JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 23, Number 2, Summer 2011

A BOUNDARY INTEGRAL EQUATIONS APPROACH FOR MIXED IMPEDANCE PROBLEMS IN ELASTICITY

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Communicated by Giovanni Monegato

ABSTRACT. Direct scattering problems for partially coated obstacles in linear elasticity lead to interior and exterior mixed impedance boundary value problems for the equations of steady-state elastic oscillations. We employ the potential method and reduce the mixed impedance problems to equivalent boundary pseudodifferential equations for arbitrary values of the oscillation parameter. We give a detailed analysis of the corresponding pseudodifferential equations which live on a proper submanifold of the boundary of the elastic body and establish uniqueness and existence results for the original mixed impedance problems for arbitrary values of the oscillation parameter; this is crucial in the study of inverse elastic scattering problems for partially coated obstacles. We also investigate regularity properties of solutions near the curves where the boundary conditions change and establish almost best Hölder smoothness results.

1. Introduction. In this paper we investigate the three-dimensional mixed impedance interior and exterior boundary value problems (BVPs) for the equations of steady-state elastic oscillations. We consider an elastic body occupying either an interior bounded domain or its complement. We assume that the simply connected boundary of this domain is divided into two parts, a Dirichlet (rigid) one and a Robin (impedance) one. On the Dirichlet part of the boundary, the displacement vector is given, while on the Robin part, a specific combination–physically expressing the proportionality relation between the displacement and the

2010 AMS *Mathematics subject classification.* Primary 74J20, 35J55, 31B10, 74G25, 74G30, 74G40.
Keywords and phrases. Elasticity, elastic oscillations, mixed boundary value problems, impedance problems, potential method, pseudodifferential equations, elastic radiation conditions.
The second author is the corresponding author.
Received by the editors on November 15, 2008, and in revised form on May 6, 2009.

DOI:10.1216/JIE-2011-23-2-183

stress vector–is given. The latter part is due to a coating on the Robin part of the boundary with a material of constant surface impedance. As usual, impedance expresses (intensity $\times$ stiffness) in relation to wave reflection, diffraction, etc., and measures the contrast between two media. A complete discussion about the notion of impedance in elasticity can be found, e.g., in [**32**]. Obstacles characterized by boundary conditions of the aforementioned type are often called *partially coated*, and there is very active current research on inverse problems for such media, mainly in acoustics and electromagnetics. Applications regarding coated obstacles include, e.g., the detection of an object in the earth from measurements of the total electric and magnetic field, the problem of a coated cable or pipe which is partially coated by a dielectric, and many others (see [**4**]). Clearly, such boundary value problems describe (from a mathematical point of view) scattering problems. For the study of the solvability of these mixed problems in elasticity we employ a boundary integral equations approach. The basic three-dimensional Dirichlet and Neumann type boundary value problems of the theory of elasticity are well investigated by Kupradze with the potential method (see [**22**] and the references therein). In particular, he formulated the radiation conditions in the elasticity theory (now called in literature *the Sommerfeld-Kupradze radiation conditions*), and proved the uniqueness theorems for steady state oscillation problems in infinite domains with compact boundaries. To establish the existence of classical solutions for smooth domains he constructed the fundamental matrix satisfying the Sommerfeld-Kupradze radiation conditions explicitly in terms of elementary functions, investigated properties of the corresponding single and double layer potentials for Hölder continuous functions, and reduced the Dirichlet and Neumann type BVPs to normally solvable singular integral equations on the boundary of the domain under consideration. Unfortunately these equations have a countable spectrum with respect to the oscillation frequency parameter. Therefore the boundary integral equations obtained are not equivalent to the original BVPs and are not solvable unconditionally for all values of the oscillation parameter. Such a situation always appears when the direct boundary integral equations method is employed in oscillation problems, i.e., when the solutions are sought in the form of either a single- or a double-layer potential. To investigate the solvability of the above integral equations one needs to find all eigenvalues and eigenfunctions (eigenvectors) of the corresponding homogeneous integral equations and their adjoint ones. Similar two dimensional problems are considered in [**3, 16, 17**] (see also the references therein).

Here we give a very naturally modified approach to remove the above handicap, and reduce the mixed impedance exterior and interior BVPs to equivalent uniquely solvable boundary pseudodifferential equations for smooth and Lipschitz domains, for arbitrary values of the oscillation frequency parameter. Moreover, for smooth domains we show uniqueness and existence in Bessel-potential (H_p^s) and Besov ($B_{p,q}^s$) spaces, and obtain almost best regularity results. For the displacement vector we establish C^α-smoothness with $\alpha \in (0,1/2)$. Notice that, in general, solutions to mixed BVPs, even for given C^∞-regular data, are not in C^α with $\alpha > 1/2$ at the collision curves, while they are infinitely differentiable elsewhere (see, e.g., [**11, 19, 28**]).

The results of the present work will be used in the study of inverse problems for partially coated obstacles. This study is in progress, and will be communicated separately.

The paper is organized as follows. In Section 2, we first formulate the interior and exterior mixed impedance problems in Sobolev-Slobodetski, Bessel potential and Besov spaces, and derive integral representations of solutions. Further, we introduce the boundary operators generated by single and double layer potentials, and describe the basic mapping and jump properties of these potentials. In Section 3, we prove the invertibility of the corresponding pseudodifferential operators for arbitrary values of the oscillation parameter and establish basic uniqueness, existence and regularity results for the original mixed impedance problems. Although the present paper treats 3D problems, the whole theory "passes" without any difficulty whatsoever to the 2D case; the form of the fundamental matrices and the Sommerfeld-Kupradze radiation condition for two dimensions are given in Section 3.6. Finally, for the reader's convenience, we include a brief appendix containing some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces, which are the main tools for proving existence theorems for mixed boundary, boundary-transmission and crack problems by the potential methods.

2. Preliminary material.

2.1. Formulation of the mixed impedance problems. We study the three-dimensional mixed impedance interior and exterior boundary value problems for equations of steady state elastic oscillations when an elastic body occupies either an interior bounded domain Ω^+ or its complement $\Omega^- = \mathbf{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^+} = \Omega^+ \cup S$. For simplicity we assume that $S = \partial\Omega^+ = \partial\Omega^-$ is simply connected. Moreover, we assume that the surface S is divided into two simply connected sub-manifolds, the so called Dirichlet part S_D and the partially coated (impedance) part S_I: $S = \overline{S_D} \cup \overline{S_I}$. Throughout the paper $n = n(x)$ denotes the outward unit normal vector at the point $x \in S$.

