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EXISTENCE RESULTS FOR NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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ABSTRACT. In this paper we discuss the existence of mild, strict and classical solutions for a class of abstract integro-differential equations in Banach spaces. Some applications to ordinary and partial integro-differential equations are considered.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space. In this paper we study the existence of mild, strict and classical solutions for a class of abstract neutral integro-differential equations with infinite delay described in the form

$$\frac{d}{dt}\left[x(t) + \int_{-\infty}^{t} N(t-s)x(s)\,ds\right] = Ax(t) + \int_{-\infty}^{t} B(t-s)x(s)\,ds + f(t, x_t), \quad t \in [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where $A, B(t)$, $t \geq 0$, are closed linear operators defined on a common domain $D(A)$ which is dense in X, $N(t)$ $(t \geq 0)$ are bounded linear operators on X, the history $x_t : (-\infty, 0] \to X$, given by $x_t(\theta) = x(t+\theta)$,

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belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f : I \times \mathcal{B} \to X$ is an appropriate function.

Our purpose in this paper is to establish the existence of solutions for the system (1.1)–(1.2) without the use of many of the strong restrictions considered in the literature. To clarify our remarks and review briefly the associated literature, we introduce the abstract neutral functional differential equation

$$\frac{d}{dt}(x(t) + g(t, x_t)) = Ax(t) + f(t, x_t), \quad t \in I = [0, a], \tag{1.3}$$

$$x_0 = \varphi, \tag{1.4}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and $f, g : [0, a] \times X \to X$ are suitable functions.

In Datko [**13**] and Adimy and Ezzinbi [**2**] some linear neutral systems similar to (1.3)–(1.4) are studied under the strong assumption that the function g is $D(A)$-valued and Ag is continuous. If A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ (the case studied by Datko), this assumption arises from the treatment of the associated integral equation

$$\begin{aligned} u(t) = T(t)(\varphi(0) + g(0, \varphi)) - g(t, u_t) - \int_0^t AT(t-s)g(s, u_s)\,ds \\ + \int_0^t T(t-s)f(s, u_s)\,ds, \end{aligned}$$

which require the integrability of the function $s \to AT(t-s)g(s, u_s)$ on $[0, t)$ for all $t \in [0, a]$. We note that, except trivial cases, the operator function $AT(\cdot)$ is not integrable in the operator topology on $[0, b]$, for $b > 0$. The same reason explains the use of a similar assumption in [**2**], which presents the case where A is a Hille-Yosida type operator.

In the papers [**24, 25, 26**] the system (1.3)–(1.4) was studied under a more general and minus restrictive assumption which can be described in the form

($\mathbf{H_g}$) There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X and $H \in L^1([0, a])$ such that $g \in C([0, a] \times \mathcal{B}, Y)$ and $\|AT(t)y\| \leq H(t)\|y\|_Y$ for all $t \geq 0$ and every $y \in Y$.

The condition $(\mathbf{H_g})$ is verified in several situations, for example in the case when $(T(t))_{t\geq 0}$ is an analytic semigroup and Y is an interpolation space between X and $D(A)$. However, it remains an important restriction on the system.

In [**4, 5, 6, 14**] (among several works) an alternative assumption has been used. In these publications it is assumed that the set $\{AT(t) : t \in (0,a]\}$ is bounded in operator topology and that $T(t)$ is compact for all $t > 0$. However, as was pointed out in [**24**], these conditions are valid if and only if A is bounded and $\dim X < \infty$, which restricts the applications to ordinary differential equations.

To finish this brief analysis on the associated literature, we note that in [**28**] the existence of solutions is studied for a class of neutral systems in the form

$$\frac{d}{dt}\left[x(t) + g(t, x(t-r_1))\right] = Ax(t) + f(t, x_t), \quad t \in [0,a],$$
$$x_0 = \varphi \in C([-r,0];X),$$

where $r_1 < r$, $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup and $g : [0,a] \times X \to X$, $f : [0,a] \times C([-r,0];X) \to X$ are appropriate functions. The results in [**28**] are proved assuming different "temporal" and "spatial" regularity type conditions on the function $\widetilde{g} : [0,r_1] \to X$ given by $\widetilde{g}(t) = g(t, \varphi(t-r))$. We note that the results in [**28**] are proved without using the above described restrictions. However, it is easy to confer that these results are not applicable for the system (1.1)–(1.2).

The purpose of this paper is to study the existence of solutions for the neutral system (1.1)–(1.2) without assuming the above restrictions. To this end, we study the existence and qualitative properties of a resolvent operator for the integro-differential system

$$\frac{d}{dt}\left[x(t) + \int_0^t N(t-s)x(s)\,ds\right] = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \quad t \geq 0, \tag{1.5}$$

$$x(0) = z \in X. \tag{1.6}$$

An extensive literature exists related to the resolvent operator for integro-differential equations. We refer the reader to the book by Gripenberg, Londen and Staffans [**20**] for the case where the underlying space X has finite dimension. For abstract integro-differential equations described on infinite dimensional spaces, we cite the book by Prüss [**34**] and the papers Da Prato et al. [**11, 12**], Grimmer et al. [**17, 18, 19**] and Lunardi [**30, 31**].

Next we review some motivations for the study of neutral functional differential equations. The literature related to ordinary neutral functional differential equations is very extensive, and we refer the reader to the Hale and Lunel book [**23**] and the references therein. Partial neutral differential equations arise, for instance, in the transmission line theory. Wu and Xia have shown in [**35**] that a ring array of identical resistively coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibits various types of discrete waves. By taking a natural limit, they obtain from this system of neutral equations a scalar partial neutral differential equation defined on the unit circle. Such a partial neutral differential equation is also investigated by Hale in [**22**] under the more general form

$$\begin{gathered}\frac{d}{dt}\mathcal{D}u_t(x) = \frac{\partial^2}{\partial x^2}\mathcal{D}u_t(x) + f(u_t)(x), \quad t \geq 0,\\ u_0 = \varphi \in C([-r,0]; C(S^1; \mathbf{R})),\end{gathered}$$

where $\mathcal{D}(\psi)(s) := \psi(0)(s) - \int_{-r}^{0}[d\eta(\theta)]\psi(\theta)(s)$ for $s \in S^1$, $\psi \in C([-r,0]; C(S^1; \mathbf{R}))$ and η is a function of bounded variation.

Abstract neutral differential equations appear in the theory of heat conduction. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature u and on its gradient ∇u. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [**21, 32**], the internal energy and the heat flux are described as functionals of u and u_x. The next integro-differential system, see [**7, 9, 10, 30**], has been frequently used to describe this phenomena,

$$\frac{d}{dt}\left[u(t,x)+\int_{-\infty}^{t}k_1(t-s)u(s,x)\,ds\right]=c\triangle u(t,x)$$
$$+\int_{-\infty}^{t}k_2(t-s)\triangle u(s,x)\,ds,$$
$$u(t,x)=0,\quad x\in\partial\Omega.$$

In this system, $\Omega\subset\mathbf{R}^n$ is open, bounded and with smooth boundary, $(t,x)\in[0,\infty)\times\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant and $k_i:\mathbf{R}\to\mathbf{R}$, $i=1,2$, are the internal energy and the heat flux relaxation, respectively. By assuming that the solution u is known on $(-\infty,0]$, we can represent this system into of an abstract system with unbounded delay described as (1.3)–(1.4)

We also note that an extensive literature exists on ordinary neutral differential equations in the theory of population dynamics, see for instance [**15**]. If in these works we consider the spatial diffusion phenomena, which arises in the natural tendency of biological populations to migrate from a high population density region to a region with minor density, then it is possible to obtain partial neutral differential systems of the form

$$\frac{d}{dt}\left[u(t,\xi)+g(t,u(t-r_1,\xi))\right]=\triangle u(t,\xi)+f(t,u(t-r_1,\xi)).$$

This paper has four sections. In Section 2 we study the existence and qualitative properties of a resolvent operator for the integro-differential system (1.5)–(1.6). In the same section, the existence of mild, strict and classical solutions for the nonhomogeneous equation associated to (1.5)–(1.6) is discussed. In Section 3, we establish some results on the existence of $\mathcal{S}$-mild, strict and classical solutions for the neutral system (1.1)–(1.2). In the last section some applications are considered.

To finish this section, we present some notations used in this paper. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z,W)$ the space of bounded linear operators from Z into W endowed with the operator norm, and we write simply $\mathcal{L}(Z)$ when $Z=W$. By $\mathcal{R}(Q)$ we denote the range of a map Q, and for a closed linear operator $P:D(P)\subseteq Z\to W$, the notation $[D(P)]$ represents the domain of P endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Pz\|_W$, $z \in D(P)$. In the case $Z = W$, the notation $\rho(P)$ stands for the resolvent set of P and $R(\lambda, P) = (\lambda I - P)^{-1}$ is the resolvent operator of P. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ and $S : [0, \infty) \to \mathcal{L}(Z, W)$, the notation $\widehat{K}$ denotes the Laplace transform of K and $S * K$ the convolution between S and K, which is defined by $S * K(t) = \int_0^t S(t-s)K(s)\,ds$.

2. Resolvent operators. In this section, we study the existence and qualitative properties of a resolvent operator for the integro-differential abstract Cauchy problem

$$\frac{d}{dt}\left[x(t) + \int_0^t N(t-s)x(s)\,ds\right] = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \tag{2.1}$$

$$x(0) = z \in X. \tag{2.2}$$

We introduce the following concept of resolvent operator for problem (2.1)–(2.2).

Definition 2.1. A one-parameter family of bounded linear operators $(\mathcal{R}(t))_{t\geq 0}$ on X is called a resolvent operator of (2.1)–(2.2) if the following conditions are verified.

(a) The function $\mathcal{R}(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous, exponentially bounded and $\mathcal{R}(0)x = x$ for all $x \in X$.

(b) For $x \in D(A)$, $\mathcal{R}(\cdot)x \in C([0, \infty), [D(A)]) \bigcap C^1((0, \infty), X)$, and

$$\frac{d}{dt}\left[\mathcal{R}(t)x + \int_0^t N(t-s)\mathcal{R}(s)x\,ds\right] = A\mathcal{R}(t)x + \int_0^t B(t-s)\mathcal{R}(s)x\,ds, \tag{2.3}$$

$$\frac{d}{dt}\left[\mathcal{R}(t)x + \int_0^t \mathcal{R}(t-s)N(s)x\,ds\right] = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)x\,ds, \tag{2.4}$$

for every $t \geq 0$.

In this work we always assume that the following conditions are verified.

(P1) The operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. In this paper, $M_0 > 0$ and $\vartheta \in (\pi/2, \pi)$ are constants such that $\rho(A) \supseteq \Lambda_\vartheta = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg(\lambda)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0|\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$.

(P2) The function $N : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\widehat{N}(\lambda)x$ is absolutely convergent for $x \in X$ and $\mathrm{Re}\,(\lambda) > 0$. There exist $\alpha > 0$ and an analytical extension of $\widehat{N}(\lambda)$ (still denoted by $\widehat{N}(\lambda)$) to Λ_ϑ such that $\|\widehat{N}(\lambda)\| \leq N_0|\lambda|^{-\alpha}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{N}(\lambda)x\| \leq N_1|\lambda|^{-1}\|x\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $x \in D(A)$.

(P3) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists a $b(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $\mathrm{Re}\,(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Lambda_{\pi/2} \to \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to Λ_ϑ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\|\,\|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

(P4) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, $\widehat{N}(\lambda)(D) \subseteq D(A)$, $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ and $\|\widehat{N}(\lambda)x\|_1 \leq C_2|\lambda|^{-\alpha}\|x\|_1$ for every $x \in D$ and all $\lambda \in \Lambda_\vartheta$.

