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APPLICATIONS OF GENERALIZED CONVOLUTIONS ASSOCIATED WITH THE FOURIER AND HARTLEY TRANSFORMS

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ABSTRACT. In this paper we present new generalized convolutions with weight-function associated with the Fourier and Hartley transforms, and consider applications. Namely, using the generalized convolutions, we construct normed rings on the space $L^1(\mathbf{R}^d)$, provide the sufficient and necessary condition for the solvability of a class of integral equations of convolution type, and receive the explicit solutions of those equations.

1. Introduction. The theory of convolutions has been studied for a long time and applied to many fields of mathematics. In recent years, many convolutions, generalized convolutions, and poly-convolutions of the well-known transforms as the Fourier, Hankel, Mellin, Laplace, and the applications of those transforms have been published (see [**2**–**7**, **11**–**13**, **18**, **27**, **33**]). Loosely speaking, each one of generalized convolutions is a new integral transform which may be an object of study; for instance, the Hilbert transform can be thought of as a convolution of $f(t)$ with the function $g(t) = 1/(\pi t)$, and the Weierstrass transform is exactly the convolution of that function with the Gaussian function $e^{-t^2/4}$. As Kakichev stated in his paper [**23**], many generalized convolutions of known transforms have not been found.

The main aims of this paper are to present generalized convolutions with the Gaussian weight-function $\gamma(x) = e^{-|x|^2/2}$ associated with the

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Hartley and Fourier transforms and consider applications. The paper is divided into three sections and is organized as follows.

In Section 2, a total of 13 new generalized convolutions is provided. In subsections 2.1 and 2.2, by using the above-mentioned convolutions, we construct normed rings on $L^1(\mathbf{R}^d)$, obtain the sufficient and necessary conditions for the solvability of a class of integral equations of convolution type and the explicit solutions of those equations, respectively. In subsection 2.3, following the solution of integral equations presented in subsection 2.2, we show that the problem about spectral radius of integral operators may be reduced to the certainly practical problem about the maximum of absolute values of a bounded continuous function on $\mathbf{R}^d$.

In Section 3, we construct further convolutions associated with the Hartley and Fourier transforms, bearing in mind their various potential applications (see [**15, 26, 29, 31, 32**]).

2. Generalized convolutions and applications. Let $\langle x, y\rangle$ denote the scalar product of $x, y \in \mathbf{R}^d$ and $|x|^2 = \langle x, x\rangle$. The Fourier and its inverse transform are defined as follows:

$$(Ff)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle} f(y)\, dy,$$
$$(F^{-1}f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{i\langle x,y\rangle} f(y)\, dy,$$

where f is a complex-valued function defined on $\mathbf{R}^d$. Throughout the paper, we use the notations: $\cos xy := \cos\langle x, y\rangle$, $\sin xy := \sin\langle x, y\rangle$ as there is no danger of confusion. There is another transform called the Hartley transform

$$(H_1 f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \operatorname{cas} xy f(y)\, dy,$$

where the integral kernel, known as the cosine-and-sine or *cas function*, is defined as $\operatorname{cas} xy := \cos xy + \sin xy$. We hereby consider additionally the transform

$$(H_2 f)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \big[\cos xy - \sin xy\big] f(y)\, dy.$$

As $(H_1f)(x) = (H_2f)(-x)$, we call H_1, H_2 the Hartley transforms (see [**2**]).

Generalized convolution with weight is a nice concept based on the so-called factorization identity. We recall a formulation of convolutions.

Let U_1, U_2, U_3 be linear spaces on the field of scalars $\mathcal{K}$, and let V be a commutative algebra on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$ and $K_3 \in L(U_3, V)$ are linear operators from U_1, U_2, U_3 to V, respectively, and δ is an element given in V.

Definition 2.1 [**11, 13**]. A bilinear map $* : U_1 \times U_2 \to U_3$ is called a convolution with weight-element δ for K_3, K_1, K_2 (in that order) if $K_3(*(f,g)) = \delta K_1(f)K_2(g)$ for any $f \in U_1$, $g \in U_2$. The bilinear form $*(f,g)$ briefly denoted by $f \underset{K_3,K_1,K_2}{\overset{\delta}{*}} g$.

If V is an algebra with unit, and if δ is the unit, we say briefly the convolution for K_3, K_1, K_2. In cases $U_1 = U_2 = U_3$ and $K_1 = K_2 = K_3$, the convolution is denoted simply by $f \underset{K_1}{\overset{\delta}{*}} g$, and by $f \underset{K_1}{*} g$ if δ is the unit of V.

Throughout this article, we consider that $U_k = L^1(\mathbf{R}^d)$ $(k = 1, 2, 3)$ with the integral by Lebesgue's mean, and V is the algebra of all real- or complex-valued measurable functions defined on $\mathbf{R}^d$.

Write $\gamma(x) := e^{-|x|^2/2}$. We define the norm of $f \in L^1(\mathbf{R}^d)$ as $\|f\|_1 = (2/(2\pi)^{d/2}) \int_{\mathbf{R}^d} |f(x)|\, dx$. Our idea for constructing the generalized convolutions below comes from the well-known convolutions of Fourier, Hartley and Weierstrass transforms.