We make here the following remark concerning the smoothness of the manifold S and the curve $\ell := \partial S_D \cap \partial S_I$. To demonstrate our approach, for simplicity, mainly we will consider two cases: either the manifold S and the curve ℓ are C^∞-smooth or they are Lipschitz. Note that for Lipschitz surfaces the components of the normal vector belong to the space of essentially bounded functions $L_\infty(S)$. It should be mentioned that the results obtained in this paper remain valid with evident reformulation also for surfaces with finite smoothness (cf., e.g., [**9, 11**]).

The mixed impedance interior and exterior boundary value problems $(\mathrm{MIP})^\pm$ read as follows: *Find a vector* $u = (u_1, u_2, u_3)^\top$ *satisfying*

(i) *the differential equation*

(2.1)
$$A(\partial,\omega)\,u(x) := \mu\,\Delta\,u(x) + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,u(x) + \varrho\,\omega^2\,u(x) = 0 \quad \text{in } \Omega^\pm,$$

(ii) *the boundary conditions*

(2.2) $$r_{S_D}\{u\}^\pm = f \quad \text{on } S_D,$$

(2.3) $$r_{S_I}[\{Tu\}^\pm + i\,\omega\,c\,\{u\}^\pm] = h \quad \text{on } S_I,$$

(iii) *in the case of the exterior problem for the domain* Ω^- *the vector* u *satisfies the Sommerfeld-Kupradze type radiation conditions at infinity, i.e.,* u *is representable as a sum of metaharmonic vectors, the so-called longitudinal* $u^{(1)} = u^{(p)}$ *and transverse parts* $u^{(2)} = u^{(s)}$, [**22**]

(2.4) $$u = u^{(1)} + u^{(2)} \quad \textit{with } \Delta\,u^{(1)} + k_1^2\,u^{(1)} = 0,\ \Delta\,u^{(2)} + k_2^2\,u^{(2)} = 0,$$

$$k_1 \equiv k_p = \omega\sqrt{\frac{\varrho}{\lambda+2\,\mu}}, \quad k_2 \equiv k_s = \omega\sqrt{\frac{\varrho}{\mu}},$$

and for sufficiently large $r = |x|$

(2.5)
$$\frac{\partial u^{(1)}(x)}{\partial r} - i\,k_1\,u^{(1)}(x) = o\,(r^{-1}), \quad \frac{\partial u^{(2)}(x)}{\partial r} - i\,k_2\,u^{(2)}(x) = o\,(r^{-1});$$

here $c > 0$ is a positive constant, λ and μ are the Lamé constants, ϱ is the density of the elastic material and $\omega \in \mathbf{R}$ is the so-called frequency parameter; further, $A(\partial, \omega)$ stands for the matrix elastic oscillation operator

$$\begin{aligned} A(\partial, \omega) &:= A(\partial) + \varrho\,\omega^2\,I_3, \\ A(\partial) &:= [\mu\,\delta_{kj}\,\Delta + (\lambda+\mu)\partial_k\,\partial_j]_{3\times 3}, \quad I_3 = [\delta_{kj}]_{3\times 3}, \end{aligned} \tag{2.6}$$

while $T(\partial, n)$ and $T(\partial, n)u$ denote the stress operator and the stress vector

(2.7)
$$T(\partial, n) := [T_{kj}(\partial, n)]_{3\times 3}, \quad T_{kj}(\partial, n) = \lambda\,n_k\,\partial_{x_j} + \mu\,n_j\,\partial_{x_k} + \mu\,\delta_{kj}\,\partial_n,$$

$$\begin{aligned} \{T(\partial, n)u\}_k = \sigma_{kj}\,n_j, \quad \sigma_{kj} &= [\lambda\,\delta_{kj}\,\mathrm{div}\,u + 2\,\mu\,e_{kj}(u)]\,n_j, \\ e_{kj} &= 2^{-1}\,(\partial_k u_j + \partial_j u_k), \end{aligned} \tag{2.8}$$

where Δ is the Laplace operator, δ_{kj} is the Kronecker delta, $\partial_j = \partial_{x_k} = \partial/\partial x_k$ denotes partial differentiation with respect to the variable x_k, n is the unit normal vector and $\partial_n = \partial/\partial n$ stands for the normal derivative, $e_{kj} = e_{kj}(u)$ and $\sigma_{kj} = \sigma_{kj}(u)$ denote the strain and stress tensors, respectively.

Here and in what follows the summation over repeated indices is meant from 1 to 3, unless stated otherwise, and the symbol $U^\top$ denotes the transpose of U. The symbols $\{\cdot\}^+$ and $\{\cdot\}^-$ denote the interior and exterior one-sided limits on $S = \partial\Omega^\pm$ from $\Omega^\pm$ respectively. We will use also the notation $\{\cdot\}_S^\pm$ for the trace operators on S. The symbol $r_{\mathcal{M}}$ denotes the restriction operator onto $\mathcal{M}$.

Note that the radiation conditions (2.5) automatically yield the following decay conditions at infinity (for details see [**34**])

$$\begin{aligned} u^{(l)}(x) = \mathcal{O}(r^{-1}), \qquad \partial_j u^{(l)}(x) - i\,k_l\,\frac{x_j}{r}u^{(l)}(x) = \mathcal{O}(r^{-2}), \\ l = 1, 2, \quad j = 1, 2, 3. \end{aligned} \tag{2.9}$$

Since solutions to mixed BVPs do not possess $C^1(\overline{\Omega^\pm})$-smoothness, in general, we look for solutions of the problems $(\mathrm{MIP})^\pm$ in the Sobolev-Slobodetski, Bessel potential and Besov spaces in order to establish almost bets regularity results.

To this end let us introduce some notation.

By L_p, W_p^r, H_p^s and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbf{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [**23, 33**]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$ and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbf{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let $\mathcal{M}_0$ be a Lipschitz surface without boundary. For a Lipschitz submanifold $\mathcal{M} \subset \mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M}) = \{g : g \in H_p^s(\mathcal{M}_0),\ \mathrm{supp}\ \ g \subset \overline{\mathcal{M}}\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M}) = \{g : g \in B_{p,q}^s(\mathcal{M}_0),\ \mathrm{supp}\, g \subset \overline{\mathcal{M}}\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M}) = \{r_{\mathcal{M}} f : f \in H_p^s(\mathcal{M}_0)\},\ B_{p,q}^s(\mathcal{M}) = \{r_{\mathcal{M}} f : f \in B_{p,q}^s(\mathcal{M}_0)\}.$$

We look for solutions of the above formulated mixed impedance interior problem in the space $W_p^1(\Omega^+)$, while solutions of the exterior problem are sought in the space $W_{p,\,\mathrm{loc}}^1(\Omega^-) \cap SK(\Omega^-)$, where $1 < p < +\infty$ and $SK(\Omega^-)$ denotes the set of functions satisfying the Sommerfeld-Kupradze radiation conditions at infinity (2.5). In the case of such a formulation, the equation (2.1) is understood in the distributional or in the weak sense, the Dirichlet type condition (2.2) is understood in the trace sense, and finally the Neumann type condition (2.3) is understood in the functional-generalized trace sense defined with the help of Green's identities.