In the sequel, for $r > 0$ and $\theta \in ((\pi/2), \vartheta)$, $\Lambda_{r,\theta} = \{\lambda \in \mathbf{C}\setminus\{0\} : |\lambda| > r,\, |\arg(\lambda)| < \theta\}$, $\Gamma_{r,\theta}, \Gamma^i_{r,\theta}$, $i = 1, 2, 3$, are the paths $\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}$, $\Gamma^2_{r,\theta} = \{re^{i\xi} : -\theta \leq \xi \leq \theta\}$, $\Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\}$ and $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise. In addition, $\Omega(F)$, $\Omega(G)$ are the sets

$$\Omega(F) = \{\lambda \in \mathbf{C} : F(\lambda) := (\lambda I + \lambda\widehat{N}(\lambda) - A)^{-1} \in \mathcal{L}(X)\},$$
$$\Omega(G) = \{\lambda \in \mathbf{C} : G(\lambda) := (\lambda I + \lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)\}.$$

Next we study some preliminary properties needed to establish existence of a resolvent operator for problem (2.1)–(2.2).

Lemma 2.1. *There exists an* $r_1 > 0$ *such that* $\Lambda_{r_1,\vartheta} \subseteq \Omega(F)$ *and the function* $F : \Lambda_{r_1,\vartheta} \to \mathcal{L}(X)$ *is analytic. Moreover,*

$$F(\lambda) = R(\lambda, A)[I + \lambda\widehat{N}(\lambda)R(\lambda, A)]^{-1}, \tag{2.5}$$

and there exist constants M_i for $i = 1, 2, 3$ such that

$$\|\lambda F(\lambda)\| \leq M_1, \tag{2.6}$$

$$\|\lambda A F(\lambda)x\| \leq M_2 \|x\|_1, \quad x \in D(A), \tag{2.7}$$

$$\|AF(\lambda)\| \leq M_3, \tag{2.8}$$

for every $\lambda \in \Lambda_{r_1,\vartheta}$.

Proof. Since $\|\lambda \widehat{N}(\lambda) R(\lambda, A)\| \leq M_0 \|\widehat{N}(\lambda)\|$ there exists a positive number r_1 such that $\|\lambda \widehat{N}(\lambda) R(\lambda, A)\| \leq 1/2$ for $\lambda \in \Lambda_{r_1,\vartheta}$. Consequently, the operator $I + \lambda \widehat{N}(\lambda) R(\lambda, A)$ has a continuous inverse with $\|(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1}\| \leq 2$. Moreover, for $x \in X$, we have

$$\begin{aligned}(\lambda I &+ \lambda \widehat{N}(\lambda) - A) R(\lambda, A)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1} x \\ &= (I + \lambda \widehat{N}(\lambda) R(\lambda, A))(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1} x = x,\end{aligned}$$

and for $x \in D(A)$,

$$\begin{aligned}R(\lambda, A)&(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1}(\lambda I + \lambda \widehat{N}(\lambda) - A)x \\ &= R(\lambda, A)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1}(I + \lambda \widehat{N}(\lambda) R(\lambda, A))(\lambda I - A)x \\ &= x,\end{aligned}$$

which shows (2.5) and that $\Lambda_{r_1,\vartheta} \subseteq \Omega(F)$. Now, from (2.5) we obtain $\mathcal{R}(F(\lambda)) \subseteq D(A)$,

$$AF(\lambda) = (\lambda R(\lambda, A) - I)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1},$$

the functions $F, AF : \Lambda_{r_1,\vartheta} \to \mathcal{L}(X)$ are analytic, and estimates (2.6) and (2.8) are valid. In addition, for $x \in D(A)$, we can write

$$\begin{aligned}\|AF(\lambda)x\| &\leq \|AR(\lambda, A)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1} x - AR(\lambda, A)x\| \\ &\quad + \|R(\lambda, A)Ax\| \\ &= \|[AR(\lambda, A) - AR(\lambda, A)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))] \\ &\quad \times (I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1} x\| + \|R(\lambda, A)Ax\| \\ &= \|\lambda AR(\lambda, A) \widehat{N}(\lambda) R(\lambda, A)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1} x\| \\ &\quad + \|R(\lambda, A)Ax\| \\ &= \|\lambda AR(\lambda, A)(I + \lambda \widehat{N}(\lambda) R(\lambda, A))^{-1} \widehat{N}(\lambda) R(\lambda, A)x\| \\ &\quad + \|R(\lambda, A)Ax\| \\ &\leq \frac{C}{|\lambda|}\|x\|_1 + \frac{M_0}{|\lambda|}\|Ax\|,\end{aligned}$$

for $|\lambda|$ sufficiently large. This proves (2.7) and completes the proof. $\square$

We need a similar result for the operators $G(\lambda)$.

Lemma 2.2. *There exists a constant $r_2 \geq r_1$ such that $\Lambda_{r_2,\vartheta} \subseteq \Omega(G)$ and*

$$G(\lambda) = F(\lambda)[I - \widehat{B}(\lambda)F(\lambda)]^{-1}, \tag{2.9}$$

for $\lambda \in \Lambda_{r_2,\vartheta}$. Moreover, the following properties hold:

(a) *The function $G : \Lambda_{r_2,\vartheta} \to \mathcal{L}(X)$ is analytic, and there exists an $M_4 > 0$ such that*

$$\|\lambda G(\lambda)\| \leq M_4, \quad \lambda \in \Lambda_{r_2,\vartheta}. \tag{2.10}$$

(b) *The space $\mathcal{R}(G(\lambda)) \subseteq D(A)$, the function $AG : \Lambda_{r_2,\vartheta} \to \mathcal{L}(X)$ is analytic, and there exist constants M_5, M_6 such that*

$$\|\lambda AG(\lambda)x\| \leq M_5\|x\|_1, \quad x \in D(A),\ \lambda \in \Lambda_{r_2,\vartheta}, \tag{2.11}$$

$$\|AG(\lambda)\| \leq M_6, \quad \lambda \in \Lambda_{r_2,\vartheta}. \tag{2.12}$$

Proof. For $\lambda \in \Lambda_{r_1,\vartheta}$ and $x \in X$,

$$\begin{aligned}\|\widehat{B}(\lambda)F(\lambda)x\| &\leq \|\widehat{B}(\lambda)R(\lambda, A)\| \\ &\qquad \times \|(I + \lambda\widehat{N}(\lambda)R(\lambda, A))^{-1}\|\,\|x\| \\ &\leq 2\|\widehat{B}(\lambda)R(\lambda, A)\|\,\|x\|.\end{aligned}$$

Hence we can assume that $\|\widehat{B}(\lambda)F(\lambda)\| \leq 1/2$ for $\lambda \in \Lambda_{r_2,\vartheta}$ and some $r_2 \geq r_1$. Consequently, $(I - \widehat{B}(\lambda)F(\lambda))$ has a continuous inverse and $\|(I - \widehat{B}(\lambda)F(\lambda))^{-1}\| \leq 2$ for every $\lambda \in \Lambda_{r_2,\vartheta}$. We have, for $x \in X$,

$$\begin{aligned}(\lambda I + \lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))F(\lambda)(I - \widehat{B}(\lambda)F(\lambda))^{-1}x \\ = (I - \widehat{B}(\lambda)F(\lambda))(I - \widehat{B}(\lambda)F(\lambda))^{-1}x = x.\end{aligned}$$

Similarly, for $x \in D(A)$, we get

$$\begin{aligned} & F(\lambda)(I-\widehat{B}(\lambda)F(\lambda))^{-1}(\lambda I+\lambda\widehat{N}(\lambda)-A-\widehat{B}(\lambda))x \\ & \quad = F(\lambda)(I-\widehat{B}(\lambda)F(\lambda))^{-1}(I-\widehat{B}(\lambda)F(\lambda))(\lambda I+\lambda\widehat{N}(\lambda)-A)x \\ & \quad = x, \end{aligned}$$

which permits us to conclude that $\Lambda_{r_2,\vartheta} \subseteq \Omega(G)$ and the relation (2.9) holds. Moreover, combining this representation with (2.6) and (2.8) we obtain (2.10) and (2.12), respectively.

On the other hand, from (2.9) we get $G(\lambda) = [I + F(\lambda)(I - \widehat{B}(\lambda)F(\lambda))^{-1}\widehat{B}(\lambda)]F(\lambda)$. For $x \in D(A)$, applying the above expression we can write

$$\begin{aligned} & AG(\lambda)x \\ & = AF(\lambda)x + AF(\lambda)(I-\widehat{B}(\lambda)F(\lambda))^{-1}\widehat{B}(\lambda)R(\lambda,A)(\lambda I-A)F(\lambda)x, \end{aligned}$$

and using (2.7) and (2.8), we estimate

$$\|AG(\lambda)x\| \leq \frac{M_2}{|\lambda|}\|x\|_1 + \frac{C}{|\lambda|}\|x\| \leq \frac{M_5}{|\lambda|}\|x\|_1$$

which proves (2.11) and completes the proof. $\square$

Remark 2.1. If $\mathcal{R}(\cdot)$ is a resolvent operator for (2.1)–(2.2), it follows from (2.4) that $\widehat{\mathcal{R}}(\lambda)(\lambda I + \lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))x = x$ for all $x \in D(A)$. Applying Lemma 2.2 and the properties of the Laplace transform we conclude that $\mathcal{R}(\cdot)$ is the unique resolvent operator for (2.1)–(2.2).

In the remainder of this section, r, θ are numbers such that $r > r_2$ and $\theta \in (\pi/2, \vartheta)$. Moreover, we denote by C a generic constant that represent any of the constants involved in the statements of Lemmas 2.1 and 2.2 as well as any other constant that arises in the estimate that follows. We now define the operator family $(\mathcal{R}(t))_{t\geq 0}$ by

$$\mathcal{R}(t) = \begin{cases} 1/(2\pi i)\int_{\Gamma_{r,\theta}} e^{\lambda t}G(\lambda)\,d\lambda & t>0, \\ I & t=0. \end{cases} \tag{2.13}$$

We will next establish that $(\mathcal{R}(t))_{t\geq 0}$ is a resolvent operator for (2.1)–(2.2).

Lemma 2.3. *The function $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$.*

Proof. If $t > 1$, from (2.13) and (2.10) we get

$$\begin{aligned}\|\mathcal{R}(t)\| &\leq \frac{C}{\pi}\int_r^\infty e^{ts\cos\theta}\,\frac{ds}{s} + \frac{C}{2\pi r}\int_{-\theta}^{\theta} e^{tr\cos\xi}\,d\xi \\ &\leq \frac{C}{\pi r|\cos\theta|} + \frac{C\theta}{\pi r}e^{rt}.\end{aligned}$$

On the other hand, using that $G(\cdot)$ is analytic on $\Lambda_{r,\theta}$, for $t \in (0,1)$ we get

$$\begin{aligned}\|\mathcal{R}(t)\| &= \left\|\frac{1}{2\pi i}\int_{\Gamma_{(r/t),\theta}} e^{\lambda t}G(\lambda)\,d\lambda\right\| \\ &\leq \frac{C}{\pi}\int_{r/t}^\infty e^{st\cos\theta}\,\frac{ds}{s} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}\,d\xi \\ &\leq \frac{C}{\pi}\int_{r}^\infty e^{u\cos\theta}\,\frac{du}{u} + \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}\,d\xi \\ &\leq \frac{C}{\pi r|\cos\theta|} + \frac{C\theta}{\pi}e^{r}\end{aligned}$$

which implies that $\{\mathcal{R}(t) : t \in (0,1)\}$ is bounded in $\mathcal{L}(X)$. This completes the proof. □

In the sequel, $\omega > 0$ is such that $\|\mathcal{R}(t)\| \leq Ce^{\omega t}$ for all $t \geq 0$. Arguing as in the proof of Lemma 2.3, but using (2.11) instead of (2.10), we can prove the following result.