Theorem 2.1. *If $f, g \in L^1(\mathbf{R}^d)$, then each one of the following integral expressions defines a generalized convolution, followed by its integral inequalities and factorization identity.*

(2.1)
$$\left(f \underset{F,H_1,H_1}{\overset{\gamma}{*}} g\right)(x) = \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(u) \times g(v) \Big[-ie^{-|x+u+v|^2/2} + e^{-|x+u-v|^2/2}$$

$$+ e^{-|x-u+v|^2/2} + ie^{-|x-u-v|^2/2}\Big]\, du\, dv,$$

$$\|f \underset{F,H_1,H_1}{\overset{\gamma}{*}} g\|_1 \leq \|f\|_1 \cdot \|g\|_1,$$

$$F(f \underset{F,H_1,H_1}{\overset{\gamma}{*}} g)(x) = \gamma(x)(H_1 f)(x)(H_1)(x);$$

(2.2)

$$(f \underset{F,H_2,H_2}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(u) \times g(v)\Big[ie^{-|x+u+v|^2/2} + e^{-|x+u-v|^2/2} + e^{-|x-u+v|^2/2} - ie^{-|x-u-v|^2/2}\Big]\, du\, dv,$$

$$\|f \underset{F,H_2,H_2}{\overset{\gamma}{*}} g\|_1 \leq \|f\|_1 \cdot \|g\|_1,$$

$$F(f \underset{F,H_2,H_2}{\overset{\gamma}{*}} g)(x) = \gamma(x)(H_2 f)(x)(H_2)(x).$$

(2.3)

$$(f \underset{F,H_1,F}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(u) \times g(v)\Big[(1-i)e^{-1/2|x+u-v|^2} + (1+i)e^{-1/2|x-u-v|^2}\Big]\, du\, dv,$$

$$\|f \underset{F,H_1,F}{\overset{\gamma}{*}} g\|_1 \leq \|f\|_1 \cdot \|g\|_1,$$

$$F(f \underset{F,H_1,F}{\overset{\gamma}{*}} g)(x) = \gamma(x)(H_1 f)(x)(Fg)(x);$$

(2.4)

$$(f \underset{F,H_2,F}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(u) \times g(v)\Big[(1+i)e^{-1/2|x+u-v|^2} + (1-i)e^{-1/2|x-u-v|^2}\Big]\, du\, dv,$$

$$\|f \underset{F,H_2,F}{\overset{\gamma}{*}} g\|_1 \leq \|f\|_1 \cdot \|g\|_1,$$

$$F(f \underset{F,H_2,F}{\overset{\gamma}{*}} g)(x) = \gamma(x)(H_2 f)(x)(Fg)(x).$$

Proof. Let us first prove the integral inequalities of these convolutions. By using the formulae $\int_{\mathbf{R}^d} e^{-|x\pm u\pm v|^2/2}dx = (2\pi)^{d/2}$ $(u,v\in\mathbf{R}^d)$, we have

$$\begin{aligned}
&\frac{2}{(2\pi)^{d/2}}\int_{\mathbf{R}^d}|(f \underset{F,H_1,H_1}{\overset{\gamma}{*}} g)|(x)\,dx\\
&\le \frac{1}{(2\pi)^{(3d)/2}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|f(u)||g(v)||i|e^{-|x+u+v|^2/2}\,du\,dv\,dx\\
&\quad+\frac{1}{(2\pi)^{(3d)/2}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|f(u)||g(v)|e^{-|x+u-v|^2/2}\,du\,dv\,dx\\
&\quad+\frac{1}{(2\pi)^{3d/2}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|f(u)||g(v)|e^{-|x-u+v|^2/2}\,du\,dv\,dx\\
&\quad+\frac{1}{(2\pi)^{(3d)/2}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|f(u)||g(v)||i|e^{-|x-u-v|^2/2}\,du\,dv\,dx\\
&=\frac{4}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|f(u)||g(v)|\,du\,dv\\
&=\|f\|_1\cdot\|g\|_1.
\end{aligned}$$

The integral inequality of (2.1) is proved. The other inequalities may be proved analogously. We shall prove the factorization identities of the convolutions. Using the identities

$$\begin{aligned}
\cos x(u-v) &= \frac{e^{i\langle x,u-v\rangle}+e^{-i\langle x,u-v\rangle}}{2},\\
\sin x(u+v) &= \frac{e^{i\langle x,u+v\rangle}-e^{-i\langle x,u+v\rangle}}{2i},\\
F\gamma = F^{-1}\gamma &= \gamma = H_1\gamma = H_2\gamma
\end{aligned} \tag{2.5}$$

(see [**25**, Theorem 7.6], or [**32**])

and changing variables, we have

$$\begin{aligned}
&\gamma(x)(H_1f)(x)(H_1g)(x)\\
&=\frac{\gamma(x)}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}f(u)g(v)\mathrm{cas}\,(xu)\mathrm{cas}\,(xv)\,du\,dv\\
&=\frac{\gamma(x)}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}f(u)g(v)[\cos x(u-v)+\sin x(u+v)]\,dudv
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2(2\pi)^{(3d)/2}} \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)[e^{i\langle x,u-v+t\rangle} \\
&\qquad + e^{i\langle x,-u+v+t\rangle} + ie^{i\langle x,-u-v+t\rangle} \\
&\qquad\qquad - ie^{i\langle x,u+v+t\rangle}]e^{-1/2|t|^2}\,du\,dv\,dt \\
&= \frac{1}{2(2\pi)^{(3d)/2}} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle}dy \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v) \\
&\qquad \times \Big[e^{-|y+u-v|^2/2} + e^{-|y-u+v|^2/2} \\
&\qquad\qquad + ie^{-|y-u-v|^2/2} - ie^{-|y+u+v|^2/2}\Big]\,du\,dv \\
&= F(f \underset{F,H_1,H_1}{\overset{\gamma}{*}} g)(x).
\end{aligned}
$$