Recall that for sufficiently regular vector functions $u, v \in [C_2(\overline{\Omega^+})]^3$ and $C^{1,\alpha}$-smooth domains we have the following Green's formula [**22**]

$$\int_S \{Tu\}^+ \cdot \{v\}^+ \, dS = \int_{\Omega^+} A(\partial, \omega) u \cdot v \, dx + \int_{\Omega^+} [E(u, \overline{v}) - \varrho\, \omega^2\, u \cdot v] \, dx,$$

where the central dot denotes the scalar product in $\mathbf{C}^3$ and
(2.10)

$$E(u,v) = \frac{3\,\lambda + 2\,\mu}{3}\;\mathrm{div}\,u\;\mathrm{div}\,v + \frac{\mu}{2}\sum_{k\neq j}(\partial_j u_k + \partial_k u_j)\,(\partial_j v_k + \partial_k v_j)$$
$$+ \frac{\mu}{3}\sum_{k,j}(\partial_k u_k - \partial_j u_j)\,(\partial_k v_k - \partial_j v_j), \quad \mu > 0,\ 3\,\lambda + 2\,\mu > 0.$$

It is evident that $E(u,\overline{u}) \geq 0$, with the equality holding only for rigid displacement vectors, i.e., for vectors $\chi(x) = [a \times x] + b$, where a and b are some constant three-dimensional complex valued vectors and $\times$ denotes the cross product (see, e.g., [**22**]).

Note that the above Green's formula can be generalized, by a standard limiting procedure, to Lipschitz domains and to vector-functions from the corresponding Sobolev-Slobodetski, Bessel potential and Besov spaces. In particular, we can generalize this formula for vector functions $u \in [W_p^1(\Omega^+)]^3$ with $A(\partial)u \in [L_p(\Omega^+)]^3$ and $v \in [W_{p'}^1(\Omega^+)]^3$ with $1/p + 1/p' = 1$, $1 < p < +\infty$,
(2.11)

$$\langle \{Tu\}^+,\ \{\overline{v}\}^+\rangle_S = \int_{\Omega^+} A(\partial,\omega)u\cdot v\,dx + \int_{\Omega^+}[E(u,\overline{v}) - \varrho\,\omega^2\,u\cdot v]\,dx,$$

where the symbol $\langle\cdot,\cdot\rangle_S$ denotes duality brackets between the adjoint spaces $[B_{p,p}^{-1/p}(S)]^3$ and $[B_{p',p'}^{1/p}(S)]^3$. Note that due to the embedding $\{v\}^+ \in [B_{p',p'}^{1/p}(S)]^3$, this relation defines the generalized stress trace functional $\{Tu\}^+ \in [B_{p,p}^{-1/p}(S)]^3$.

For the vector $u \in [W_{p,\ \mathrm{loc}}^1(\Omega^-)]^3$ with $A(\partial)u \in [L_{p,\ \mathrm{loc}}(\Omega^-)]^3$ the functional $\{Tu\}^-$ is determined quite similarly by formula
(2.12)

$$\langle \{Tu\}^-, \{\overline{v}\}^-\rangle_S = -\int_{\Omega^-} A(\partial,\omega)u\cdot v\,dx - \int_{\Omega^-}[E(u,\overline{v}) - \varrho\,\omega^2\,u\cdot v]\,dx,$$

with $v \in [W_{p',\ \mathrm{comp}}^1(\Omega^-)]^3$.

Since v has a compact support and $\{v\}_S^- \in [B_{p',p'}^{1/p}(S)]^3$, the right hand side of (2.12) is well defined and defines the functional $\{Tu\}^- \in [B_{p,p}^{-1/p}(S)]^3$.

The Neumann condition (2.3) is understood in the functional sense just described.

We note that due to the strong ellipticity of the operator $A(\partial)$, every solution of the equation (2.1) is actually $C^\infty-$regular in $\Omega^\pm$, in view of the interior regularity property (see, e.g., [**13**]).

2.2. General integral representations and properties of potentials. Denote by $\Gamma(x,\omega)$ and $\Gamma(x)$ the matrices of fundamental solutions of the differential operators $A(\partial,\omega)$ and its principal part $A(\partial)$, $A(\partial,\omega)\Gamma(x,\omega) = I_3\,\delta(x)$ and $A(\partial)\Gamma(x) = I_3\,\delta(x)$, where $\delta(x)$ is the Dirac delta function. These matrices read as follows (see [**22**, Chapter 2, Section 1], [**24**, Chapter VIII, Section 130])

$$\Gamma(x,\omega) = [\Gamma_{kj}(x,\omega)]_{3\times 3},$$

$$\Gamma_{kj}(x,\omega) = \sum_{l=1}^{2} (\delta_{kj}\alpha_l + \beta_l\,\partial_k\,\partial_j)\,\frac{e^{ik_l\,|x|}}{|x|},$$

$$\alpha_l = -\frac{\delta_{2l}}{4\,\pi\,\mu}, \qquad \beta_l = \frac{(-1)^{l+1}}{4\,\pi\,\varrho\,\omega^2},$$

$$\Gamma(x) = [\Gamma_{kj}(x)]_{3\times 3}, \qquad \Gamma_{kj}(x) = \frac{\delta_{kj}\,\lambda'}{|x|} + \frac{\mu'\,x_k\,x_j}{|x|^3},$$

$$\lambda' = -\frac{\lambda+3\,\mu}{8\,\pi\,\mu\,(\lambda+2\,\mu)}, \qquad \mu' = -\frac{\lambda+\mu}{8\,\pi\,\mu\,(\lambda+2\,\mu)}.$$

We have the following relations:

(2.13)

$$\begin{aligned}
&\Gamma(x,\omega) = \Gamma(-x,\omega) = [\Gamma(x,\omega)]^\top, \qquad \Gamma(x) = \Gamma(-x) = [\Gamma(x)]^\top,\\
&|\Gamma(x,\omega)| \le |x|^{-1}\,c(\lambda,\mu), \qquad |\Gamma(x,\omega)-\Gamma(x)| \le |\omega|\,c(\lambda,\mu),\\
&|\partial_j\Gamma(x,\omega)-\partial_j\Gamma(x)| \le |\omega|^2\,c(\lambda,\mu),\\
&|\partial_j\partial_l\Gamma(x,\omega)-\partial_j\partial_l\Gamma(x)| \le |x|^{-1}\,c(\lambda,\mu,\omega),
\end{aligned}$$

showing that the matrix of statics $\Gamma(x)$ is the principal singular homogeneous part of the matrix of oscillations $\Gamma(x,\omega)$. It is evident that the entries of $\Gamma(x,\omega)$ and $\Gamma(x)$ are real analytic functions in $\mathbf{R}^3\setminus\{0\}$ and, moreover, $\Gamma(x,\omega)$ satisfies the Sommerfeld-Kupradze radiation condition at infinity.