Lemma 2.4. *The operator valued function $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$.*

Proof. From Lemma 2.2, $G : \Lambda_{r,\vartheta} \to \mathcal{L}([D(A)])$ is analytic and $\|G(\lambda)\|_1 \leq C|\lambda|^{-1}$. Now, the assertion can be proved arguing as in the proof of Lemma 2.3 but using (2.11) instead of (2.10). □

Lemma 2.5. *The function $\mathcal{R} : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous.*

Proof. It is clear from (2.13) that $\mathcal{R}(\cdot)x$ is continuous at $t > 0$ for every $x \in X$. Next we establish the continuity at $t = 0$. By using that $1/(2\pi i)\int_{\Gamma_{r,\theta}} \lambda^{-1}e^{\lambda t}\,d\lambda = 1$, for $x \in D(A)$ and $0 < t \leq 1$ we get

$$\begin{aligned}\mathcal{R}(t)x - x &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} \left(e^{\lambda t}G(\lambda)x - \lambda^{-1}e^{\lambda t}x\right)\,d\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{-1}G(\lambda)(\lambda\widehat{N}(\lambda) - A \\ &\quad - \widehat{B}(\lambda))x\,d\lambda.\end{aligned}$$

Furthermore, it follows from (2.10) and assumptions (P2) and (P3) that

$$\|e^{\lambda t}\lambda^{-1}G(\lambda)(\lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))x\| \leq e^r C\left(\frac{1}{|\lambda|^{1+\alpha}} + \frac{1}{|\lambda|^2}\right)\|x\|_1,$$

for $\lambda \in \Gamma_{r,\theta}$. From the Lebesgue dominated convergence theorem we infer that

$$\lim_{t\to 0^+}(\mathcal{R}(t)x - x) = -\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} \lambda^{-1}G(\lambda)(\lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))x\,d\lambda. \tag{2.14}$$

Now let $C_{L,\theta}$ be the curve $Le^{i\xi}$ for $-\theta \leq \xi \leq \theta$. From the Cauchy's theorem we obtain that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} \lambda^{-1}G(\lambda)(\lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))x\,d\lambda \\ &\qquad = \lim_{L\to\infty}\frac{1}{2\pi i}\int_{C_{L,\theta}} \lambda^{-1}G(\lambda)(\lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))x\,d\lambda.\end{aligned}$$

Combining this equality with the estimate

$$\left\|\int_{C_{L,\theta}} \lambda^{-1}G(\lambda)(\lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))x\,d\lambda\right\| \leq C\theta\left(\frac{1}{L^\alpha} + \frac{1}{L}\right)\|x\|_1,$$

we can affirm that $\lim_{t\to 0^+}(\mathcal{R}(t)x - x) = 0$ for all $x \in D(A)$, which completes the proof since $D(A)$ is dense in X and $\mathcal{R}(\cdot)$ is bounded on $[0, 1]$. $\square$

The following two results can be proved with an argument similar to the one used in the proof of the preceding lemmas. For the sake of brevity we include only an outline of the proof.

Lemma 2.6. *The function $\mathcal{R} : [0, \infty) \to \mathcal{L}([D(A)])$ is strongly continuous.*

Proof. It follows from (2.11) that the integral in

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\,\theta}} e^{\lambda t} AG(\lambda)\, d\lambda, \quad t > 0,$$

is absolutely convergent in $\mathcal{L}([D(A)], X)$ and defines a linear operator $S(t) \in \mathcal{L}([D(A)], X)$. Using that A is closed, we can affirm that $S(t) = A\mathcal{R}(t)$. For $x \in D$, proceeding as in the proof of Lemma 2.5, we have

$$A\mathcal{R}(t)x - Ax = -\frac{1}{2\pi i} \int_{\Gamma_{r,\,\theta}} e^{\lambda t} \lambda^{-1} AG(\lambda)(\lambda \widehat{N}(\lambda) - A - \widehat{B}(\lambda))x\, d\lambda.$$

Using now that $(\lambda \widehat{N}(\lambda) - A - \widehat{B}(\lambda))x \in D(A)$, the inequality (2.11), Lemma 2.4, the assumption (P4) and proceeding as in the proof of Lemma 2.5, we can conclude that $A\mathcal{R}(t)x - Ax \to 0$ as $t \to 0$.

The above remarks show that $\|\mathcal{R}(t)x - x\|_1 \to 0$ as $t \to 0$ for all $x \in D(A)$, since D is dense in $[D(A)]$ and $\mathcal{R}(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$. □

Next we set $\delta = \min\{\vartheta - (\pi/2), \pi - \vartheta\}$.

Lemma 2.7. *The function $\mathcal{R} : (0, \infty) \to \mathcal{L}(X)$ has an analytic extension to Λ_δ, and*

$$\mathcal{R}'(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \lambda e^{\lambda z} G(\lambda)\, d\lambda, \quad z \in \Lambda_\delta. \tag{2.15}$$

Proof. It is not difficult to see that the integral $1/(2\pi i) \int_{\Gamma_{r,\theta}} e^{\lambda z} G(\lambda)\, d\lambda$, is absolutely convergent in $\mathcal{L}(X)$ for $|\arg z| < \delta$. This property allows us to define the extension $\mathcal{R}(z)$ by this integral. Similarly, the integral on the right hand side of (2.15) is also absolutely convergent in $\mathcal{L}(X)$ for $|\arg z| < \delta$, which implies that $\mathcal{R}'(z)$ verifies (2.15). $\square$

Lemma 2.8. *For every* $\lambda \in \mathbf{C}$ *with* $\mathrm{Re}\,(\lambda) > \omega$, $\widehat{\mathcal{R}}(\lambda) = G(\lambda)$.

Proof. Using that $G(\cdot)$ is analytic on $\Lambda_{r,\theta}$, and that the integrals involved in the calculus that follows are absolutely convergent, we have

$$\begin{aligned}\widehat{\mathcal{R}}(\lambda) &= \int_0^\infty e^{-\lambda t}\mathcal{R}(t)\,dt\\ &= \int_0^\infty \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}} e^{-(\lambda-\gamma)t}G(\gamma)\,d\gamma\,dt\\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\,\theta}} (\lambda-\gamma)^{-1}G(\gamma)\,d\gamma\\ &= \lim_{L\to\infty}\frac{1}{2\pi i}\int_{C_{L,\,\theta}} (\lambda-\gamma)^{-1}G(\gamma)\,d\gamma\\ &= G(\lambda). \qquad \square\end{aligned}$$

Theorem 2.1. *The function* $\mathcal{R}(\cdot)$ *is a resolvent operator for the system* (2.1)–(2.2).

Proof. Let $x \in D(A)$. From Lemma 2.8, for $\mathrm{Re}\,(\lambda) > \omega$, $\widehat{\mathcal{R}}(\lambda)[\lambda I + \lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda)]x = x$, which implies

$$\widehat{\mathcal{R}}(\lambda)x = \frac{1}{\lambda}x - \widehat{\mathcal{R}}(\lambda)\widehat{N}(\lambda)x + \frac{1}{\lambda}\widehat{\mathcal{R}}(\lambda)A + \frac{1}{\lambda}\widehat{\mathcal{R}}(\lambda)\widehat{B}(\lambda)x.$$

Applying [**3**, Proposition 1.6.4, Corollary 1.6.5] we get

$$\begin{aligned}\mathcal{R}(t)x = x &- \int_0^t \mathcal{R}(t-s)N(s)x\,ds + \int_0^t \mathcal{R}(s)Ax\,ds\\ &+ \int_0^t\int_0^s \mathcal{R}(s-\xi)B(\xi)x\,d\xi\,ds\end{aligned}$$

which in turn implies that

$$\begin{aligned}\mathcal{R}(t)x &+ \int_0^t \mathcal{R}(t-s)N(s)x\,ds \\ &= x + \int_0^t \mathcal{R}(s)Ax\,ds + \int_0^t \int_0^s \mathcal{R}(s-\xi)B(\xi)x\,d\xi\,ds\end{aligned}$$

and

$$\frac{d}{dt}\left[\mathcal{R}(t)x + \int_0^t \mathcal{R}(t-s)N(s)x\,ds\right] = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)x\,ds.$$

Moreover, by Lemma 2.7 we infer that $\mathcal{R}(\cdot)x \in C^1((0,\infty), X)$.

Arguing as above but using the equality $[\lambda I + \lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda)]\widehat{\mathcal{R}}(\lambda)x = x$, we obtain that (2.3) holds. The proof is now complete. □

Lemma 2.9. *Let $x \in D(A)$. Then the function $f(t) = \int_0^t \mathcal{R}(t-s)N(s)x\,ds$ is continuously differentiable on $(0,\infty)$. Furthermore, if $N(\cdot)$ is a bounded variation function on an interval $[0,\varepsilon]$ for some $\varepsilon > 0$ and $x \in D$, then f is continuously differentiable on $[0,\infty)$.*

Proof. Let $g(t) = 1/(2\pi i)\int_{\Gamma_{r,\theta}} e^{\lambda t} G(\lambda)\widehat{N}(\lambda)x\,d\lambda$. It follows from (P2) that this integral is absolutely convergent. Moreover, since $\widehat{g}(\lambda) = G(\lambda)\widehat{N}(\lambda)x = \widehat{\mathcal{R}}(\lambda)\widehat{N}(\lambda)x = \widehat{f}(\lambda)$ for $\operatorname{Re}(\lambda) > \omega$ and $g'(t) = 1/(2\pi i)\int_{\Gamma_{r,\theta}} \lambda e^{\lambda t} G(\lambda)\widehat{N}(\lambda)x\,d\lambda$ for $t > 0$, we can conclude that $g(t) = f(t)$ for $t \geq 0$ and $f \in C^1((0,\infty), X)$.

Now let $x \in D$. From the definition of $f(\cdot)$ we obtain that $f'(0) = N(0)x$. Furthermore, since $\|\widehat{N}(\lambda)x\| \leq C/|\lambda|^{\alpha}\|x\|_1$ and

$$\begin{aligned}\|(\lambda G(\lambda) - I)\widehat{N}(\lambda)x\| &= \|(\lambda\widehat{N}(\lambda) - \widehat{B}(\lambda) - A)G(\lambda)\widehat{N}(\lambda)x\| \\ &\leq \frac{C}{|\lambda|^{1+\alpha}}\|x\|_1,\end{aligned}$$

by using Cauchy's theorem we can modify the integration path, and applying the complex inversion theorem for the Laplace transform, we can write

$$\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda G(\lambda)-I)\widehat{N}(\lambda)x\,d\lambda = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}(\lambda G(\lambda)-I)\widehat{N}(\lambda)x\,d\lambda = f'(t)-N(t)x,$$

for $a \in \mathbf{R}$ sufficiently large and $t < \varepsilon$. Since the function $(\lambda G(\lambda) - I)\widehat{N}(\lambda)x$ is absolutely integrable, from the Lebesgue dominated convergence theorem we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}(\lambda G(\lambda)-I)\widehat{N}(\lambda)x\,d\lambda = \lim_{t\to 0^+} f'(t)-N(0)x.$$

Repeating the preceding argument, and taking the limit in the above expression as a goes to infinity, we obtain that $\lim_{t\to 0^+} f'(t)-N(0)x = 0$, which completes the proof that f' is continuous on $[0,\infty)$. □

Remark 2.2. It follows from Lemma 2.9, and modifying slightly the argument used in the proof of Theorem 2.1, that if $N(\cdot)$ is a bounded variation function on an interval $[0,\varepsilon]$ for some $\varepsilon > 0$ and $x \in D$, then $\mathcal{R}(\cdot)x \in C^1([0,\infty),X)$. This property leads us to introduce the space E consisting of vectors $x \in X$ such that $\mathcal{R}(\cdot)x \in C([0,\infty),[D(A)]) \cap C^1([0,\infty),X)$. It is clear that $E \subseteq D(A)$ and $d/dt\mathcal{R}(t)x|_{t=0} = Ax - N(0)x$ for $x \in E$.

We next denote by $[D(-A)^\beta]$ the domain of the operator $(-A)^\beta$ endowed with the graph norm $\|\cdot\|_\beta$. To simplify the exposition, with respect to the fractional powers of A, we assume that $A^{-1} \in \mathcal{L}(X)$. It follows from [**33**, Theorem 2.6.10] that

$$\|(-A)^\beta R(\lambda,A)\| \le \frac{C_\beta}{|\lambda|^{1-\beta}}, \quad 0<\beta<1,\ \lambda\in\Lambda_\vartheta. \tag{2.16}$$

We introduce the following condition.