The factorization identity in (2.1) is proved and that in (2.2) may be proved similarly. We shall prove the factorization identity in (2.3). Using Euler's formulae and (2.5), we have

$$
\begin{aligned}
&\gamma(x)(H_1 f)(x)(Fg)(x) \\
&\qquad = \frac{\gamma(x)}{(2\pi)^d} \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)(\cos xu + \sin xu)e^{-i\langle x,v\rangle}\,du\,dv \\
&\qquad = \frac{\gamma(x)}{2(2\pi)^d} \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)[(1-i)e^{i\langle x,u-v\rangle} \\
&\qquad\qquad + (1+i)e^{-i\langle x,u+v\rangle}]\,du\,dv \\
&\qquad = \frac{1}{2(2\pi)^{3d/2}} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle}dy \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \\
&\qquad\qquad \times g(v)\Big[(1-i)e^{-1/2|y+u-v|^2} \\
&\qquad\qquad\qquad + (1+i)e^{-1/2|y-u-v|^2}\Big]\,du\,dv \\
&\qquad = F(f \underset{F,H_1,F}{\overset{\gamma}{*}} g)(x).
\end{aligned}
$$

The factorization identity in (2.4) is proved analogously. The theorem is proved. □

In the next three subsections, the applications of the above-mentioned convolutions to normed rings on $L^1(\mathbf{R}^d)$, to the integral equations of convolutions type, and to the problem about spectral radius of the integral operators are considered.

2.1. Normed ring structures on $L^1(\mathbf{R}^d)$. Convolution transforms were of special interest to a great number of authors, as they have many applications in pure and applied mathematics (see [**12, 14, 29, 30, 32, 33**]). Practically, generalized convolution is considered to be a tool for multi-dimensional filtering tasks; theoretically, it is a new transform which can be an object of study. By Theorem 2.1, we have the fact that, for any f (or g) fixed in $L^1(\mathbf{R}^d)$, those convolution transforms are continuous operators from $L^1(\mathbf{R}^d)$ into itself. In the theory of normed rings, the multiplication of two elements can be a convolution.

In this section, we present the normed ring structures on $L^1(\mathbf{R}^d)$ that might be used in theories of Banach *-algebra (see [**19**]). We should recall the concept of a normed ring.

Definition 2.2 [**21**]. A vector space V with a ring structure and a vector norm is called normed ring if $\|vw\| \leq \|v\|\|w\|$, for all $v, w \in V$. If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Theorem 2.2. *The space* $X := L^1(\mathbf{R}^d)$, *equipped with each of the convolution multiplications* (2.1)–(2.4), *becomes a normed ring having no unit. Moreover,*

(a) *For convolutions* (2.1), (2.2), *X is commutative.*

(b) *For convolutions* (2.3), (2.4), *X is non-commutative.*

Proof. The proof of the theorem is divided into two steps.

Step 1. *X has a normed ring structure with no unit.* It is clear that $L^1(\mathbf{R}^d)$, equipped with each of the convolution multiplications listed above, has a normed ring structure. We will prove that this normed ring X has no unit. For brevity, let us use the common symbol $*$ for the above convolution multiplications. Suppose that there exists an element $e \in X$ such that $f = f * e = e * f$ for every $f \in X$. Choosing $f = \gamma$, we have $\gamma = \gamma * e = e * \gamma$. By the factorization identities of convolutions (2.1)–(2.4), we have $F(\gamma) = \gamma(T_k\gamma)(T_je)$, where $T_k, T_j \in \{H_1, H_2, F\}$ (note that it may be $T_j = T_k = H_1$, etc.). By formula (2.5), we have $\gamma = \gamma^2 T_j e$. Since $\gamma(x) \neq 0$ for every $x \in \mathbf{R}^d$, $\gamma(x)(T_je)(x) = 1$ for every $x \in \mathbf{R}^d$. But this contradicts the fact that $\lim_{x\to\infty} \gamma(x)(T_je)(x) = 0$ is derived from the Riemann-Lebesgue lemma (see [**25, 32**]).

Step 2. *The commutativity and non-commutativity convolutions.* Obviously, convolutions (2.1) and (2.2) are commutative. We shall prove the non-commutativity of (2.3), (2.4). Choose $\delta_0(x) = -2(\partial\gamma(x))/(\partial x_1) = 2x_1\gamma(x)$. Note that $\delta_0 \in L^1(\mathbf{R}^d)$.

Lemma 2.1. *The following formula holds*:

$$F\delta_0 = -i\delta_0, \qquad H_2\delta_0 = -\delta_0, \qquad H_1\delta_0 = \delta_0.$$

Proof of the lemma. Integrating by parts on the variable y_1, we have

$$\begin{aligned}
(F\delta_0)(x) &= \frac{-2}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle} \left(\frac{\partial\gamma(y)}{\partial y_1}\right) dy \\
&= \frac{-2ix_1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-i\langle x,y\rangle} \gamma(y)\, dy \\
&= -2ix_1(F\gamma)(x) = -2ix_1\gamma(x) = -i\delta_0(x), \\
(H_2\delta_0)(x) &= \frac{-2}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \cos(xy) \left(\frac{\partial\gamma(y)}{\partial y_1}\right) dy \\
&\quad + \frac{2}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \sin(xy) \left(\frac{\partial\gamma(y)}{\partial y_1}\right) dy \\
&= \frac{-2x_1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \sin(xy)\gamma(y)\, dy \\
&\quad + \frac{-2x_1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \cos(xy)\gamma(y)\, dy \\
&= -2x_1(H_1\gamma)(x) = -\delta_0(x), \\
(H_1\delta_0)(x) &= \frac{-2}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \cos(xy) \left(\frac{\partial\gamma(y)}{\partial y_1}\right) dy \\
&\quad + \frac{-2}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \sin(xy) \left(\frac{\partial\gamma(y)}{\partial y_1}\right) dy \\
&= \frac{-2x_1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \sin(xy)\gamma(y)\, dy \\
&\quad + \frac{2x_1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \cos(xy)\gamma(y)\, dy \\
&= 2x_1(H_2\gamma)(x) = 2x_1\gamma(x) = \delta_0(x).
\end{aligned}$$