By standard arguments, applying the corresponding Green's formulae, we can derive the following integral representation of solutions to the equation (2.1)

$$W(\{u\}^+)(x) - V(\{Tu\}^+)(x) = \begin{cases} u(x) & \text{in } \Omega^+,\\ 0 & \text{in } \Omega^-, \end{cases} \tag{2.14}$$

where V and W are the single and double layer potentials

$$V(g)(x) := \int_S \Gamma(x-y,\omega)\, g(y)\, dS_y, \quad x \in \mathbf{R}^3 \setminus S,$$
$$W(h)(x) := \int_S [T(\partial_y, n(y))\Gamma(x-y,\omega)]^\top\, h(y)\, dS_y, \quad x \in \mathbf{R}^3 \setminus S,$$

$g = (g_1, g_2, g_3)^\top$ and $h = (h_1, h_2, h_3)^\top$ being the densities of the corresponding potentials.

Quite similarly, for a radiating solution of the equation (2.1) we have the representation (see [**22, 27**])

$$-W(\{u\}^-)(x) + V(\{Tu\}^-)(x) = \begin{cases} 0 & \text{in } \Omega^+, \\ u(x) & \text{in } \Omega^-. \end{cases} \tag{2.15}$$

These representations remain valid for solutions of the class $W^1_p(\Omega^+)$ and $W^1_{p,\,\mathrm{loc}}(\Omega^-) \cap SK(\Omega^-)$. From these representation formulae it is evident that any solution to the equation (2.1) is actually an analytic vector function of the real variable $x \in \Omega^\pm$ in the corresponding domains. Further, if $u \in W^1_p(\Omega^+) \cap W^1_{p,\,\mathrm{loc}}(\Omega^-) \cap SK(\Omega^-)$ solves the equation (2.1) in Ω^+ and Ω^-, then by adding of formulae (2.14) and (2.15) we get

$$u(x) = W(\{u\}^+ - \{u\}^-)(x) - V(\{Tu\}^+ - \{Tu\}^-)(x) \quad \text{in } \Omega^+ \cup \Omega^-,$$

which shows that if on some open part $S_1 \subset S$ of the common boundary S of the adjacent domains Ω^+ and Ω^- the jumps of the Cauchy data equal to zero, i.e., $r_{S_1}[\{u\}^+ - \{u\}^-] = 0$ and $r_{S_1}[\{Tu\}^+ - \{Tu\}^-] = 0$, then the vector function $\widetilde{u}$ defined by the equality

$$\widetilde{u} := \begin{cases} u(x) & \text{for } x \in \Omega^+, \\ u(x) & \text{for } x \in \Omega^-, \\ \{u\}^+ & \text{for } x \in S_1, \end{cases} \tag{2.16}$$

is a real analytic vector in the connected domain $\mathbf{R}^3 \setminus \overline{S_2}$ with $S_2 = S \setminus \overline{S_1}$. Therefore if $\widetilde{u}$ vanishes either in Ω^+ or in Ω^-, then it vanishes in $\mathbf{R}^3$.

Further we introduce the boundary operators generated by the single and double layer potentials. For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

(2.17)
$$(\mathcal{H}\, g)(x) := \int_S \Gamma(x-y,\omega)\, g(y)\, dS_y, \quad x \in S,$$

(2.18)
$$(\mathcal{K}\, g)(x) := \int_S \big[T(\partial_x, n(x))\Gamma(x-y,\omega)\big]\, g(y)\, dS_y, \quad x \in S,$$

(2.19)
$$(\mathcal{K}^*\, h)(x) := \int_S \big[T(\partial_y, n(y))\, \Gamma(x-y,\omega)\big]^\top\, h(y)\, dS_y, \quad x \in S,$$

(2.20)
$$(\mathcal{L}h)(x) := \big\{T(\partial_x, n(x))W(h)(x)\big\}^\pm, \quad x \in S.$$

The boundary operators $\mathcal{H}$ and $\mathcal{L}$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}$ and $\mathcal{K}^*$ are mutually adjoint singular integral operators–pseudodifferential operators of order 0 (for details see [**1, 2, 7, 17, 18, 19, 22**]).

We will denote the potentials constructed by the matrix $\Gamma(x-y)$, and the corresponding boundary operators, by the same symbols as above but equipped with the subscript 0, e.g., $V_0(g)\; W_0(h)$, $(\mathcal{H}_0 g), \dots, (\mathcal{L}_0 h)$.

Now we describe the basic mapping and jump properties of the above introduced layer potentials. They can be found in [**9, 10, 17, 18, 19**]. Note that the main ideas for generalization to the scale of Bessel potential and Besov spaces are based on the duality and interpolation technique, and are described in [**30**] using the theory of pseudodifferential operators on smooth manifolds without boundary, while in [**1, 2, 6, 14, 25, 26**], for general Lipschitz boundaries.