(P5) The operator $\widehat{N}(\lambda) : [D(-A)^\beta] \to [D(-A)^\beta]$ for $\lambda \in \Lambda_\vartheta$, and $\partial\widehat{N}(\lambda)\partial_\beta \to 0$ as $|\lambda| \to \infty$ uniformly for $\lambda \in \Lambda_\vartheta$.

Proposition 2.1. *Let condition* (P5) *hold. Then there exist constants* $C, r > 0$ *such that*

$$\max\{\|AF(\lambda)x\|, \|AG(\lambda)x\|, \|\widehat{B}(\lambda)G(\lambda)x\|\} \leq C|\lambda|^{-\beta}\|(-A)^{\beta}x\|,$$

for every $x \in [D(-A)^{\beta}]$ *and* $\lambda \in \Lambda_{r,\theta}$.

Proof. Let $x \in [D(-A)^{\beta}]$. It follows from (2.5) that

$$\begin{aligned}
&AF(\lambda)x \\
&= AR(\lambda, A)[I + \lambda\widehat{N}(\lambda)R(\lambda, A)]^{-1}x \\
&= -(-A)^{1-\beta}R(\lambda, A)(-A)^{\beta}[I + \lambda\widehat{N}(\lambda)R(\lambda, A)]^{-1}(-A)^{-\beta}(-A)^{\beta}x.
\end{aligned}$$

Since

$$(-A)^{\beta}\lambda\widehat{N}(\lambda)R(\lambda, A)(-A)^{-\beta} = (-A)^{\beta}\widehat{N}(\lambda)(-A)^{-\beta}\lambda R(\lambda, A),$$

combining these relations with (2.16) and assumption (P5), we obtain the desired estimate for $\|AF(\lambda)x\|$. Moreover, from (2.9), we have

$$AG(\lambda)x - AF(\lambda)x = AF(\lambda)[I - \widehat{B}(\lambda)F(\lambda)]^{-1}\widehat{B}(\lambda)A^{-1}AF(\lambda)x$$

and

$$\begin{aligned}
\|AG(\lambda)x\| &\leq \|AG(\lambda)x - AF(\lambda)x\| + \|AF(\lambda)x\| \\
&\leq C\|AF(\lambda)x\| + \|AF(\lambda)x\|,
\end{aligned}$$

from which we can complete the proof. □

Theorem 2.2. *Assume that condition* (P5) *is fulfilled, and* $b(\cdot)$ *is locally bounded on* $(0, \infty)$. *Then for* $\beta \in (0, 1)$ *and* $t > 0$, *the operator* $\mathcal{R}(t) \in \mathcal{L}([D(-A)^{\beta}], [D(A)])$ *with* $\|\mathcal{R}(t)\| \leq Ct^{\beta-1}$ *for a positive constant* C. *Furthermore, equation* (2.3) *holds for* $x \in D((-A)^{\beta})$ *and* $t > 0$.

Proof. The first assertion follows from (2.13) and Proposition 2.1. To show the second assertion, we proceed as in the proof of Theorem 2.1 for $x \in [D(-A)^{\beta}]$ in order to obtain

$$\begin{aligned}
&\mathcal{R}(t)x + \int_0^t N(s)\mathcal{R}(t-s)x\,ds \\
&\qquad = x + \int_0^t A\mathcal{R}(s)x\,ds + \int_0^t \int_0^s B(\xi)\mathcal{R}(s-\xi)x\,d\xi\,ds.
\end{aligned}$$

Proceeding as in [**19**, Theorem 3.3], using the estimate for $\|\widehat{B}(\lambda)G(\lambda)x\|$ in Proposition 2.1, we infer that the function $t \to \int_0^t B(s)\mathcal{R}(t-s)x\,ds$ is continuous for $t > 0$ and that (2.3) holds for $t > 0$. $\square$

Theorem 2.3. *Assume that* $B(t) \in \mathcal{L}([D(-A)^\beta], X)$ *for some* $0 \le \beta < 1$ *and* $\|B(t)\| \le b_\beta(t)$, *where the function* $b_\beta(\cdot) \in L^1_{\text{loc}}(0,\infty)$ *is locally bounded on* $(0,\infty)$. *Then equation* (2.3) *holds for every* $x \in X$ *and* $t > 0$.

Proof. Using expression (2.13), and arguing as in the proof of Lemma 2.3 we obtain $\|(-A)^\beta \mathcal{R}(t)x\| \le Ct^{-\beta}\|x\|$, $t > 0$. Hence, $\|B(t-s)\mathcal{R}(s)x\| \le b_\beta(t-s)Cs^{-\beta}\|x\|$, for all $x \in X$. This implies the function $\int_0^t B(s)\mathcal{R}(t-s)x\,ds$ is continuous for $t > 0$ and that (2.3) holds for $t > 0$. $\square$

2.1. On the non-homogeneous system. In the remainder of this section we discuss existence and regularity of solutions of

(2.17)
$$\frac{d}{dt}\left[x(t)+\int_0^t N(t-s)x(s)\,ds\right] = Ax(t) + \int_0^t B(t-s)x(s)\,ds + f(t), \quad t \in [0,a],$$

(2.18)
$$x(0) = z \in X,$$

where $f \in L^1([0,a], X)$. In the sequel, $\mathcal{R}(\cdot)$ is the operator function defined by (2.13). We begin by introducing the following concept of classical solution.

Definition 2.2. A function $x : [0,b] \to X$, $0 < b \le a$, is called a classical solution of (2.17)–(2.18) on $[0,b]$ if $x \in C([0,b],[D(A)]) \cap C^1((0,b],X)$, the condition (2.18) holds and the equation (2.17) is verified on $[0,a]$. If, further, $x \in C([0,b],[D(A)]) \cap C^1([0,b],X)$ the function x is said to be a strict solution of (2.17)–(2.18) on $[0,b]$.

In Theorem 2.4 below, we establish a variation of constants formula for the solutions of (2.17)–(2.18).

Theorem 2.4. *Let $z \in D(A)$. Assume that $f \in C([0,a],X)$ and $x(\cdot)$ is a classical solution of* (2.17)–(2.18) *on* $[0,a]$. *Then*

$$x(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)\,ds, \quad t \in [0,a]. \tag{2.19}$$

Proof. For $\varepsilon > 0$, we consider $t \geq \varepsilon$. We define

$$\begin{aligned} w(t) &= \mathcal{R}(\varepsilon)x(t-\varepsilon) - \mathcal{R}(t)z - \int_0^{t-\varepsilon} \mathcal{R}(t-s)f(s)\,ds \\ &= \int_0^{t-\varepsilon} \frac{\partial}{\partial s}\left[\mathcal{R}(t-s)x(s)\right]\,ds \\ &\quad - \int_0^{t-\varepsilon} \mathcal{R}(t-s)f(s)\,ds \\ &= \int_0^{t-\varepsilon} \left[-\mathcal{R}'(t-s)x(s) + \mathcal{R}(t-s)x'(s)\right]\,ds \\ &\quad - \int_0^{t-\varepsilon} \mathcal{R}(t-s)f(s)\,ds. \end{aligned}$$

Substituting $\mathcal{R}'(t-s)$ and $x'(s)$ obtained from (2.4) and (2.17), we get

$$\begin{aligned} w(t) = &- \int_0^{t-\varepsilon} \left(\mathcal{R}(t-s)Ax(s) + \int_0^{t-s} \mathcal{R}(t-s-\xi)B(\xi)x(s)\,d\xi\right) ds \\ &+ \int_0^{t-\varepsilon} H'(t-s)\,ds + \int_0^{t-\varepsilon} \mathcal{R}(t-s)Ax(s)\,ds \\ &+ \int_0^{t-\varepsilon} \mathcal{R}(t-s)\int_0^s B(s-\xi)x(\xi)\,d\xi\,ds \\ &- \int_0^{t-\varepsilon} \mathcal{R}(t-s)\frac{\partial}{\partial s}\int_0^s N(s-\xi)x(\xi)\,d\xi\,ds, \\ = &\ 0, \end{aligned}$$

where $H(t) = \int_0^t \mathcal{R}(t-\xi)N(\xi)x(s)\,d\xi$. Taking the limit as $\varepsilon \to 0$ we obtain (2.19). □

An immediate consequence of the above theorem is the uniqueness of classical solutions.

Corollary 2.1. *If u, v are classical solutions of* (2.17)–(2.18) *on* $[0, b]$*, then $u = v$ on* $[0, b]$.

Motivated by (2.19), we introduce the following concept.

Definition 2.3. A function $u \in C([0, a], X)$ is called a mild solution of (2.17)–(2.18) if

$$u(t) = \mathcal{R}(t)z + \int_0^t \mathcal{R}(t-s)f(s)\,ds, \quad ; t \in [0, a].$$

Next we will study several conditions under which a mild solution of (2.17)–(2.18) is a classical solution. We begin with the following lemma.

Lemma 2.10. *Let $V : [0, \infty) \to \mathcal{L}(X)$ be the function defined by $V(t)x = \int_0^t \mathcal{R}(s)x\,ds$. Then $\mathcal{R}(V(t)) \subseteq D(A)$ for all $t \geq 0$ and $AV(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous.*

Proof. Let $x \in D(A)$. From the definition of $\mathcal{R}(\cdot)$ we have that $A\mathcal{R}(\cdot)x$ is continuous on $[0, \infty)$, so that $V(t)x \in D(A)$ and $AV(t)x = \int_0^t A\mathcal{R}(s)x\,ds$. Moreover, from (2.3) we get

$$\begin{aligned} AV(t)x = \mathcal{R}(t)x - x &+ \int_0^t N(t-s)\mathcal{R}(s)x\,ds \\ &- \int_0^t B(t-s)V(s)x\,ds, \end{aligned}$$

and hence,

$$\begin{aligned} \|AV(t)x\| &\leq C_1(t)\|x\| + \int_0^t b(t-s)\|V(s)x\|_1\,ds \\ &\leq C_1(t)\|x\| + \int_0^t b(t-s)\|AV(s)x\|\,ds, \end{aligned}$$

where $C_1(\cdot)$ is a continuous function independent of x and $b(\cdot)$ is the function introduced in condition **(P3)**. From Gronwall-Bellman's lemma we infer that $\|AV(t)x\| \leq C(t)\|x\|$, where $C(\cdot)$ is a continuous function independent of x.

Let $x \in X$ and $(x_n)_{n\in\mathbf{N}}$ be a sequence in $D(A)$ such that $x_n \to x$ as $n \to \infty$. Consequently, $V(t)x_n \to V(t)x$ as $n \to \infty$ and $(AV(t)x_n)_n$ is a Cauchy sequence. Since A is closed, we obtain that $V(t)x \in D(A)$. Moreover, $B(t-s)V(s)$ is a bounded linear operator and $B(t-s)V(s)x_n \to B(t-s)V(s)x$ as $n \to \infty$. In view of $\|B(t-s)V(s)x_n\| \leq b(t-s)\|V(s)x_n\|_1$, from the Lebesgue dominated convergence theorem we can affirm that $\int_0^t B(t-s)V(s)x_n\,ds \to \int_0^t B(t-s)V(s)x\,ds$ as $n \to \infty$.