The claim is proved. $\square$

We shall prove the non-commutativity of the convolution multiplications in (2.3) and (2.4). Indeed, by using Lemma 2.1 we obtain $F(\delta_0 \underset{F,H_1,F}{\overset{\gamma}{*}} \gamma) = \gamma^2\delta_0$, $F(\gamma \underset{F,H_1,F}{\overset{\gamma}{*}} \delta_0) = -i\gamma^2\delta_0$, $F(\delta_0 \underset{F,H_2,F}{\overset{\gamma}{*}} \gamma) = -\gamma^2\delta_0$, $F(\gamma \underset{F,H_2,F}{\overset{\gamma}{*}} \delta_0) = -i\gamma^2\delta_0$. Therefore, convolution multiplications (2.3) and (2.4) are non-commutative. Theorem 2.2 is proved. $\square$

2.2. Integral equations of convolution type. Consider the integral equation

(2.6)
$$\lambda\varphi(x) + \frac{2}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\Big[k_1(u)e^{-|x+u-v|^2/2} + k_2(u)e^{-|x-u-v|^2/2}\Big] \times \varphi(v)\,du\,dv = p(x),$$

where $\lambda \in \mathbf{C}$ is predetermined, k_1, k_2, p are in $L^1(\mathbf{R}^d)$, and φ is to be determined. In what follows, the functional equality $f(x) = g(x)$ means that it is valid for almost every $x \in \mathbf{R}^d$. However, if the functions f, g are continuous, it should be emphasized that the identity $f(x) = g(x)$ is true for every $x \in \mathbf{R}^d$.

In equation (2.6), the function

$$K(x,v) = \frac{2}{(2\pi)^d}\int_{\mathbf{R}^d}\Big[k_1(u)e^{-|x-u+v|^2/2}+k_2(u)e^{-|x-u-v|^2/2}\Big]\,du \tag{2.7}$$

is considered to be the kernel. It is well known that the d-dimension Gaussian function is of the form $q(x) = 1/(\sqrt{2\pi\sigma^2})^{d/2}e^{-|x-u|^2/(2\sigma^2)}$. The integral equations with Gaussian kernels have applications in physics, medicine and biology (see [**8, 10**]).

In what follows, we write $\check{f}(x) := f(-x)$. Clearly, $f \in L^1(\mathbf{R}^d)$ if and only if so is $\check{f}$. Put:

$$A(x) := \lambda + 2\gamma(x)(F\mathbf{K})(x), \quad \text{where } \mathbf{K} = \check{k}_1 + k_2. \tag{2.8}$$

Theorem 2.3. *Assume that $A(x) \neq 0$ for every $x \in \mathbf{R}^d$, and one of the following conditions is satisfied*:

(i) $Fp/A \in L^1(\mathbf{R}^d)$.

(ii) $\lambda \neq 0$, and $Fp \in L^1(\mathbf{R}^d)$.

Then equation (2.6) *has a solution in* $L^1(\mathbf{R}^d)$ *if and only if* $F^{-1}(Fp/A) \in L^1(\mathbf{R}^d)$. *If this is the case, then the solution is given by* $\varphi = F^{-1}(Fp/A) \in L^1(\mathbf{R}^d)$.

Proof. Let us first assume that (i) is fulfilled. By using (2.3) and (2.4),

$$\frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)e^{-|x+u-v|^2/2}\,du\,dv = \frac{1}{4}[(1+i)(f \underset{F,H_1,F}{\overset{\gamma}{*}} g)(x) + (1-i)(f \underset{F,H_2,F}{\overset{\gamma}{*}} g)(x)], \tag{2.9}$$

$$\frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)e^{-|x-u-v|^2/2}\,du\,dv = \frac{1}{4}[(1-i)(f \underset{F,H_1,F}{\overset{\gamma}{*}} g)(x) + (1+i)(f \underset{F,H_2,F}{\overset{\gamma}{*}} g)(x)], \tag{2.10}$$

for all $f, g \in L^1(\mathbf{R}^d)$. By the factorization identities of those convolutions we obtain

$$F\left(\frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} e^{-|x+u-v|^2/2} f(v)g(u)\,du\,dv\right) = \frac{\gamma(x)}{4}\Big[(1+i)(H_1f)(x)(Fg)(x) + (1-i)(H_2f)(x)(Fg)(x)\Big], \tag{2.11}$$

$$F\left(\frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} e^{-|x-u-v|^2/2} f(v)g(u)\,du\,dv\right) = \frac{\gamma(x)}{4}\Big[(1-i)(H_1f)(x)(Fg)(x) + (1+i)(H_2f)(x)(Fg)(x)\Big], \tag{2.12}$$

for $f, g \in L^1(\mathbf{R}^d)$.