Theorem 2.1. *Let S be $C^\infty-$smooth and $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbf{R}$. The operators*

$$V : \big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_p^{s+1+1/p}(\Omega^+)\big]^3$$
$$\Big[\big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H^{s+1+1/p}_{p,\,\mathrm{loc}}(\Omega^-) \cap SK(\Omega^-)\big]^3\Big],$$

$$
\begin{aligned}
&: \left[B^s_{p,t}(S)\right]^3 \longrightarrow \left[B^{s+1+1/p}_{p,t}(\Omega^+)\right]^3 \\
&\quad \Big[\left[B^s_{p,t}(S)\right]^3 \longrightarrow \left[B^{s+1+1/p}_{p,t,\ \mathrm{loc}}(\Omega^-)\cap SK(\Omega^-)\right]^3\Big], \\
W &: \left[B^s_{p,p}(S)\right]^3 \longrightarrow \left[H^{s+1/p}_{p}(\Omega^+)\right]^3 \\
&\quad \Big[\left[B^s_{p,p}(S)\right]^3 \longrightarrow \left[H^{s+1/p}_{p,\ \mathrm{loc}}(\Omega^-)\cap SK(\Omega^-)\right]^3\Big], \\
&: \left[B^s_{p,t}(S)\right]^3 \longrightarrow \left[B^{s+1/p}_{p,t}(\Omega^+)\right]^3 \\
&\quad \Big[\left[B^s_{p,t}(S)\right]^3 \longrightarrow \left[B^{s+1/p}_{p,t,\ \mathrm{loc}}(\Omega^-)\cap SK(\Omega^-)\right]^3\Big],
\end{aligned}
$$

are continuous. If S is Lipschitz, then the operators

$$
\begin{aligned}
V &: \left[H^{-1/2}_2(S)\right]^3 \longrightarrow \left[H^1_2(\Omega^+)\right]^3 \\
&\quad \Big[\left[H^{-1/2}_2(S)\right]^3 \longrightarrow \left[H^1_{2,\ \mathrm{loc}}(\Omega^-)\cap SK(\Omega^-)\right]^3\Big], \\
W &: \left[H^{1/2}_2(S)\right]^3 \longrightarrow \left[H^1_2(\Omega^+)\right]^3 \\
&\quad \Big[\left[H^{1/2}_2(S)\right]^3 \longrightarrow \left[H^1_{2,\ \mathrm{loc}}(\Omega^-)\cap SK(\Omega^-)\right]^3\Big],
\end{aligned}
$$

are continuous.

Theorem 2.2. *Let S be $C^\infty-$smooth and $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbf{R}$ and*

$$g \in \left[B^{-1/p}_{p,t}(S)\right]^3, \qquad h \in \left[B^{1-1/p}_{p,t}(S)\right]^3.$$

Then

$$
\begin{aligned}
&\left\{V(g)\right\}^+ = \left\{V(g)\right\}^- = \mathcal{H}\,g \quad \text{on } S, \\
&\left\{T(\partial,n)V(g)\right\}^\pm = \left[\,\mp\, 2^{-1}I_3 + \mathcal{K}\right] g, \quad \text{on } S, \\
&\left\{W(h)\right\}^\pm = \left[\,\pm\, 2^{-1}I_3 + \mathcal{K}^*\right] h \quad \text{on } S.
\end{aligned}
$$

Moreover,

$$\left\{\mathcal{T}(\partial,n)W(h)\right\}^+ = \left\{\mathcal{T}(\partial,n)W(h)\right\}^- = \mathcal{L}h \quad \text{on } S.$$

The same relations hold for a Lipschitz boundary S and $p = t = 2$.

Theorem 2.3. (i) *Let* S *be* $C^\infty-$*smooth and* $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbf{R}$. *The operators*

$$\begin{aligned} \mathcal{H} : \big[H_p^s(S)\big]^3 &\longrightarrow \big[H_p^{s+1}(S)\big]^3 \\ \Big[\big[B_{p,t}^s(S)\big]^3 &\longrightarrow \big[B_{p,t}^{s+1}(S)\big]^3\Big], \\ \pm 2^{-1} I_3 + \mathcal{K},\ \pm 2^{-1} I_3 + \mathcal{K}^* : \big[H_p^s(S)\big]^3 &\longrightarrow \big[H_p^s(S)\big]^3 \\ \Big[\big[B_{p,t}^s(S)\big]^3 &\longrightarrow \big[B_{p,t}^s(S)\big]^3\Big], \\ \mathcal{L} : \big[H_p^{s+1}(S)\big]^3 &\longrightarrow \big[H_p^s(S)\big]^3 \\ \Big[\big[B_{p,t}^{s+1}(S)\big]^3 &\longrightarrow \big[B_{p,t}^s(S)\big]^3\Big], \end{aligned}$$

are continuous and Fredholm with zero index.

(ii) *If* S *is Lipschitz, then the operators*

$$\begin{aligned} \mathcal{H} : \big[H_2^{-1/2}(S)\big]^3 &\longrightarrow \big[H_2^{1/2}(S)\big]^3, \\ \pm 2^{-1} I_3 + \mathcal{K} : \big[H_2^{-1/2}(S)\big]^3 &\longrightarrow \big[H_2^{-1/2}(S)\big]^3, \\ \pm 2^{-1} I_3 + \mathcal{K}^* : \big[H_2^{1/2}(S)\big]^3 &\longrightarrow \big[H_2^{1/2}(S)\big]^3, \\ \mathcal{L} : \big[H_2^{1/2}(S)\big]^3 &\longrightarrow \big[H_2^{-1/2}(S)\big]^3, \end{aligned}$$

are continuous and Fredholm with zero index, and moreover, there are positive constants C_1 *and* C_2 *such that*

$$\langle \mathcal{L} g,\, g\rangle_S \ge C_1\, ||g||^2_{[H_2^{1/2}(S)]^3} - C_2\, ||g||^2_{[H_2^{0}(S)]^3}$$
$$\text{for all } g \in [H_2^{1/2}(S)]^3,$$

where the symbol $\langle\cdot,\cdot\rangle_S$ *denotes duality brackets between the adjoint spaces* $[H_2^{-1/2}(S)]^3$ *and* $[H_2^{1/2}(S)]^3$.

(iii) *The following operator equalities hold in appropriate function spaces*:

$$\begin{aligned} \mathcal{K}^*\, \mathcal{H} &= \mathcal{H}\, \mathcal{K}, \\ \mathcal{L}\, \mathcal{K}^* &= \mathcal{K}\, \mathcal{L}, \\ \mathcal{L}\, \mathcal{H} &= -4^{-1} I_3 + \big[\mathcal{K}\big]^2, \\ \mathcal{H}\, \mathcal{L} &= -4^{-1} I_3 + \big[\mathcal{K}^*\big]^2. \end{aligned} \tag{2.21}$$

The above theorems hold true for the potentials V_0 and W_0, and for the operators $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{K}_0^*$ and $\mathcal{L}_0$.

2.3. Auxiliary problems. Here we study the following auxiliary impedance (Robin type) interior and exterior boundary value problems.