Using the resolvent equation (2.3) with x_n instead of x, we get

$$AV(t)x_n \longrightarrow \mathcal{R}(t)x - x + \int_0^t N(t-s)\mathcal{R}(s)x\,ds - \int_0^t B(t-s)V(s)x\,ds$$

as $n \to \infty$, which permits us to conclude that

$$AV(t)x = \mathcal{R}(t)x - x + \int_0^t N(t-s)\mathcal{R}(s)x\,ds - \int_0^t B(t-s)V(s)x\,ds.$$

Since $t \mapsto \int_0^t B(t-s)V(s)x\,ds$ is continuous, we infer that $AV(\cdot)x \in C([0,\infty),X)$. This completes the proof. $\square$

Theorem 2.5. *Let $z \in D(A)$ and $f \in C([0,a],[D(A)])$. Then the mild solution $x(\cdot)$ of (2.17)–(2.18) is a classical solution on $[0,a]$. Further, if $z \in E$, then $x(\cdot)$ is a strict solution on $[0,a]$.*

Proof. Let $u : [0,a] \to X$ be the function given by

$$u(t) = \int_0^t \mathcal{R}(t-s)f(s)\,ds. \tag{2.20}$$

It is easy to see that $u \in C([0,a],[D(A)]) \cap C^1([0,a],X)$ and

$$u'(t) = \int_0^t \mathcal{R}'(t-s)f(s)\,ds + f(t), \quad t \in [0,a] \tag{2.21}$$

$$\frac{d}{dt}\left(\int_0^t N(t-s)u(s)\,ds\right) = \int_0^t N(t-s)u'(s)\,ds, \quad t \in [0,a]. \tag{2.22}$$

To abbreviate the expressions that follow, we set $H(t) = \int_0^t N(t-\xi)\mathcal{R}(\xi)f(s)\,d\xi$. Using equalities (2.21), (2.22) and the resolvent equation (2.3) we find that

$$
\begin{aligned}
&\frac{d}{dt}(u + N * u)(t) - Au(t) - (B * u)(t) - f(t)\\
&\quad = u'(t) + \frac{d}{dt}(N * u)(t) - \int_0^t A\mathcal{R}(t-s)f(s)\,ds\\
&\qquad - \int_0^t B(t-s)\int_0^s \mathcal{R}(s-\xi)f(\xi)\,d\xi\,ds - f(t)\\
&\quad = \int_0^t \mathcal{R}'(t-s)f(s)\,ds + \int_0^t N(t-s)u'(s)\,ds\\
&\qquad - \int_0^t A\mathcal{R}(t-s)f(s)\,ds\\
&\qquad - \int_0^t B(t-s)\int_0^s \mathcal{R}(s-\xi)f(\xi)\,d\xi\,ds\\
&\quad = \int_0^t \bigg(A\mathcal{R}(t-s)f(s)\,ds + \int_0^{t-s} B(t-s-\xi)\mathcal{R}(\xi)f(s)\,d\xi\bigg)\,ds\\
&\qquad - \int_0^t H'(t-s)\,ds\\
&\qquad + \int_0^t N(t-s)\bigg(\int_0^s \mathcal{R}'(s-\xi)f(\xi)\,d\xi + f(s)\bigg)\,ds\\
&\qquad - \int_0^t A\mathcal{R}(t-s)f(s)\,ds\\
&\qquad - \int_0^t \int_0^s B(t-s)\mathcal{R}(s-\xi)f(\xi)\,d\xi\,ds\\
&\quad = - \int_0^t \int_0^{t-s} N(t-s-\xi)\mathcal{R}'(\xi)f(s)\,d\xi\,ds\\
&\qquad - \int_0^t N(t-s)f(s)\,ds\\
&\qquad + \int_0^t N(t-s)\int_0^s \mathcal{R}'(s-\xi)f(\xi)\,d\xi\,ds\\
&\qquad + \int_0^t N(t-s)f(s)\,ds,
\end{aligned}
$$

$$= 0,$$

which implies that $u(\cdot)$ is a classical solution of problem (2.17)–(2.18) on $[0, a]$ corresponding to the initial condition $u(0) = 0$. Since the mild solution is $x(t) = \mathcal{R}(t)z + u(t)$, the assertions follow from Theorem 2.1 and Remark 2.2. $\square$

Theorem 2.6. *Let $z \in D(A)$, and let $f \in W^{1,1}([0, a], X)$. Then the mild solution $x(\cdot)$ of problem (2.17)–(2.18) is a classical solution on $[0, a]$. Further, if $z \in E$, then $x(\cdot)$ is a strict solution on $[0, a]$.*

Proof. Proceeding as in the proof of Theorem 2.5, we may assume that $z = 0$. Let $u(\cdot)$ be the function given by (2.20). Applying [**3**, Proposition 1.3.6], we can assert that functions $u(\cdot)$ and $N * u(\cdot)$ are of class C^1 on $[0, a]$ and that

$$u'(t) = \int_0^t \mathcal{R}(t-s) f'(s)\, ds + \mathcal{R}(t) f(0),$$
$$\frac{d}{dt}\left(\int_0^t N(t-s) u(s)\, ds \right) = \int_0^t N(t-s) u'(s)\, ds + N(t) u(0),$$

for each $t \in [0, a]$. Using these expressions, and arguing as in the proof of Theorem 2.5, we can establish that $u(\cdot)$ is a classical solution of problem (2.17)–(2.18) on $[0, a]$ with initial condition $u(0) = 0$. We omit the details. $\square$

Corollary 2.2. *Let $z \in D(A)$ and $f \in C([0, a], X)$. Let $u(\cdot)$ be the mild solution of problem (2.17)–(2.18). If $u \in C([0, a], [D(A)])$, then $u(\cdot)$ is a classical solution on $(0, a]$.*

Proof. Since $\mathcal{R}(\cdot)z \in C([0, a], [D(A)])$, without loss of generality we can assume that $z = 0$ and u is the function given by (2.20). Consequently,

$$\int_0^t u(s)\, ds = \int_0^t V(t-\xi) f(\xi)\, d\xi.$$

Since $AV(\cdot)$ is strongly continuous, from the above equality we get that

$$A \int_0^t u(s)\, ds = \int_0^t AV(t-\xi) f(\xi)\, d\xi.$$

Let now f_n, $n \in \mathbf{N}$, be a continuously differentiable function such that $f_n \to f$ as $n \to \infty$ uniformly on $[0,a]$. From Theorem 2.6 we know that $u_n = \mathcal{R} * f_n$, $n \in \mathbf{N}$, is a classical solution of problem (2.17)–(2.18) with f_n instead of f. Applying the initial remark for u_n instead of u and Lemma 2.10, we infer that

$$\left\| \int_0^t u_n(s)\,ds - \int_0^t u(s)\,ds \right\|_1 \to 0, \quad n \to \infty$$

uniformly on $[0,a]$. Furthermore,

$$\begin{aligned}\int_0^t \int_0^s B(s-\xi)u_n(\xi)\,d\xi\,ds &= \int_0^t \int_0^s B(\xi)u_n(s-\xi)\,d\xi\,ds \\ &= \int_0^t B(\xi) \int_0^{t-\xi} u_n(s)\,ds\,d\xi.\end{aligned}$$

Using the preceding property and assumption (P3) we obtain that

$$\begin{aligned}\int_0^t \int_0^s B(s-\xi)u_n(\xi)\,d\xi\,ds &\longrightarrow \int_0^t B(\xi) \int_0^{t-\xi} u(s)\,ds\,d\xi \\ &= \int_0^t \int_0^s B(s-\xi)u(\xi)\,d\xi\,ds.\end{aligned}$$

Since u_n is a classical solution on $[0,a]$ of problem (2.17)–(2.18) with f_n instead of f, we can write

$$\begin{aligned}u_n(t) + \int_0^t N(t-s)u_n(s)\,ds = A &\int_0^t u_n(s)\,ds \\ &+ \int_0^t \int_0^s B(s-\xi)u_n(\xi)\,d\xi\,ds \\ &+ \int_0^t f_n(s)\,ds,\end{aligned}$$

for every $t \in [0,a]$ and all $n \in \mathbf{N}$. Since A is closed, and taking the limit in the above expression as $n \to \infty$, we get

$$\begin{aligned}A \int_0^t u(s)\,ds = u(t) &+ \int_0^t N(t-s)u(s)\,ds \\ &- \int_0^t \int_0^s B(s-\xi)u(\xi)\,d\xi\,ds - \int_0^t f(s)\,ds,\end{aligned}$$

and, in view of that the function $Au(s)$ is continuous,

$$\int_0^t Au(s)\,ds = u(t) + \int_0^t N(t-s)u(s)\,ds \\ - \int_0^t \int_0^s B(s-\xi)u(\xi)\,d\xi\,ds - \int_0^t f(s)\,ds,$$

and this expression permits us to conclude the proof that $u(\cdot)$ is a classical solution. $\square$

For functions f with values in $[D(-A)^\beta]$, and proceeding as in Grimmer and Pritchard [**17**], we can establish the following properties of mild solutions.

Theorem 2.7. *Assume that hypotheses of Theorem* 2.2 *are fulfilled. Let* $z \in [D(-A)^\beta]$, *and let* $f \in C([0,a],[D(-A)^\beta])$ *for* $0 < \beta < 1$. *Let* $u(\cdot)$ *be the mild solution of problem* (2.17)–(2.18). *Then* $u \in C([0,a],X) \cap C^1((0,a],X)$ *and* (2.17) *holds for* $t > 0$.

Theorem 2.8. *Assume that hypotheses of Theorem* 2.3 *are fulfilled. Let* $z \in [D(-A)^\beta]$, *and let* $f \in C([0,a],[D(-A)^\beta])$ *for* $0 < \beta < 1$. *Let* $u(\cdot)$ *be the mild solution of problem* (2.17)–(2.18). *Then* $u \in C([0,a],X) \cap C^1([0,a],X)$ *and* (2.17) *holds for* $t > 0$.

3. Existence results for neutral equations. In this section we study the existence of solutions for the neutral system (1.1)–(1.2). Here, $(\mathcal{R}(t))_{t\geq 0}$ is the resolvent operator defined in (2.13) and we use an axiomatic definition of the phase space $\mathcal{B}$ which is similar to those in [**29**]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms.

(**A**) If $x : (-\infty,\sigma+b) \to X$, $b > 0, \sigma \in \mathbf{R}$, is continuous on $[\sigma,\sigma+b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$.

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma, \sigma + b)$ into $\mathcal{B}$.

(**B**) The space $\mathcal{B}$ is complete.

Remark 3.3. In the remainder of this section, $\|\mathcal{R}\|_\infty, M^a$ and K^a are the constants $\|\mathcal{R}\|_\infty = \sup_{s\in[0,a]} \|\mathcal{R}(s)\|$, $M^a = \sup_{s\in[0,a]} M(s)$ and $K^a = \sup_{s\in[0,a]} K(s)$. In addition, we introduce the function $y : (-\infty, a] \to X$ defined by $y(t) = \varphi(t)$ for $t \leq 0$ and $y(t) = \mathcal{R}(t)\varphi(0)$ for $t \in [0,a]$.

Example 3.1. The phase space $C_r \times L^p(\rho, X)$. Let $r \geq 0$, $1 \leq p < \infty$, and let $\rho : (-\infty, -r] \to \mathbf{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [**29**]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p \, d\theta\right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho, X)$ satisfies axioms (A), (A-1) and (B). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^0 \rho(\theta)\, d\theta\right)^{1/2}$, for $t \geq 0$. See [**29**, Theorem 1.3.8] for details.

To obtain our desired results, we introduce the following conditions.

(H1) The function $f : [0,a] \times \mathcal{B} \to X$ verifies the following conditions.

(i) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for every $t \in [0,a]$, and for every $\psi \in \mathcal{B}$, the function $f(\cdot, \psi) : [0,a] \to X$ is strongly measurable.

(ii) There exist $m_f \in C([0,a],[0,\infty))$ and a continuous non-decreasing function $\Omega_f : [0,\infty) \to (0,\infty)$ such that $\partial f(t,\psi)\partial \leq m_f(t)\Omega_f(\partial\psi\partial_{\mathcal{B}})$, for all $(t,\psi) \in [0,a] \times \mathcal{B}$.

(H2) The function $f : [0,a] \times \mathcal{B} \to X$ is continuous and there is $L_f \in L^1([0,a],\mathbf{R}^+)$ such that

$$\|f(t,\psi_1) - f(t,\psi_2)\| \leq L_f(t)\|\psi_1 - \psi_2\|, \quad t \in [0,a], \psi_1, \psi_2 \in \mathcal{B}.$$

Motivated by the results in Section 2, we introduce the following concepts of solutions for the neutral system (1.1)–(1.2).