Necessity. Suppose that equation (2.6) has a solution $\varphi \in L^1(\mathbf{R}^d)$. Applying F to both sides of the equation, and using (2.11), (2.12) and the identities $H_1 = [(1+i)/2]F + [(1-i)/2]F^{-1}$, $H_2 = [(1-i)/2]F + [(1+i)/2]F^{-1}$, we obtain $A(x)(F\varphi)(x) = (Fp)(x)$, where $F\varphi$ is the unknown function, $A(x)$ is determined by (2.8). Since $A(x) \neq 0$ for every $x \in \mathbf{R}^d$, $F\varphi = Fp/A$. As $Fp/A \in L^1(\mathbf{R}^d)$, we apply the inverse Fourier transform to obtain $\varphi = F^{-1}(Fp/A)$. The necessity is proved.

Sufficiency. Consider $\varphi := F^{-1}(Fp/A)$. It implies that $\varphi \in L^1(\mathbf{R}^d)$. Applying the inverse Fourier transform, $F\varphi = (Fp/A)$. We thus have $A(x)(F\varphi)(x) = (Fp)(x)$. Using the factorization identities of convolutions (2.3), (2.4). we get

$$F\Big[\lambda\varphi + (1+i)(k_1 \underset{F,H_1,F}{\overset{\gamma}{*}} \varphi) + (1-i)(k_1 \underset{F,H_2,F}{\overset{\gamma}{*}} \varphi) + (1-i)(k_2 \underset{F,H_1,F}{\overset{\gamma}{*}} \varphi) + (1+i)(k_2 \underset{F,H_2,F}{\overset{\gamma}{*}} \varphi)\Big] = Fp.$$

Equivalently,

$$F[\lambda\varphi(x) + \frac{2}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}[k_1(u)e^{-|x+u-v|^2/2} + k_2(u)e^{-|x-u-v|^2/2}\Big] \times \varphi(v)\,du\,dv = (Fp)(x).$$

By the uniqueness theorem of F, φ fulfills equation (2.6) for almost every $x \in \mathbf{R}^d$.

Now we assume that (ii) is fulfilled. It is known that (2.6) is an integral equation of the first kind if $\lambda = 0$, and that of the second kind if $\lambda \neq 0$. For the second kind, item (a) in Proposition 2.1 serves as a clear illustration of the assumption that $A(x) \neq 0$ for all $x \in \mathbf{R}^d$, and item (b) is useful for proving item (ii).

Proposition 2.1. (a) *If $\lambda \neq 0$, then $A(x) \neq 0$ for every x outside a ball with a finite radius.*

(b) *Assume that $\lambda \neq 0$ and $A(x) \neq 0$ for every $x \in \mathbf{R}^d$. Then $Fp \in L^1(\mathbf{R}^d)$ if and only if $Fp/A \in L^1(\mathbf{R}^d)$.*

Proof of Proposition 2.1. (a) By the Riemann-Lebesgue lemma, the function $A(x)$ is continuous on $\mathbf{R}^d$ and $\lim_{|x|\to\infty} A(x) = \lambda$. Item (a) follows from $\lambda \neq 0$ and the continuity of $A(x)$.

(b) Assume $Fp \in L^1(\mathbf{R}^d)$. By the continuity of A and $\lim_{|x|\to\infty} A(x) = \lambda \neq 0$, there exist $R > 0$ and $\epsilon_1 > 0$ so that $\inf_{|x|>R} |A(x)| > \epsilon_1$. Since

A is continuous and does not vanish in the compact set $S(0,R) = \{x \in \mathbf{R}^d : |x| \leq R\}$, there exists an $\epsilon_2 > 0$ so that $\inf_{|x|\leq R}|A(x)| > \epsilon_2$. We then have $\sup_{x\in\mathbf{R}^d}(1/|A(x)|) \leq \max\{(1/\epsilon_1),(1/\epsilon_2)\} < \infty$. It follows that the function $1/|A(x)|$ is continuous and bounded on $\mathbf{R}^d$. Since $Fp \in L^1(\mathbf{R}^d)$ we have $(Fp/A) \in L^1(\mathbf{R}^d)$.

Conversely, from the assumption $(Fp/A) \in L^1(\mathbf{R}^d)$, and the function $1/A(x)$ is continuous and bounded on $\mathbf{R}^d$ we can deduce $Fp \in L^1(\mathbf{R}^d)$. The proposition is proved.

The assertion of Theorem 2.3 for item (ii) now is an immediate consequence of item (b) of Proposition 2.1 and item (i) of this theorem just proved. The proof of Theorem 2.3 is complete. □

Comparison. In constructing generalized convolutions, other authors [**15, 16, 18, 22**] solved their integral equations. Due to using the Wiener-Lévy theorem, however, those works obtained only sufficient conditions for the solvability and implicit solutions of equations for the case $\lambda \neq 0$.

The second term on the left side of (2.6) is not any known generalized convolution. By means of a pair of two convolutions associated with the Fourier and Hartley transforms, those convolutions work out the sufficient and necessary condition for the solvability and the explicit solution of the equation.

Remark 2.1. (1) In the general theory of integral equations, the requirement that $A(x) \neq 0$ for every $x \in \mathbf{R}^d$ is the normally solvable condition of the equation (see [**16, 23**]). In the case $\lambda \neq 0$, the assumption $Fp \in L^1(\mathbf{R}^d)$ as in item (ii) of Theorem 2.3 seems to be simpler and easier to check, which is quite fair.

(2) Convolutions (2.1) and (2.2) are commutative, while convolutions (2.3) and (2.4) are non-commutative. Nevertheless, convolutions (2.3), (2.4) are helpful for solving equation (2.6).

(3) The second term on the right side of (2.8) is a continuous and bounded function on $\mathbf{R}^d$, and it vanishes at infinity. If $|\lambda|$ is sufficiently large, then $A(x) \neq 0$ for every $x \in \mathbf{R}^d$.