Problem $(\mathrm{I})^+$: *Find a vector* $u = (u_1, u_2, u_3)^\top \in [W_p^1(\Omega^+)]^3$, $1 < p < +\infty$, *satisfying*

(i) *the differential equation*

$$A(\partial, \omega)\, u(x) = 0 \quad \text{in } \Omega^+, \tag{2.22}$$

(ii) *the boundary condition*

$$\{Tu\}^+ + i\,\omega\, c\,\{u\}^+ = h \quad \text{on } S, \quad h \in [B_{p,p}^{-1/p}(S)]^3. \tag{2.23}$$

Problem $(\mathrm{I})^-$: *Find a vector* $u = (u_1, u_2, u_3)^\top \in [W^1_{p,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$, $1 < p < +\infty$, *satisfying*

(i) *the differential equation*

$$A(\partial, \omega)\, u(x) = 0 \quad \text{in } \Omega^-, \tag{2.24}$$

(ii) *the boundary condition*

$$\{Tu\}^- + i\,\omega\, c\,\{u\}^- = h \quad \text{on } S, \quad h \in [B_{p,p}^{-1/p}(S)]^3. \tag{2.25}$$

We start with a uniqueness theorem.

Theorem 2.4. *Let S be Lipschitz. For $p = 2$ the homogeneous problems* $(\mathrm{I})^\pm$ *have only the trivial solution.*

Proof. Let $u \in [W_2^1(\Omega^+)]^3$ be a solution to the homogeneous problem $(\mathrm{I})^+$. Since $A(\partial)u \in [L_2(\Omega^+)]^3$ we can apply Green's formula (2.11) with $v = u$, which in view of the homogeneous impedance condition (2.23) leads to the relation

$$\int_{\Omega^+} [E(u, \overline{u}) - \varrho\, \omega^2 |u|^2]\, dx + i\,\omega\, c \int_S |\{u\}^+|^2\, dS = 0.$$

Separating the imaginary part we get $\{u\}^+ = 0$ on S and, consequently, $\{Tu\}^+ = 0$ on S due to the homogeneous impedance condition (2.23). Then by (2.14) we conclude $u = 0$ in Ω^+.

Now, let $u \in [W_2^1(\Omega^-)]^3 \cap SK(\Omega^-)$ be a solution to the homogeneous problem $(\mathrm{I})^-$. Further, let R be a sufficiently large number, such that $\Omega^+ \subset B(0,R) := \{y \in \mathbf{R}^3 : |y| < R\}$. We use the notation $\Omega_R^- := \Omega^- \cap B(0,R)$ and $\Sigma_R := \partial B(0,R)$. We can write Green's formula (2.11) in Ω_R^- with $v = u$, as

$$\int_{\Omega_R^-} [E(u,\overline{u}) - \varrho\, \omega^2 |u|^2]\, dx - i\,\omega\, c \int_S |\{u\}^-|^2\, dS - \int_{\Sigma_R} T(\partial, n) u \cdot u\, d\Sigma_R = 0.$$

Again, by separating the imaginary part we get

$$\omega\, c \int_S |\{u\}^-|^2\, dS + \Im \int_{\Sigma_R} T(\partial, n) u \cdot u\, d\Sigma_R = 0. \tag{2.26}$$

With the help of (2.9) we can rewrite (2.26) as

$$\omega\, c \int_S |\{u\}^-|^2\, dS + \Im \int_{\Sigma_R} \sum_{l,q=1}^{2} i k_l\, T(\widehat{x}, \widehat{x})\, u^{(l)} \cdot u^{(q)}\, d\Sigma_R + \mathcal{O}(R^{-1}) = 0, \tag{2.27}$$

where $\widehat{x} = x/|x|$ and $\widehat{x} = n(x)$ is the exterior unit normal vector at the point $x \in \Sigma_R$.

Note that due to (2.6) and (2.7), we have $T(\widehat{x}, \widehat{x}) = A(\widehat{x}) = [\mu\, \delta_{kj} + (\lambda+\mu)\, \widehat{x}_k\, \widehat{x}_j]_{3\times 3}$. On the other hand, the matrix $A(\widehat{x})$ is positive definite and for sufficiently large $|x|$

$$u^{(l)} = \frac{e^{i\, k_l\, |x|}}{|x|}\, u_\infty^{(l)}(\widehat{x}) + \mathcal{O}(|x|^{-2}),$$

where $u_\infty^{(l)}(\widehat{x})$ is the so-called *far field pattern* of the metaharmonic vector function $u^{(l)}$ (see, e.g., [**5, 34**]). Therefore from (2.27) it follows

$$\begin{aligned} \omega\, c &\int_S |\{u\}^-|^2\, dS \\ &+ \Im \int_{\Sigma_R} \sum_{l,q=1}^{2} i k_l\, A(\widehat{x})\, u_\infty^{(l)}(\widehat{x}) \cdot u_\infty^{(q)}(\widehat{x})\, \frac{e^{i\,(k_l - k_q)\, R}}{R^2}\, d\Sigma_R \\ &+ \mathcal{O}(R^{-1}) = 0, \end{aligned}$$

and, consequently,

$$\begin{aligned}
&\omega\, c \int_S |\{u\}^-|^2\, dS \\
&+ \int_{\Sigma_1} \Big(k_1 A(\widehat{x})\, u_\infty^{(1)}(\widehat{x}) \cdot u_\infty^{(1)}(\widehat{x}) + k_2 A(\widehat{x})\, u_\infty^{(2)}(\widehat{x}) \cdot u_\infty^{(2)}(\widehat{x}) \Big)\, d\Sigma_1 \\
&+ \Im \int_{\Sigma_1} \sum_{l \neq q} i\, k_l\, A(\widehat{x})\, u_\infty^{(l)}(\widehat{x}) \cdot u_\infty^{(q)}(\widehat{x})\, e^{i\,(k_l - k_q)\,R}\, d\Sigma_1 + \mathcal{O}(R^{-1}) = 0.
\end{aligned}$$

Take into consideration that $k_1 \neq k_2$ and integrate the last equality from t to $2t$ with respect to R and divide the result by t to obtain

$$\begin{aligned}
&\omega\, c \int_S |\{u\}^-|^2\, dS \\
&\quad + \int_{\Sigma_1} \Big(k_1\, A(\widehat{x})\, u_\infty^{(1)}(\widehat{x}) \cdot u_\infty^{(1)}(\widehat{x}) + k_2\, A(\widehat{x})\, u_\infty^{(2)}(\widehat{x}) \cdot u_\infty^{(2)}(\widehat{x}) \Big)\, d\Sigma_1 \\
&\quad \Im \Bigg[\sum_{l \neq q} \frac{e^{i\,(k_l - k_q)\,2t} - e^{i\,(k_l - k_q)\,t}}{i\,(k_l - k_q)\,t} \\
&\qquad\qquad \cdot \int_{\Sigma_1} i\, k_l\, A(\widehat{x})\, u_\infty^{(l)}(\widehat{x}) \cdot u_\infty^{(q)}(\widehat{x})\; d\Sigma_1 \Bigg] + \mathcal{O}(t^{-1}) = 0.
\end{aligned}$$