Definition 3.4. A function $u : (-\infty, b] \to X$, $0 < b \leq a$, is called a classical solution of the neutral system (1.1)–(1.2) on $[0,b]$ if $u_0 = \varphi$, $u_{|_{[0,b]}} \in C([0,b],[D(A)]) \cap C^1((0,b],X)$ and (1.1) is verified on $(0,b]$. If $u_{|_{[0,b]}} \in C([0,b],[D(A)]) \cap C^1([0,b],X)$ and (1.1) is verified on $[0,b]$, then $u(\cdot)$ is said to be a strict solution of (1.1)–(1.2) on $[0,b]$.

In the remainder of this work, φ is a fixed function in $\mathcal{B}$ and $f_i : [0,a] \to X$, $i = 1,2$, will be the function defined by $f_1(t) = -\int_{-\infty}^0 N(t-s)\varphi(s)\,ds$ and $f_2(t) = \int_{-\infty}^0 B(t-s)\varphi(s)\,ds$.

Definition 3.5. A function $u : (-\infty, b] \to X$ is called an S-mild solution of the neutral system (1.1)–(1.2) on $[0,b]$ if $u_0 = \varphi$, f_1 is differentiable on $[0,b]$, $f_1' \in L^1([0,b],X)$, $u_{|_{[0,a]}} \in C([0,b],X)$ and

$$\begin{aligned} u(t) = \mathcal{R}(t)\varphi(0) &+ \int_0^t \mathcal{R}(t-s)f(s,u_s)\,ds \\ &+ \int_0^t \mathcal{R}(t-s)(f_1'(s) + f_2(s))\,ds, \quad t \in [0,b]. \end{aligned}$$

The proof of the next result is standard, however we include it for completeness.

Theorem 3.9. *Assume that* **(H2)** *is fulfilled,* $f_1 \in W^{1,1}([0,a],X)$ *and* $f_2 \in L^1([0,a],X)$*. Then there exists a unique* $\mathcal{S}$*-mild solution of* (1.1)–(1.2) *on* $[0,b]$ *for some* $0 < b \leq a$*.*

Proof. Let $b \in [0,a]$ be such that $\Theta = \|\mathcal{R}\|_\infty K^b \|L_f\|_{L^1([0,b])} < 1$. On the space

$$\mathbf{Z}(b) = \{u : (-\infty, b] \longrightarrow X;\ u_0 = \varphi,\ u_{|_{[0,b]}} \in C([0,b], X)\},$$

endowed with the metric $d(u,v) = \sup_{s\in[0,b]} \|u(s) - v(s)\|$, we define the map $\Gamma : \mathbf{Z}(b) \to \mathbf{Z}(b)$ by $\Gamma u(\theta) = \varphi(\theta)$ for $\theta \leq 0$ and

$$\begin{aligned}\Gamma u(t) = \mathcal{R}(t)\varphi(0) + \int_0^t \mathcal{R}(t-s) f(s, u_s)\, ds \\ + \int_0^t \mathcal{R}(t-s)(f_1'(s) + f_2(s))\, ds, \quad t \in [0,b].\end{aligned}$$

It is easy to see that $\Gamma\mathbf{Z}(b) \subset \mathbf{Z}(b)$. Moreover, for $u, v \in \mathbf{Z}(b)$ we have

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &\leq \|\mathcal{R}\|_\infty K^b \int_0^t L_f(s) \sup_{0\leq\xi\leq s} \|u(\xi) - v(\xi)\|\, ds \\ &\leq \Theta\, d(u,v),\end{aligned}$$

which implies that Γ is a contraction on $\mathbf{Z}(b)$ and there exists a unique S-mild solution $u(\cdot)$ of (1.1)–(1.2) on $[0,b]$. This completes the proof. □

Arguing as in the proof of Theorem 3.9, we obtain the following result.

Proposition 3.2. *Assume $f \in C([0,a] \times \mathcal{B}, X)$ and for all $r > 0$ there is an $L_r > 0$ such that*

$$\|f(t,\psi_1) - f(t,\psi_2)\| \leq L_r \|\psi_1 - \psi_2\|, \quad t \in [0,r],\ \psi_i \in B_r(\varphi, \mathcal{B}).$$

Then there exists a unique $\mathcal{S}$-mild solution of (1.1)–(1.2) *on* $[0,b]$*, for some* $0 < b \leq a$.

To establish our next existence result, we need the following lemma.

Lemma 3.11. *If $R(\lambda_0, A)$ is compact for some $\lambda_0 \in \rho(A)$, then $\mathcal{R}(t)$ is compact for all $t > 0$.*

Proof. It follows from Lemmas 2.1 and 2.2 that $G(\lambda)$ is compact for all $\lambda \in \Lambda_{r,\vartheta}$. The assertion is now a consequence of (2.13). □

Theorem 3.10. *Assume condition* **(H1)** *holds,* $\mathcal{R}(t)$ *is compact for all* $t > 0$, $f_1 \in W^{1,1}([0,a],X)$ *and* $f_2 \in L^1([0,a],X)$. *Then there exists an* $\mathcal{S}$*-mild solution of* (1.1)–(1.2) *on* $[0,b]$ *for some* $0 < b \le a$.

Proof. Let $0 < b \le a$ be such that

$$K^a \|\mathcal{R}\|_\infty \int_0^b m_f(s)\, ds < \int_c^\infty \frac{1}{\Omega_f(s)}\, ds, \tag{3.1}$$

where $c = (M^a + K^a\|\mathcal{R}\|_\infty H)\|\varphi\|_{\mathcal{B}} + K^a\|\mathcal{R}\|_\infty \|f_1' + f_2\|_{L^1([0,b],X)}$.

On the space $\mathcal{W}(b) = \{u : (-\infty, b] \to X;\, u_0 \in \mathcal{B},\, u_{|_{[0,b]}} \in C([0,b],X)\}$, endowed with the norm $|u|_{\mathcal{W}(b)} = |u_0|_{\mathcal{B}} + \sup_{\theta\in[0,b]} |u(\theta)|$, we define the map $\Gamma : \mathcal{W}(b) \to \mathcal{W}(b)$ by $(\Gamma u)_0 = \varphi$ and

$$\begin{aligned}\Gamma u(t) = \mathcal{R}(t)\varphi(0) &+ \int_0^t \mathcal{R}(t-s) f(s,u_s)\, ds \\ &+ \int_0^t \mathcal{R}(t-s)(f_1'(s) + f_2(s))\, ds, \quad t \in [0,b].\end{aligned}$$

In the sequel, we prove that Γ verifies the conditions of the Leray-Schauder alternative theorem ([**16**, Theorem 6.5.4]). Initially, we point out that a direct application of the Lebesgue dominated convergence theorem permits us to conclude that Γ is continuous.

We next establish an *a priori* estimate for the solutions of the integral equation $u = \lambda\Gamma u$ for $\lambda \in (0,1)$. Let u^λ be a solution of $u = \lambda\Gamma u$, $\lambda \in (0,1)$, and let $\alpha^\lambda(\cdot)$ be the function defined by $\alpha^\lambda(s) = M^a\|\varphi\|_{\mathcal{B}} + K^a \sup_{\xi\in[0,s]} \partial u^\lambda(\xi)\partial$. By noting that $\|u_s^\lambda\|_{\mathcal{B}} \le \alpha^\lambda(s)$, we get

$$\begin{aligned}\alpha^\lambda(t) \le (M^a &+ K^a\|\mathcal{R}\|_\infty H)\|\varphi\|_{\mathcal{B}} \\ &+ K^a\|\mathcal{R}\|_\infty \|f_1' + f_2\|_{L^1([0,b],X)} \\ &+ K^a\|\mathcal{R}\|_\infty \int_0^t m_f(s)\Omega_f(\alpha^\lambda(s))\, ds.\end{aligned} \tag{3.2}$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, we obtain that

$$\beta_\lambda'(t) \le K^a\|\mathcal{R}\|_\infty m_f(t)\Omega_f(\beta_\lambda(t))$$

and hence,

$$\int_c^{\beta_\lambda(t)} \frac{1}{\Omega_f(s)}\, ds \le K^a \|\mathcal{R}\|_\infty \int_0^b m_f(s)\, ds.$$

This inequality and (3.1) permit us to conclude that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded, which shows that $\{u^\lambda : \lambda \in (0,1)\}$ is bounded in $\mathcal{W}(b)$.

On the other hand, from [**27**, Lemma 3.1] it follows that Γ is completely continuous. Applying now [**16**, Theorem 6.5.4], we infer that Γ has fixed point $u \in \mathcal{W}(b)$ and note the existence of an $\mathcal{S}$-mild solution on $[0,b]$. This completes the proof. □

Next we discuss the existence of classical solutions. The first result is an immediate consequence of Theorem 2.5.

Proposition 3.3. *Assume that $u(\cdot)$ is an $\mathcal{S}$-mild solution of (1.1)–(1.2) on $(0,b]$, $\varphi(0) \in D(A)$ and that the function $f(t) = f(t,u_t)+f_1'(t)+f_2(t)$ belongs to $C([0,a],[D(A)])$. Then $u(\cdot)$ is a classical solution on $[0,b]$. Further, if $\varphi(0) \in E$ then $u(\cdot)$ is a strict solution.*

To prove our next result, we introduce some additional notations and properties. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. For a differentiable function $g : [0,a] \times W \to Z$ we denote by $Dg(t,w) : \mathcal{R} \times W \to Z$ the derivative of $g(\cdot)$ at (t,w). We decompose

$$Dg(t,w)(h,w_1) = hD_1 g(t,w) + D_2 g(t,w)(w_1),$$

and we set

$$\begin{aligned} g(t_2,w_2) - g(t_1,w_1) = (t_2 - t_1)D_1 g(t_1,w_1) + D_2 g(t_1,w_1)(w_2 - w_1) \\ + \varrho(t_2,t_1,w_2,w_1), \end{aligned}$$

where $\|\varrho(t_2,t_1,w_2,w_1)\|_Z/[|t_2-t_1| + \|w_2-w_1\|_W] \to 0$ as $|t_2-t_1| + \|w_2-w_1\|_W \to 0$. Moreover, for $x : [0,a] \to Z$ and $h \in \mathcal{R}$, we represent by $\partial_h x(\cdot)$ the function given by $\partial_h x(t) = [x(t+h)-x(t)]/h$.

The following property is an immediate consequence of [**1**, Proposition 2.4.7].

Lemma 3.12. *Let $g \in C^1(W, Z)$, $x : [0, a] \to W$ a Lipschitz continuous function and*

$$\varrho(s, h) = g(x(s + h)) - g(x(s)) - Dg(x(s))(x(s + h) - x(s)).$$

Then $\varrho(s, h)/h \to 0$ as $h \to 0$ uniformly on $s \in [0, a]$.

Remark 3.4. We emphasize that the last result does not require x to be differentiable.

Now we introduce the operators $S(t) : \mathcal{B} \to \mathcal{B}$ given by

$$[S(t)\psi](\theta) = \begin{cases} \psi(0) & -t \leq \theta \leq 0, \\ \psi(t + \theta), & \theta \leq -t. \end{cases}$$

It follows from the axioms of the phase space that $(S(t))_{t \geq 0}$ is a C_0-semigroup on $\mathcal{B}$.

We denote by $\mathcal{B}_{Lip}$ the subspace of $\mathcal{B}$ consisting of functions ψ for which there exists $L_\psi \geq 0$ such that $\|S(h)\psi - \psi\|_{\mathcal{B}} \leq L_\psi h$, for all $h \geq 0$.

We consider the following axiom for the phase space $\mathcal{B}$ ([**29**]).

(C2) If a uniformly bounded sequence $(\psi^n)_{n \in \mathbf{N}}$ of continuous functions from $(-\infty, 0]$ to X with compact support converges to a function ψ in the compact-open topology, then $\psi \in \mathcal{B}$ and $\|\psi^n - \psi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.