2.3. Examples of spectral radius of the integral operators. Based on the solvability of equation (2.6) we can determine spectral radius of the integral operator defined in a specific space.

Following the idea of Cherskij [**9**], we denote $\mathcal{X} := L^1(\mathbf{R}^d) \cap F(L^1(\mathbf{R}^d))$. Then $\mathcal{X}$ is a normed linear space (see [**1**]). Let $\mathscr{K}$ denote the integral operator with the kernel $K(x, v)$ defined by (2.7) as

$$(\mathscr{K}\varphi)(x) = \int_{\mathbf{R}^d} K(x, v)\varphi(v)\, dv.$$

Theorem 2.4. *$\mathscr{K}$ is a continuous operator from $\mathcal{X}$ into itself.*

Proof. Let us first prove that F is a continuous linear operator from $\mathcal{X}$ into itself. Let $\varphi \in \mathcal{X}$. There exists a $\varphi_0 \in L^1(\mathbf{R}^d)$ such that $\varphi = F\varphi_0$. By the inverse theorem of F, $\varphi_0 = F^{-1}\varphi = F\check{\varphi} \in L^1(\mathbf{R}^d)$. It follows that $\varphi_0 \in \mathcal{X}$. Consequently, there exists a $\varphi_1 \in L^1(\mathbf{R}^d)$ such that $\varphi_0 = F\varphi_1$. So, $F\varphi_1 = F\check{\varphi}$. By the uniqueness theorem, $\check{\varphi} = \varphi_1$, by which $\varphi = \check{\varphi}_1$ (this is suitable for the known identity of F in $L^2(\mathbf{R}^d)$: $F^2\varphi = \check{\varphi}$, see [**25**, Theorem 7.7, page 187]). Hence, $F\varphi = F\check{\varphi}_1 = \check{\varphi}_0 \in L^1(\mathbf{R}^d)$. Thus, $F\varphi \in \mathcal{X}$, by which $F(\mathcal{X}) \subset \mathcal{X}$. Similarly, F^{-1} is a continuous linear operator from $\mathcal{X}$ into itself, too. We now prove the theorem. It suffices to prove that, if $\varphi \in \mathcal{X}$, then $\mathscr{K}\varphi \in \mathcal{X}$. By Theorem 2.1, the functions on the right sides of (2.9) and (2.10) belong to $L^1(\mathbf{R}^d)$, by which those on the left sides belong to that space too. Replacing f by k_1 and g by φ in (2.9), and f by k_2 and g by φ in (2.10) and adding together (2.9) and (2.10), we derive $\mathscr{K}\varphi \in L^1(\mathbf{R}^d)$. We will prove that $\mathscr{K}\varphi$ is an image of a function belonging to $L^1(\mathbf{R}^d)$. Put $\mathscr{K}\varphi := h \in L^1(\mathbf{R}^d)$. It follows that $F(\mathscr{K}\varphi) = Fh$. By the *necessity* part in the proof of Theorem 2.3, $2\gamma(F\mathbf{K})(F\varphi) = Fh$. Since the function $\gamma F\mathbf{K}$ is continuous, bounded, and vanishes at infinity, we have $2\gamma(F\mathbf{K})(F\varphi) \in L^1(\mathbf{R}^d)$, by which $Fh \in L^1(\mathbf{R}^d)$. We thus have $\mathscr{K}\varphi = h = F(F^{-1}h) \in F(L^1(\mathbf{R}^d))$ as $F^{-1}h \in L^1(\mathbf{R}^d)$. The theorem is proved. □

Put $R_0 := \max_{x\in\mathbf{R}^d} |2\gamma(x)(F\mathbf{K})(x)| < \infty$. The following proposition serves as the conclusions in Examples 2.1, 2.2 and Claim 2.1.

Proposition 2.2. *If $\lambda > R_0$, and if $p \in \mathcal{X}$, then $F^{-1}(Fp/A) \in L^1(\mathbf{R}^d)$.*

Proof. As indicated above, $F^{-1}(\mathcal{X}) \subset \mathcal{X}$. Instead of proving $F^{-1}(Fp/A) \in L^1(\mathbf{R}^d)$, we prove $(Fp/A) \in \mathcal{X}$. As proved in item (b) of Proposition 2.1, $(Fp/A) \in L^1(\mathbf{R}^d)$.

We shall determine $g \in L^1(\mathbf{R}^d)$ such that $(Fp/A) = Fg$, i.e., $(Fp/\lambda)[1 - (F(\gamma * \mathbf{K}))/(\lambda + F(\gamma * \mathbf{K}))] = Fg$, where $\gamma * \mathbf{K}$ is denoted the Fourier convolution. By the Wiener-Lévy theorem, there exists an $h \in L^1(\mathbf{R}^d)$ such that $[(F(\gamma * \mathbf{K}))/(\lambda + F(\gamma * \mathbf{K}))] = Fh$ (see [**1, 28**]). Therefore, $(Fp/\lambda)(1-Fh) = Fg$, by which $F(p - p * h)/\lambda = Fg$. Using the uniqueness theorem of F, we obtain $g = (p - p * h)/\lambda \in L^1(\mathbf{R}^d)$. So, we have $(Fp/A) = Fg$, where $g = (p - p * h)/\lambda \in L^1(\mathbf{R}^d)$. Thus, $(Fp/A) \in \mathcal{X}$. The proposition is proved. □

Proposition 2.2 means that if equation (2.6) is considered in $\mathcal{X}$, and if $\lambda > R_0$, then the necessary and sufficient condition in Theorem 2.3 disappears, and then the operator $\lambda I + \mathcal{K}$ is invertible in $\mathcal{X}$.