Since the last two summands tend to zero as $t \to +\infty$, we arrive at the equation

$$\begin{aligned}
&\omega\, c \int_S |\{u\}^-|^2\, dS \\
&+ \int_{\Sigma_1} \Big(k_1\, A(\widehat{x})\, u_\infty^{(1)}(\widehat{x}) \cdot u_\infty^{(1)}(\widehat{x}) + k_2\, A(\widehat{x})\, u_\infty^{(2)}(\widehat{x}) \cdot u_\infty^{(2)}(\widehat{x}) \Big)\, d\Sigma_1 = 0.
\end{aligned}$$

Now, since $c > 0$, $\operatorname{sign} \omega = \operatorname{sign} k_l$, $l = 1, 2$, and the matrix $A(\widehat{x})$ is positive definite, we get

(2.28) $$\{u\}^- = 0 \quad \text{on } S,$$

(2.29) $$u_\infty^{(l)}(\widehat{x}) = 0, \quad l = 1, 2, \text{ for all } \widehat{x} \in \Sigma_1.$$

The equality (2.28) implies $\{Tu\}^- = 0$ on S due to the homogeneous impedance condition (2.25). Therefore, by the representation formula (2.15) we finally obtain $u = 0$ in Ω^-.

Note that the same result for $u = u^{(1)} + u^{(2)}$ follows also from the equalities (2.29) since they yield $u^{(l)}(x) = 0$, $l = 1, 2$, in Ω^- due to the well known Rellich-Vekua theorem for metaharmonic functions (see, e.g., [**5, 34**]). $\square$

To study the existence of solutions of the above formulated auxiliary boundary value problems we proceed as follows. We look for a solution of the BVP problem $(\mathrm{I})^-$ in the form

$$u(x) = W(g)(x) + i\,\varkappa\,V(g)(x), \quad x \in \Omega^-, \tag{2.30}$$

where $\varkappa$ is a nonzero real number and $g \in (g_1, g_2, g_3)^\top \in [B^{1-1/p}_{p,p}(S)]^3$ is an unknown density.

We assume that $1 < p < +\infty$ if S is a smooth surface, and that $p = 2$ if S is a Lipschitz manifold.

Clearly, $u \in W^1_{p,\,\mathrm{loc}}(\Omega^-) \cap SK(\Omega^-)$ by Theorem 2.1 and solves the equation (2.24). In view of Theorem 2.2, the boundary condition (2.25) leads to the equation

$$\mathcal{L}\,g + i\,\varkappa\,[2^{-1}\,I_3 + \mathcal{K}]\,g + i\,\omega\,c\,[-2^{-1}\,I_3 + \mathcal{K}^* + i\,\varkappa\,\mathcal{H}]\,g = h \quad \text{on } S,$$

where $\mathcal{L}$, $\mathcal{K}$, $\mathcal{K}^*$ and $\mathcal{H}$ are given by equalities (2.17)–(2.20). Let

$$\mathcal{P} := \mathcal{L} + i\,\varkappa\,[2^{-1}\,I_3 + \mathcal{K}] + i\,\omega\,c\,[-2^{-1}\,I_3 + \mathcal{K}^* + i\,\varkappa\,\mathcal{H}]. \tag{2.31}$$

By Theorem 2.3, when S is a $C^\infty-$regular surface, we have the following mapping property for the pseudodifferential (singular integro-differential) operator (2.31)

$$\begin{gathered} \mathcal{P} : [H^{s+1}_p(S)]^3 \longrightarrow [H^s_p(S)]^3 \qquad \Big[[B^{s+1}_{p,t}(S)]^3 \longrightarrow [B^s_{p,t}(S)]^3\Big], \\ s \in \mathbf{R}, \quad 1 < p < +\infty, \quad 1 \le t \le +\infty. \end{gathered} \tag{2.32}$$

In the case when S is Lipschitz, we have to take $s = -1/2$ and $p = t = 2$ in (2.32). It can easily be seen that, for both smooth and Lipschitz cases, the principal singular part of the operator $\mathcal{P}$ is the first summand $\mathcal{L}$ since the operators

$$\mathcal{P} - \mathcal{L} : [H^{s+1}_p(S)]^3 \to [H^s_p(S)]^3 \qquad \Big[[B^{s+1}_{p,t}(S)]^3 \to [B^s_{p,t}(S)]^3\Big]$$

for $S \in C^\infty$,

$$\mathcal{P} - \mathcal{L} : [H^{1/2}_2(S)]^3 \to [H^{-1/2}_2(S)]^3 \quad \text{for Lipschitz } S,$$

are compact for all $s \in \mathbf{R}$, $1 < p < +\infty$, $1 \le t \le +\infty$ due to the mapping properties described in Theorem 2.2 and the compactness of the embeddings $H_p^{s_1}(S) \subset H_p^{s_2}(S)$ and $B_{p,t}^{s_1}(S) \subset B_{p,t}^{s_2}(S)$ with $s_1 > s_2$ for smooth S, and $H_2^{1/2}(S) \subset H_2^{-1/2}(S)$ for Lipschitz S.

Theorem 2.5. *Let S be $C^\infty-$smooth and $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbf{R}$. Then the operator* (2.32) *is invertible.*

Proof. By Theorem 2.3 the operator (2.32) is Fredholm with zero index. Therefore, in order to establish the invertibility of the operator, we have to prove that it has the trivial kernel.