We know from (**29**) that if the axiom (C2) holds, then the space of continuous and bounded functions $C_b((-\infty, 0], X)$ is continuously included in $\mathcal{B}$, so that, there exists a $\gamma > 0$ such that $\|\psi\|_{\mathcal{B}} \leq \gamma \|\psi\|_\infty$, for all $\psi \in C_b((-\infty, 0], X)$. As an example, we mention that if the function ρ is integrable on $(-\infty, -r]$, then the space $C_r \times L^p(\rho, X)$ defined in Example 3.1 satisfies axiom (C2).

The next property is established using a standard argument based on the phase space axioms and the Gronwall-Bellman lemma. We omit the proof.

Lemma 3.13. *Assume that* **(H2)** *is fulfilled, $\varphi \in \mathcal{B}_{Lip}$ and $\varphi(0) \in E$. If $u(\cdot)$ is an $\mathcal{S}$-mild solution of* (1.1)–(1.2) *on* $[0, b]$ *and $f_1' + f_2$ is*

Lipschitz on $[0, b]$*, then* $u(\cdot)$ *and the function* $s \to u_s$ *are Lipschitz on* $[0, b]$*.*

In the next result, $UC_b((-\infty, 0], X)$ is the space of uniformly continuous and bounded functions from $(-\infty, 0]$ into X.

Theorem 3.11. *Assume* $\mathcal{B}$ *satisfies axiom* **(C2)**, $f \in C^1([0, a] \times \mathcal{B}, X)$, $f_1' + f_2 \in C^1([0, a], X)$, $\varphi \in C_b((-\infty, 0], X)$ *with* $\varphi' \in UC_b((-\infty, 0], X)$ $\varphi(0) \in E$ *and* $\varphi'(0) = A\varphi(0) + f(0, \varphi) + f_1'(0) + f_2(0) - N(0)\varphi(0)$*. Then there exists a classical solution of* (1.1)–(1.2) *on* $[0, b]$ *for some* $0 < b \le a$*.*

Proof. By Proposition 3.2 there exits an $\mathcal{S}$-mild solution $u(\cdot)$ of (1.1)–(1.2) on $[0, b_1]$ for some $0 < b_1 \le a$. Using axiom (C2) and the fact φ' is bounded, we obtain that $\varphi \in \mathcal{B}_{Lip}$ which from Lemma 3.13 implies that $u(\cdot)$ and $t \to u_t$ are Lipschitz on $[0, b_1]$.

Consider the initial value problem

$$
\begin{aligned}
(3.3) \qquad w(t) &= \frac{d}{dt}\mathcal{R}(t)\varphi(0) + \mathcal{R}(t)g(0, \varphi) \\
&\quad + \int_0^t \mathcal{R}(t-s)D_1 g(s, u_s)\, ds \\
&\quad + \int_0^t \mathcal{R}(t-s)D_2 g(s, u_s)(w_s)\, ds, \\
(3.4) \qquad w_0 &= \varphi',
\end{aligned}
$$

where $g(s, u_s) = f(s, u_s) + f_1'(s) + f_2(s)$. From the contraction mapping principle, there exists $0 < b \le b_1$ and a unique solution $w \in C((-\infty, b], X)$ of (3.3)–(3.4).

Next, we show that $u' = w$ on $[0, b]$. For $t \in [0, b)$ and $h > 0$ with $t + h \in [0, b]$, we get

$$
\begin{aligned}
&\|\partial_h u(t) - w(t)\| \\
&\quad \le \left\| \partial_h \mathcal{R}(t)\varphi(0) - \frac{d}{dt}\mathcal{R}(t)\varphi(0) \right\| \\
&\qquad + \left\| \frac{1}{h}\int_0^h \mathcal{R}(t+h-s)g(s, u_s)\, ds - \mathcal{R}(t)g(0, \varphi) \right\|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left\| \int_0^t \mathcal{R}(t-s)[\partial_h g(s,u_s) - D_1 g(s,u_s) - D_2 g(s,u_s) w_s]\, ds \right\| \\
&= \Lambda_1(t,h) + \Lambda_2(t,h) \\
&\quad + \left\| \int_0^t \mathcal{R}(t-s)[\partial_h g(s,u_s) - D_1 g(s,u_s) - D_2 g(s,u_s) w_s]\, ds \right\|.
\end{aligned}
$$

Since $\varphi(0) \in E$, $\Lambda_1(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$. Similarly, from the properties of the functions $\mathcal{R}(\cdot)$ and $s \to g(s,u_s)$, we infer that $\Lambda_2(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$.

On the other hand,

$$
\frac{g(s+h,u_{s+h}) - g(s,u_s)}{h} - D_1 g(s,u_s) - D_2 g(s,u_s)\left[\frac{u_{s+h}-u_s}{h}\right] = \frac{\varrho(s,u_{s+h}-u_s)}{h}
$$

so that

$$
\begin{aligned}
&\partial_h g(s,u_s) - D_1 g(s,u_s) - D_2 g(s,u_s) w_s \\
&\qquad = \frac{\varrho(s,u_{s+h}-u_s)}{h} + D_2 g(s,u_s)\left[\frac{u_{s+h}-u_s}{h} - w_s\right].
\end{aligned}
$$

Using that the function $t \to u_t$ is Lipschitz on $[0,b]$, and applying Lemma 3.12 we can affirm that $\varrho(s,u_{s+h}-u_s)h^{-1} \to 0$ as $h \to 0$ uniformly for $s \in [0,b]$. Since $D_2 g(s,u_s)$ is bounded on $[0,b]$, we can abridge these properties as

$$
\|\partial_h u(t) - w(t)\| \le \Lambda_3(t,h) + C_1 \int_0^t \left\| \frac{u_{s+h}-u_s}{h} - w_s \right\|_{\mathcal{B}} ds, \tag{3.5}
$$

where $\Lambda_3(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$ and $C_1 > 0$ is independent of t and h.

Next we establish that $\|(u_h - \varphi)/h - \varphi'\|_{\mathcal{B}} \to 0$ as $h \to 0$. At first we note that for $-h \le \theta \le 0$

$$
\begin{aligned}
&\frac{u(h+\theta) - \varphi(\theta)}{h} - \varphi'(\theta) \\
&\quad = \frac{h+\theta}{h}\left[\frac{\mathcal{R}(h+\theta)\varphi(0) - \varphi(0)}{h+\theta} - A\varphi(0) + N(0)\varphi(0)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[A\varphi(0) - N(0)\varphi(0) \\
&\quad + \frac{1}{h+\theta}\int_0^{h+\theta} \mathcal{R}(h+\theta-s)g(s,u_s)\,ds - \varphi'(0)\Big] \\
&+ \frac{\theta}{h}\Big[A\varphi(0) - N(0)\varphi(0) + \frac{1}{h+\theta}\int_0^{h+\theta} \mathcal{R}(h+\theta-s)g(s,u_s)\,ds \\
&\quad + \frac{\varphi(0)-\varphi(\theta)}{\theta}\Big] + \varphi'(0) - \varphi'(\theta)
\end{aligned}
$$

which permits us to infer that $(u(h+\theta)-\varphi(\theta))/h - \varphi'(\theta)$ converges to zero as $h \to 0$ since $\varphi(0) \in E$. Moreover, for $\theta \leq -h$,

$$\frac{u(h+\theta)-\varphi(\theta)}{h} - \varphi'(\theta) = \frac{\varphi(h+\theta)-\varphi(\theta)}{h} - \varphi'(\theta) \to 0, \quad \text{as } h \to 0,$$

uniformly on θ since $\varphi' \in UC_b((-\infty,0],X)$. Now, the assertion follows from the inequality

$$\Big\| \frac{u_h - \varphi}{h} - \varphi' \|_{\mathcal{B}} \leq \gamma \sup_{\theta \leq 0} \Big\| \frac{u(h+\theta)-\varphi(\theta)}{h} - \varphi'(\theta) \Big\|.$$

Using now the axioms of $\mathcal{B}$ we have

$$
\begin{aligned}
\Big\| \frac{u_{t+h}-u_t}{h} - w_t \Big\|_{\mathcal{B}} &\leq K^a \max_{0\leq s\leq t} \Big\| \frac{u(s+h)-u(s)}{h} - w(s) \Big\| \\
&\quad + M^a \| \frac{u_h - \varphi}{h} - \varphi' \|_{\mathcal{B}}.
\end{aligned}
$$

From the preceding results, and combining with the estimate (3.5), we can write

$$\text{(3.6)} \quad \Big\| \frac{u_{t+h}-u_t}{h} - w_t \Big\|_{\mathcal{B}} \leq \Lambda_4(t,h) + C_2 \int_0^t \Big\| \frac{u_{s+h}-u_s}{h} - w_s \Big\|_{\mathcal{B}} ds,$$

where $\Lambda_4(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$ and $C_2 \geq 0$ is independent of t and h. Applying the Gronwall-Bellman lemma we infer that $\|(u_{t+h}-u_t/h) - w_t\|_{\mathcal{B}} \to 0$ as $h \to 0$. Consequently, the functions $t \mapsto u_t$ and $t \mapsto g(t,u_t)$ are continuously differentiable on $[0,b]$, and using now axiom (A)(ii) we obtain $u' = w$ on $[0,b]$.

Finally, from Theorem 2.6 and the above remarks it follows that $u(\cdot)$ is a strict solution of (1.1)–(1.2) on $[0, b]$. $\square$

4. Applications. In this section we consider some applications. At first, we consider the particular case in which $\dim X < \infty$. The literature on neutral integro-differential systems with $x(t) \in \mathbf{R}^n$ is extensive and, in this case, our results are easily applicable since the operators A, $B(t)$ are bounded. As a practical application, we consider the neutral equation

$$(4.1) \quad \frac{d}{dt}\left[u(t) - \lambda \int_{-\infty}^{t} C(t-s)u(s)\,ds\right] = Au(t) + \lambda \int_{-\infty}^{t} B(t-s)u(s)\,ds - p(t) + q(t),$$

which arises in the study of the dynamics of income, employment, value of capital stock and cumulative balance of payment, see [**8**] for details. In this system, λ is a real number, the state $u(t) \in \mathbf{R}^n$, $C(\cdot), B(\cdot)$ are $n \times n$ matrix continuous functions, A is a constant $n \times n$ matrix, $p(\cdot)$ represents the government intervention and $q(\cdot)$ the private initiative.

To treat system (4.1), we assume that the solution $u(\cdot)$ is known on $(-\infty, 0]$ and we take $\mathcal{B} = C_0 \times L^p(\rho, X)$ with $X = \mathbf{R}^n$, see Example 3.1. In the next results, $\varphi \in \mathcal{B}$, $f_i : [0, a] \to X$, $i = 1, 2$, are defined by $f_1(t) = -\lambda \int_{-\infty}^{0} C(t-s)\varphi(s)\,ds$ and $f_2(t) = \int_{-\infty}^{0} B(t-s)\varphi(s)\,ds$, and Λ_ϑ is the set defined in Section 2.

Proposition 4.4. *Assume $f_1 \in W^{1,1}([0, a], X)$, $f_2 \in L^1([0, a], X)$ and the following conditions are verified.*

(a) *$\widehat{C}(\lambda)x$ is absolutely convergent for $x \in X$ and $\mathrm{Re}\,(\lambda) > 0$, there are $\vartheta_1 \in (\pi/2, \pi)$, $\alpha \geq 1$ and $N_1 \geq 0$ such that $\widehat{C}(\lambda)$ is analytic on Λ_{ϑ_1} and $\|\widehat{C}(\lambda)\| \leq N|\lambda|$ for every $\lambda \in \Lambda_{\vartheta_1}$.*

(b) *The operator function $B(\cdot)$ is strongly continuous in X, and there are $\vartheta_2 \in (\pi/2, \pi)$ and $b(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$ such that $\|B(t)\|_{\mathcal{L}(X)} \leq b(t)$ for all $t > 0$, $\widehat{B} : \Lambda_{\vartheta_2} \to \mathcal{L}(X)$ is analytical and $\|\widehat{B}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.*

Then there exists an $\mathcal{S}$-mild solution of problem (4.1) on $[0, b]$ for some $0 < b \leq a$. If, in addition, $f_1' + f_2 \in C([0, a], X)$ then $u(\cdot)$ is a classical solution.