Recall that the spectral radius of a continuous linear operator is the supremum among the absolute values of the elements in its spectrum. Let $\rho(A)$ denote the spectral radius of an operator A defined in $\mathcal{X}$.

Example 2.1. Consider $k_1(x) = k_2(x) = \gamma(x)$. We have $A(x) = \lambda + 4e^{-|x|^2/2}$. Let $\lambda \in \mathbf{C} \setminus \{[-4, 0)\}$ be given. As $0 < e^{-|x|^2/2} \leq 1$, $A(x) \neq 0$ for every $x \in \mathbf{R}^d$. By Theorem 2.3 and Proposition 2.2, $\rho(\mathcal{K}) = 4$.

Example 2.2. Consider that the cases $k_1(x) = \Phi_\alpha(x)$, $k_2(x) = \Phi_\beta(x)$ are given Hermite functions, where $\alpha, \beta \in \mathbf{N}^d$ are multi-indices (see [**24, 29, 32**]). Since $F\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha, F\Phi_\beta = (-i)^{|\beta|}\Phi_\beta$ (see [**29, 32**]), we have $A(x) = \lambda + 2\gamma(x)[(-i)^{|\alpha|}\Phi_\alpha(x) + (-i)^{|\beta|}\Phi_\beta(x)]$. By Theorem 2.3 and Proposition 2.2, we have $\rho(\mathcal{K}) = R_0$.

In the case $d = 1$, the function $\gamma\{(-i)^{|\alpha|}\Phi_\alpha + (-i)^{|\beta|}\Phi_\beta\}$ is rapidly decreasing, and it has a finite number of real critical points which are not greater than $\max\{|\alpha|, |\beta|\}$, where $|\alpha|, |\beta|$ are degrees of the appropriate Hermite polynomials. So, the determination of R_0 is reduced to the solution of an algebraic equation.

In the general case, the maximum absolute values of $A(x)$ exist as the function $2\gamma F\mathbf{K}$ is bounded and continuous on $\mathbf{R}^d$ and vanishes at infinity. By Theorem 2.3 and Proposition 2.2, the operator $\lambda I + \mathscr{K}$ is invertible in $\mathcal{X}$, provided $|\lambda| > \max_{x\in\mathbf{R}^d}\{|2\gamma(x)(F\mathbf{K})(x)|\}$. So, the problem about spectral radius in the theory of functional analysis is reduced to the well-known practical problem in analysis.

Claim 2.1. *The problem about the spectral radius of the integral operator $\mathscr{K}$ (defined in $\mathcal{X}$) is reduced to the problem about the maximum of absolute values of a determinable function on $\mathbf{R}^d$.*

3. Other generalized convolutions. We now construct further generalized convolutions associated with the Fourier and Hartley transforms where the Gaussian function appears in the kernels (see [**20**]).

Theorem 3.1 *If $f, g \in L^1(\mathbf{R}^d)$, then each of the transforms below defines a generalized convolution followed by its norm inequality and factorization identity.*

$$\begin{aligned}(f \overset{\gamma}{\underset{F,H_2,H_1}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \\ \times g(v)\Big[e^{-|x+u+v|^2/2} + ie^{-|x+u-v|^2.2} \\ - ie^{-|x-u+v|^2/2} + e^{-|x-u-v|^2/2}\Big]\,du\,dv,\end{aligned} \tag{3.1}$$

$$\|f \overset{\gamma}{\underset{F,H_2,H_1}{*}} g\|_1 \le \|f\|_1 \cdot \|g\|_1,$$

$$F(f \overset{\gamma}{\underset{F,H_2,H_1}{*}} g)(x) = \gamma(x)(H_2 f)(x)(H_1)(x);$$

$$\begin{aligned}(f \overset{\gamma}{\underset{F,H_1,H_2}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \\ \times g(v)\Big[e^{-|x+u+v|^2/2} - ie^{-|x+u-v|^2/2} \\ + ie^{-|x-u+v|^2/2} + e^{-|x-u-v|^2/2}\Big]\,du\,dv,\end{aligned} \tag{3.2}$$

$$\|f \overset{\gamma}{\underset{F,H_1,H_2}{*}} g\|_1 \le \|f\|_1 \cdot \|g\|_1,$$

$$F(f \overset{\gamma}{\underset{F,H_1,H_2}{*}} g)(x) = \gamma(x)(H_1 f)(x)(H_2)(x);$$

$$(f \overset{\gamma}{\underset{F,F,H_1}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)$$

$$\times g(v)\Big[(1-i)e^{-|x-u+v|^2/2} + (1+i)e^{-|x-u-v|^2/2}\Big]\,du\,dv, \tag{3.3}$$

$$\|f \underset{F,F,H_1}{\overset{\gamma}{*}} g\|_1 \le \|f\|_1 \cdot \|g\|_1,$$

$$F(f \underset{F,F,H_1}{\overset{\gamma}{*}} g)(x) = \gamma(x)(Ff)(x)(H_1 g)(x);$$

$$(f \underset{F,F,H_2}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \times g(v)\Big[(1+i)e^{-|x-u+v|^2/2} + (1-i)e^{-|x-u-v|^2/2}\Big]\,du\,dv, \tag{3.4}$$

$$\|f \underset{F,F,H_2}{\overset{\gamma}{*}} g\|_1 \le \|f\|_1 \cdot \|g\|_1,$$

$$F(f \underset{F,F,H_2}{\overset{\gamma}{*}} g)(x) = \gamma(x)(Ff)(x)(H_2 g)(x);$$

