_\lambda(u)-\lambda(\ln\lambda)^p$$

and

$$\phi_1(t)=\phi(t)+\lambda(\ln\lambda)^p\frac{1}{\beta}t^\beta$$

satisfy Corollary 2.21 and $g_{1,\lambda}(u)\sim u(\ln u)^p$ as $u\to\infty$. Hence, the solution blows up in finite time if and only if $0<\beta<p$.

These two examples show that the blow-up behavior of the solution is independent of the value of $g(u)$ in a neighborhood of zero.

Example 3.7. In (3.2), let $g(u)=(u+1)(\ln(u+1))^p$ and $k(z)=k_{\beta,\lambda}(z)=\exp((\lambda z)^{\beta-1})-1$ with $\lambda>0$ and $\beta\geqslant 1$. Since $k(z)=z^{\beta-1}k_1(z)$ with $k_1(z)=[k(z)/z^{\beta-1}]$ satisfies Corollary 2.21, the solution blows up in finite time if and only if $0<\beta<p$. Specifically, in the case of $p=2$, the solution blows up in finite time when $k(z)=\exp(z)-1$ but does not blow up in finite time when $k(z)=\exp(z^2)-1$, while, for $\beta_2>\beta_1$,

$$k_{\beta_2,\lambda}(z)\geqslant k_{\beta_1,\lambda}(z)\quad\text{for } z\geqslant\frac{1}{\lambda}.$$

This example shows that the blow-up behavior only depends upon the value of the kernel in a neighborhood of zero.

Example 3.8. In (3.2), let $k(z) = k_\omega(z) = 1 - \cos(\omega z)$ and $g(u)$ be increasing, positive functions satisfying $\lim_{u\to\infty}[g(u)/u] = \infty$. In order to apply Corollary 2.21, we have to write $k(z)$ as $z^2 k_1(z)$, where $k_1(z) = [1-\cos(\omega z)]/z^2$. Therefore the solution blows up in finite time for any given $\omega > 0$ if and only if, for some $U > 0$,

$$\int_U^\infty \left(\frac{u}{g(u)}\right)^{1/3} \frac{\mathrm{d}u}{u} < \infty.$$

This example shows that the solution still blows up in finite time when the kernel is nonnegative and be highly oscillatory with respect to a positive number even if an infinite number of zeros exist.

3.2. Applications. We now apply our results to Volterra integro-differential equations of Hammerstein-type,

$$\begin{aligned} u'(t) &= \phi(t) + \int_{t_0}^t k(t-s)G(s,u(s))\,\mathrm{d}s, \\ u(t_0) &= u_0, \end{aligned} \tag{3.3}$$

where $t_0 \in \mathbf{R}$, $u_0 > 0$, $\phi(t) \geqslant 0$, $k(z) = z^{\beta-1}k_1(z)$, $\beta > 0$, $k_1(z)$ satisfies conditions in Theorems 2.12 and 2.20 and $G(s,u)$ satisfies conditions (G1) and (G2). It is obvious that the solution $u(t)$ satisfies

$$\begin{aligned} u(t) &= u_0 + \int_{t_0}^t \phi(s)\,\mathrm{d}s + \int_{t_0}^t \int_{t_0}^s k(s-r)G(r,u(r))\,\mathrm{d}r\,\mathrm{d}s \\ &= u_0 + \int_{t_0}^t \phi(s)\,\mathrm{d}s + \int_{t_0}^t \int_r^t k(s-r)\,\mathrm{d}s G(r,u(r))\,\mathrm{d}r \\ &= \overline{\phi}(t) + \int_{t_0}^t \overline{k}(t-s)G(s,u(s))\,\mathrm{d}s, \end{aligned}$$

where $\overline{\phi}(t) := u_0 + \int_{t_0}^t \phi(s)\,\mathrm{d}s$ and $\overline{k}(z) := \int_0^z k(r)\,\mathrm{d}r$. Hence, conditions (P) and (K) hold. From Remark 2.26, we only need to estimate the $\overline{k}(z)$

in a neighborhood of zero. Since $k_1(z)$ satisfies conditions in Theorems 2.12 and 2.20, a $\delta > 0$ exists such that

$$0 < k_{\inf}^1(\delta) \leqslant k_1(z) \leqslant k_{\sup}^1(\delta) \quad \text{for all } z \in [0, \delta].$$

Thus, for all $z \in [0, \delta]$,

$$k_{\inf}^1(\delta) z^\beta \leqslant \overline{k}(z) \leqslant k_{\sup}^1(\delta) z^\beta.$$

As a result, from Corollary 2.21, we have

Theorem 3.9. *Assume that* $\phi(t) \geqslant 0$, $k(z) = z^{\beta-1}k_1(z)$, $\beta > 0$, $k_1(z)$ *satisfies conditions in Theorems* 2.12 *and* 2.20, *and* $G(s,u)$ *satisfies conditions* (G1) *and* (G2)*. Then the solution of* (3.3) *blows up in finite time if and only if a* $t^* > 0$ *exists such that*

$$\overline{\phi}(t^*) + \min_{u \in [0,\infty)} \left(\int_{t_0}^{t} \overline{k}(t-s) G(s,u) \, \mathrm{d}s - u \right) > 0$$

and

$$\int_U^\infty \left(\frac{u}{G(t^*, u)} \right)^{1/(\beta+1)} \frac{\mathrm{d}u}{u} < \infty \quad \textit{for some } U > 0.$$

Remark 3.10. Consider an ordinary differential equation with higher order

$$\begin{aligned} u^{(\beta)}(t) &= G(t, u(t)), && t \geqslant 0, \\ u^{(i)}(0) &= u_0^i, && i = 0, 1, \ldots, \beta - 1, \end{aligned} \tag{3.4}$$

where $\beta \geqslant 1$ is an integer, $G(t,u)$ satisfies conditions (G1) and (G2) and $u_0^i > 0$, $i = 0, 1, \ldots, \beta - 1$. Since the solution of (3.4) satisfies

$$u(t) = \phi(t) + \int_0^t k(t-s) G(s, u(s)) \, \mathrm{d}s,$$

where $\phi(t) = \sum_{i=0}^{\beta-1} 1/(i!) u_0^i t^i$ and $k(z) = [1/(\beta-1)!] z^{\beta-1}$, it follows from Corollary 2.20 that the solution of (3.4) blows up in finite time if and only if a $t^* > 0$ exists such that

$$\int_U^\infty \left(\frac{u}{G(t^*, u)} \right)^{1/\beta} \frac{\mathrm{d}u}{u} < \infty \quad \text{for some } U > 0.$$

4. Concluding remarks. The various results on finite-time blow-up or global existence of solutions for Volterra integral equations of the form (1.1) and (1.2) lead one to ask whether these results remain valid, or how they have to be modified, if (1.1) is replaced by

$$u(t) = \phi(t) + \int_{\theta(t)}^{t} k(t-s)G(s, u(s))\, ds, \quad t > 0, \tag{4.1}$$

where the *delay function* θ is of the form

$$\theta(t) = qt, \quad \text{with } 0 < q < 1,$$

or

$$\theta(t) = t - \tau, \quad \text{with } \tau > 0.$$

In the latter case, equation (4.1) has to be complemented by the initial condition

$$u(t) = \psi(t), \quad -\tau \leqslant t \leqslant 0,$$

where ψ is a given (continuous) function.

We shall present the corresponding blow-up analysis in a sequel to this paper.

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