Proof. We note that if $\dim(X) < \infty$, then A is the infinitesimal generator of an analytic semigroup in X and there are positive constants r, M_0 such that

$$\rho(A) \supseteq \Lambda_{\vartheta,r} = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg(\lambda)| < \vartheta, |\lambda| \geq r\}$$

and

$$\|R(\lambda, A)\| \leq M_0 |\lambda|^{-1}$$

for all $\lambda \in \Lambda_{\vartheta,r}$. Proceeding as in Section 2, it is easy to see that under these conditions there exists and analytic resolvent $(\mathcal{R}(t))_{t\geq 0}$ for system (4.1) with $p = q = 0$. Now the assertions follow directly from Theorem 3.10 and Proposition 3.3, respectively. $\square$

To finish this section, we apply our results to study a neutral integro-differential equation which arise in the theory of heat conduction in fading memory materials. Consider the system

$$\begin{aligned} &\frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-\infty}^{t} (t-s)^{\alpha} e^{-\omega(t-s)} u(s,\xi)\, ds\right] \\ &\quad = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_{-\infty}^{t} e^{-\gamma(t-s)} \frac{\partial^2 u(s,\xi)}{\partial \xi^2}\, ds + \int_{-\infty}^{t} a(t-s) u(s,\xi)\, ds, \end{aligned} \tag{4.2}$$

$$u(t,\pi) = u(t,0) = 0, \tag{4.3}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \tag{4.4}$$

for $(t,\xi) \in [0,a] \times [0,\pi]$, $\theta \leq 0$. In this system, $\alpha \in (0,1)$, ω, γ are positive numbers, and $a : [0,\infty) \to \mathbf{R}$ is an appropriated function. Moreover, we have identified $\varphi(\theta)(\xi) = \varphi(\theta,\xi)$.

To represent this system in the abstract form (1.1)–(1.2), we choose the spaces $X = L^2([0,\pi])$ and $\mathcal{B} = C_0 \times L^2(\rho, X)$, see Example 3.1 for details. We also consider the operators $A, B(t) : D(A) \subseteq X \to X$ and $B(t) : X \to X$ given by $Ax = x''$, $B(t)x = e^{-\gamma t} Ax$ for $x \in D(A) = \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$ and $N(t)y = t^{\alpha} e^{-\omega t} y$ for $y \in X$.

The operator A is the infinitesimal generator of an analytic semigroup on X and $\rho(A) = \mathbf{C} \setminus \{-n^2 : n \in \mathbf{N}\}$. As a consequence, for all

$\vartheta \in (\pi/2, \pi)$ there exists an $M_\vartheta > 0$ such that $\|R(\lambda, A)\| \le M_\vartheta |\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$. Moreover, it is easy to see that the conditions (P2)–(P4) in Section 2 are satisfied with $\widehat{N}(\lambda) = [\Gamma(\alpha+1)]/(\lambda+\omega)^{\alpha+1}$, $b(t) = e^{-\gamma t}$ and $D = C_0^\infty([0,\pi])$, where Γ is the gamma function and $C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions that vanish at $\xi = 0$ and $\xi = \pi$. In addition, from the expression for $\widehat{N}(\lambda)$ it follows that $E = D(A)$.

Under the above conditions and notations, we can represent the system

$$\frac{\partial}{\partial t}\left[x(t,\xi) + \int_0^t (t-s)^\alpha e^{-\omega(t-s)} x(s,\xi)\, ds\right] = \frac{\partial^2 x(t,\xi)}{\partial \xi^2} + \int_0^t e^{-\gamma(t-s)} \frac{\partial^2 x(s,\xi)}{\partial \xi^2}\, ds, \tag{4.5}$$

$$x(t,\pi) = x(t,0) = 0, \tag{4.6}$$

$$x(0)(\xi) = z(\xi), \tag{4.7}$$

in the abstract form

$$\frac{d}{dt}\left[x(t) + \int_0^t N(t-s)x(s)\, ds\right] = Ax(t) + \int_0^t B(t-s)x(s)\, ds, \tag{4.8}$$

$$x(0) = z \in X. \tag{4.9}$$

Proposition 4.5 below is a consequence of the developments in Section 2.

Proposition 4.5. *There exists an operator resolvent for* (4.8)–(4.9).

We next consider the problem of the existence of mild solutions for the system (4.2)–(4.4). To this end, we introduce the following conditions.

(a) The function $a(\cdot)$ is continuous and $L_f = (\int_{-\infty}^0 |a(-s)|^2/\rho(s)\, ds)^{1/2} < \infty$.

(b) The functions φ, $A\varphi$ belong to $\mathcal{B}$ and the expressions $\sup_{t\in[0,a]}[\int_{-\infty}^0 (t-\tau)^{2\alpha}/[\rho(-\tau)]e^{2\omega\tau}\, d\tau]^{1/2}$ and $(\int_{-\infty}^0 e^{-2\gamma\tau}/[\rho(-\tau)]\, d\tau)^{1/2}$ are finite.

Under the conditions (**a**) and (**b**), the functions $f : [0,a] \times \mathcal{B} \to X$, $f_i : [0,a] \to X$, $i = 1,2$, given by

$$\begin{aligned} f(t,\psi)(\xi) &= \int_{-\infty}^{0} a(-s)\psi(s,\xi)\,ds, \\ f_1(t)(\xi) &= \int_{-\infty}^{0} (t-s)^{\alpha} e^{-\omega(t-s)}\varphi(s,\xi)\,ds, \\ f_2(t)(\xi) &= \int_{-\infty}^{0} e^{-\gamma(t-s)} A\varphi(s,\xi)\,ds, \end{aligned}$$

are well defined, which permit us to re-write the system (4.2)–(4.4) in the abstract form

$$\begin{aligned} &\frac{d}{dt}\Big[x(t) + \int_0^t N(t-s)x(s)\,ds + f_1(t)\Big] \\ &\qquad = Ax(t) + \int_0^t B(t-s)x(s)\,ds \\ &\qquad\quad + f_2(t) + f(t,x_t), \quad t \in [0,a], \end{aligned} \tag{4.10}$$

$$x_0 = \varphi \in \mathcal{B}. \tag{4.11}$$

In the next result, which is a direct consequence of Theorem 3.9, we said that a function $u \in C([0,a];X)$ is an $\mathcal{S}$-mild solution of (4.2)–(4.4) if $u(\cdot)$ is a mild solution of the associated abstract system (4.10)–(4.11).

Proposition 4.6. *Assume that the above conditions are fulfilled. If any of the following conditions is verified,*

(i) $\sup_{t\in[0,a]}[\int_{-\infty}^{0} 1/[\rho(\tau)][e^{-\omega(t-\tau)}/[(t-\tau)^{1-\alpha}]]^2 d\tau]^{1/2} < \infty$,

(ii) $\varphi' \in C((-\infty,0],X) \cap \mathcal{B}$,

then there exists a unique $\mathcal{S}$-mild solution of (4.2)–(4.4) *on* $[0,b]$ *for some* $0 < b \le a$.

Proof. From condition **(a)** it is easy to see that f is a bounded linear operator with $\|f\|_{\mathcal{L}(\mathcal{B},X)} \le L_f$, and from condition **(b)** it follows that f_1 and f_2 are continuous functions.

If condition **(i)** is valid, then f_1 is differentiable and

$$f_1'(t)(\xi) = \int_{-\infty}^{0} [(t-s)^{\alpha-1} + \omega(t-s)^{\alpha}] e^{-\omega(t-s)} \varphi(s,\xi)\, ds, \quad \forall\, (t,\xi) \in [0,a] \times [0,\pi]. \tag{4.12}$$

Moreover, using this expression and **(i)** we can prove that $f_1 \in C^1([0,a], X)$. Similarly, if **(ii)** is verified, then f_1 is differentiable,

$$f_1'(t)(\xi) = \int_{-\infty}^{0} (t-s)^{\alpha} e^{-\omega(t-s)} \varphi'(s,\xi)\, ds - e^{\omega t} t^{\alpha} \varphi(0,\xi), \quad \forall\, (t,\xi) \in [0,a] \times [0,\pi], \tag{4.13}$$

and using this representation we can prove that $f_1 \in C^1([0,a], X)$.

Now, from Theorem 3.9 we can assert that there exists a unique $\mathcal{S}$-mild solution for the system (4.2)–(4.4) on $[0,b]$ for some $0 < b \leq a$. ◻

To finish this section, we establish two results on the existence of classical solutions. In the next propositions, we state that a function $u \in C([0,b]; X)$ is a classical solution of (4.2)–(4.4) if $u(\cdot)$ is a classical solution of the associated abstract system (4.10)–(4.11).

Proposition 4.7. *Assume the assumptions* **(a)**, **(b)** *and the condition* **(i)** *of Proposition* 4.6 *are verified. Suppose, in addition,* $\varphi(0,\cdot) = 0$, φ *is differentiable,* $\varphi' \in UC_b((-\infty,0], X)$ *and* $\varphi'(0) = f(0,\varphi) + f_1'(0) + f_2(0)$. *Then there exists a classical solution* $u(\cdot)$ *of* (4.2)–(4.4) *on* $[0,b]$ *for some* $0 < b \leq a$.

Proof. From the proof of Proposition 4.6 we know that $f_1 \in C^1([0,a]; X)$ and $f_2 \in C([0,a]; X)$. Moreover, a straightforward procedure permit to prove that $f_2 \in C^1([0,a]; X)$ and

$$f_2'|(t)(\xi) = \int_{-\infty}^{0} \omega e^{-\omega(t-s)} \varphi(s,\xi)\, ds, \quad (t,\xi) \in [0,a] \times [0,\pi].$$

Using that φ is differentiable, condition **(i)** of Proposition 4.6, the fact that $\varphi(0) = 0$ and the representation (4.12) we obtain that

$$f_1''(t)(\xi) = \int_{-\infty}^{0} [(t-s)^{\alpha-1} + \omega(t-s)^{\alpha}] e^{-\omega(t-s)} \varphi'|(s,\xi)\, ds,$$

$$\text{for all } (t, \xi) \in [0, a] \times [0, \pi],$$

which permits us to prove that $f_1 \in C^2([0, a]; X)$.

The assertion is now a direct consequence of Theorem 3.11. $\square$

Now we consider a slight variant of Proposition 4.7.

Proposition 4.8. *Assume the assumptions* **(a)***,* **(b)** *and condition* **(ii)** *of Proposition* 4.6 *are valid. Suppose, in addition,* $\varphi(0, \cdot) = 0$*,* φ *is a function of class* C^2*,* $\varphi' \in UC_b((-\infty, 0], X)$*,* $\varphi'' \in \mathcal{B}$ *and* $\varphi'(0) = f(0, \varphi) + f_1'(0) + f_2(0)$*. Then there exists a classical solution* $u(\cdot)$ *of* (4.2)–(4.4) *on* $[0, b]$ *for some* $0 < b \leq a$*.*

Proof. Arguing as in the proof of Propositions 4.7, we obtain that

$$\begin{aligned}
f_2'(t)(\xi) &= \int_{-\infty}^{0} \omega e^{-\omega(t-s)} \varphi(s, \xi) ds\,, \\
f_1'(t)(\xi) &= \int_{-\infty}^{0} (t-s)^{\alpha} e^{-\omega(t-s)} \varphi'(s, \xi)\, ds \\
f_1''(t)(\xi) &= \int_{-\infty}^{0} (t-s)^{\alpha} e^{-\omega(t-s)} \varphi''(s, \xi)\, ds - e^{\omega t} t^{\alpha} \varphi'(0, \xi),
\end{aligned}$$

for all $(t, \xi) \in [0, a] \times [0, \pi]$. Using that φ and φ'' belong to $\mathcal{B}$ and the condition **(b)**, we can prove that f_1'' and f_2' are continuous functions. The proof can be completed now by applying Theorem 3.11. $\square$

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