$$(f \underset{H_1,H_1,F}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \times g(v)\Big[e^{-|x-u-v|^2/2} + e^{-|x-u+v|^2/2} - ie^{-|x+u-v|^2/2} + ie^{-|x+u+v|^2/2}\Big]\,du\,dv, \tag{3.5}$$

$$H_1(f \underset{H_1,H_1,F}{\overset{\gamma}{*}} g)(x) = \gamma(x)(H_1 f)(x)(Fg)(x);$$

$$(f \underset{H_1,H_2,F}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \times g(v)\Big[ie^{-|x-u+v|^2/2} - ie^{-|x-u-v|^2/2} + e^{-|x+u-v|^2/2} + e^{-|x+u+v|^2/2}\Big]\,du\,dv, \tag{3.6}$$

$$H_1(f \underset{H_1,H_2,F}{\overset{\gamma}{*}} g)(x) = \gamma(x)(H_2 f)(x)(Fg)(x);$$

$$(f \underset{H_1,F,H_1}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \times g(v)\Big[e^{-|x-u-v|^2/2} - ie^{-|x-u+v|^2/2} + e^{-|x+u-v|^2/2} + ie^{-|x+u+v|^2/2}\Big]\,du\,dv, \tag{3.7}$$

$$H_1(f \overset{\gamma}{\underset{H_1,F,H_1}{*}} g)(x) = \gamma(x)(Ff)(x)(H_1 g)(x);$$

$$(f \overset{\gamma}{\underset{H_1,F,H_2}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \times g(v)\Big[e^{-|x-u+v|^2/2} - ie^{-|x-u-v|^2/2} + ie^{-|x+u-v|^2/2} + e^{-|x+u+v|^2/2}\Big]\,du\,dv, \tag{3.8}$$

$$H_1(f \overset{\gamma}{\underset{H_1,F,H_2}{*}} g)(x) = \gamma(x)(Ff)(x)(H_2 g)(x);$$

$$(f \overset{\gamma}{\underset{H_1,F,F}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u) \times g(v)\Big[(1-i)e^{-|x-u-v|^2/2} + (1+i)e^{-|x+u+v|^2/2}\Big]\,du\,dv, \tag{3.9}$$

$$H_1(f \overset{\gamma}{\underset{H_1,F,F}{*}} g)(x) = \gamma(x)(Ff)(x)(Fg)(x).$$

This theorem can be proved in the same way as Theorem 2.1. For brevity, let us only give a sketch of the proof. The norm inequalities can be proved similar to the respective inequality in (2.1). So, it is sufficient to give an outline of the proof of the factorization identities.

Proof of the factorization identity in (3.1). By using convolution (2.1), we have

$$\gamma(x)(H_2 f)(x)(H_1)(x) = \gamma(x)(H_1 f(-u))(x)(H_1)(x) = F(f(-u) \overset{\gamma}{\underset{F,H_1,H_1}{*}} g)(x),$$

which is, by replacing the variable $-u$ by u, equal to $F(f \overset{\gamma}{\underset{F,H_2,H_1}{*}} g)(x)$, which gives the desired identity. The proof of (3.2) is analogous.

Proof of the factorization identity in (3.3). By using convolution (2.3), we have

$$\gamma(x)(Ff)(x)(H_1 g)(x) = \gamma(x)(H_1 g)(x)(Ff)(x) = F(g \overset{\gamma}{\underset{F,H_1,F}{*}} f)(x),$$

which is, permuting u and v, equal to $F(f \overset{\gamma}{\underset{F,F,H_1}{*}} g)(x)$, which proves (3.3).

The generalized convolutions from (3.4) to (3.8) may be proved analogously.

Proof of the factorization identity in (3.9). We have

$$\begin{aligned}
&\gamma(x)(Ff)(x)(Fg)(x)\\
&= \frac{\gamma(x)}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)(\cos xu - i\sin xu)\\
&\qquad\qquad \times (\cos xv - i\sin xv)\,du\,dv\\
&= \frac{\gamma(x)}{(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)\\
&\times g(v)\Big[\cos xu\cos xv - i\cos xu\sin xv - i\sin xu\cos xv - \sin xu\sin xv\Big]dudv\\
&= \frac{(1-i)\gamma(x)}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)[\cos x(u+v)+\sin x(u+v)]\,du\,dv\\
&\quad + \frac{(1+i)\gamma(x)}{2(2\pi)^d}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)g(v)[\cos x(-u-v)+\sin x(-u-v)]\,du\,dv\\
&= \frac{1}{2(2\pi)^{(3d)/2}}\int_{\mathbf{R}^d}(\cos xy+\sin xy)\,dy\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(u)\\
&\qquad\qquad \times g(v)\Big[(1-i)e^{-|y-u-v|^2/2}+(1+i)e^{-|y+u+v|^2/2}\Big]\,du\,dv\\
&= H_1(f \overset{\gamma}{\underset{H_1,F,F}{*}} g)(x),
\end{aligned}$$

as desired.

Remark 3.1. Each one of the convolution transforms in Theorems 2.1 and 3.1 is a convolution related to the Gaussian function. If f is given in $L^1(\mathbf{R}^d)$, then those transforms are continuous linear operators from $L^1(\mathbf{R}^d)$ onto themselves, and their norms are equal to $\|f\|_1$. Therefore, those convolution transforms may be an object of study, and may bring about further applications.

Finally, Theorem 3.2 may be proved similarly as Theorem 2.2.

Theorem 3.2. *The space* $L^1(\mathbf{R}^d)$*, equipped with each of the convolutive multiplications from* (3.1) *to* (3.9)*, becomes a normed ring having no unit. Only the convolution in* (3.9) *is commutative and the others are non-commutative.*

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