First, we assume that $s = -1/2$ and $p = 2$, and let $g \in [H_2^{1/2}(S)]^3$ be a solution of the homogeneous equation $\mathcal{P}\, g = 0$. Construct the vector function u by the formula (2.30). It is easy to see that $u \in [H^1_{2,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$ and satisfies the homogenous boundary condition (2.25), i.e., u solves the BVP $(\mathrm{I})^-$. Consequently, $u = 0$ in Ω^- in view of Theorem 2.4. Further, it is clear that $u \in [H_2^1(\Omega^+)]^3$ and solves the equation (2.22) in Ω^+. Taking into account the jump relations of single and double layer potentials described in Theorem 2.2, we easily derive that $\{Tu\}^+ - \{Tu\}^- = -i\,\varkappa\, g$ and $\{u\}^+ - \{u\}^- = g$ on S, i.e.,

$$\{Tu\}^+ + i\,\varkappa\,\{u\}^+ = 0 \quad \text{on } S.$$

From Green's equality (2.11) with $v = u$ by separating the imaginary part we get $\{u\}^+ = 0$ on S, whence $\{Tu\}^+ = 0$ on S follows. Therefore $u = 0$ in Ω^+ due to the general integral representation formula (2.14), which implies $g = \{u\}^+ - \{u\}^- = 0$. Thus the operator

$$\mathcal{P} : H_2^{1/2}(S) \longrightarrow H_2^{-1/2}(S)$$

has the trivial kernel and is invertible. Taking into account that $H_2^{1/2}(S) = B_{2,2}^{1/2}(S)$ and $H_2^{-1/2}(S) = B_{2,2}^{-1/2}(S)$, we conclude that the operators (2.32) are invertible as well, since they have the same kernels for all $s \in \mathbf{R}$, $1 < p < +\infty$ and $1 \le t \le +\infty$ due to the general theory of pseudodifferential equations on manifolds without boundary. □

Theorem 2.6. *Let S be Lipschitz. Then the operator*

$$\mathcal{P} : H_2^{1/2}(S) \longrightarrow H_2^{-1/2}(S) \tag{2.33}$$

is invertible.

Proof. By Theorem 2.3 (ii) and in view of the compactness of the embedding $H_2^{1/2}(S) \subset H_2^{-1/2}(S)$ we see that (2.33) is a Fredholm operator with zero index. By the word for word arguments applied in the proof of Theorem 2.5 we show that the kernel of the operator (2.33) is trivial, which completes the proof. □

In the sequel, when dealing with the operator $\mathcal{P}$ we shall assume that it is the mapping defined either by (2.32) for smooth S, or by (2.33) for Lipschitz S. Now we can formulate the following basic assertion.

Theorem 2.7. *Let S be smooth. If $u \in [W^1_{p,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$, $1 < p < +\infty$, solves the homogeneous equation* (2.24)*, then it is uniquely representable in the form*

$$u(x) = W(\mathcal{P}^{-1}h)(x) + i\,\varkappa\,V(\mathcal{P}^{-1}h)(x), \quad x \in \Omega^-, \tag{2.34}$$

where $\mathcal{P}^{-1}$ is the inverse to the operator $\mathcal{P}$ and

$$h := \{Tu\}^- + i\,\omega\,c\,\{u\}^- \in [B^{-1/p}_{p,p}(S)]^3. \tag{2.35}$$

In the case of a Lipschitz S the same assertion holds with $p = 2$.

Proof. It follows from Theorems 2.1–2.6. Indeed, it is easy to verify that the vector h defined by (2.35) is correctly defined in view of Theorem 2.1. On the other hand, with the help of Theorems 2.2, 2.3, 2.5 and 2.6, we see that the vector

$$v(x) = W(\mathcal{P}^{-1}h)(x) + i\,\varkappa\,V(\mathcal{P}^{-1}h)(x), \quad x \in \Omega^- \tag{2.36}$$

solves the BVP $(\mathrm{I})^-$ (see (2.24), (2.25)) with h defined by (2.35). Further, we note that if the vector (2.36) vanishes in Ω^-, then $h = 0$, which follows again from Theorems 2.1 and 2.2 leading to (2.35). To complete the proof we have to show that $u(x) = v(x)$ in Ω^-. To this end let us note that both vectors u and v solve the same BVP $(\mathrm{I})^-$. Therefore it remains to prove that the homogeneous BVP $(\mathrm{I})^-$ has only the trivial solution. Until now we have the uniqueness result only for $p = 2$ due to Theorem 2.4, which covers the Lipschitz case. In the case when S is smooth, for arbitrary $p \in (1, +\infty)$, we can write the general integral representation formula (2.15) for a solution $w := u - v$ to the homogenous BVP $(\mathrm{I})^-$, (2.24)–(2.25) with $h = 0$. Due to the homogeneous boundary condition (2.25) we arrive at the formula

$$w(x) = W(g)(x) + i\,\omega\,c\,V(g)(x) \quad \text{in } \Omega^-,$$

where $g := -\{w\}^- \in [B^{1-1/p}_{p,p}(S)]^3$. Since $\{Tw\}^- + i\,\omega\,c\,\{w\}^- = 0$ on S, we get the following pseudodifferential equation with respect to g

$$\mathcal{P}\,g = 0 \quad \text{on } S,$$

where $\mathcal{P}$ is defined by (2.31) with $\varkappa = \omega\,c \neq 0$. Therefore by Theorem 2.5 we conclude that $g = 0$ on S, and consequently $w = 0$ in Ω^-. □

Corollary 2.8. *The problem* $(\mathrm{I})^-$ *is uniquely solvable and its solution is representable in the form* (2.34)*.*

Quite analogously we can prove similar results for the problem $(\mathrm{I})^+$ (see (2.22), (2.23)). Instead, let us look for a solution to the problem $(\mathrm{I})^+$ in the form

$$u(x) = V(g)(x), \quad x \in \Omega^+,$$

where g is an unknown density. The boundary condition (2.23) leads then to the following boundary integral equation

$$[-2^{-1}\,I_3 + \mathcal{K} + i\,\omega\,c\,\mathcal{H}]\,g = h \quad \text{on } S.$$

Let

$$\mathcal{Q} := -2^{-1}\,I_3 + \mathcal{K} + i\,\omega\,c\,\mathcal{H}.$$

For a C^∞-regular surface S, by Theorem 2.3 we have the following mapping property

$$\text{(2.37)}\qquad \begin{gathered} \mathcal{Q} : [H^s_p(S)]^3 \longrightarrow [H^s_p(S)]^3 \qquad \Big[[B^s_{p,t}(S)]^3 \longrightarrow [B^s_{p,t}(S)]^3\Big], \\ s \in \mathbf{R}, \quad 1 < p < +\infty, \quad 1 \le t \le +\infty. \end{gathered}$$

For a Lipschitz S we have to take $s = -1/2$ and $p = t = 2$ in (2.37). It can easily be seen that for both (smooth and Lipschitz) cases the principal singular part of the operator $\mathcal{Q}$ is the first summand $-2^{-1} I_3 + \mathcal{K}$.

Theorem 2.9. *Let S be $C^\infty-$smooth and $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbf{R}$. Then the operator* (2.37) *is invertible.*

Proof. Due to Theorem 2.3 the operator (2.37) is Fredholm with zero index. Let us show that its kernel is trivial. With the help of the uniqueness Theorem 2.4 and the jump relations for the single layer potential, it can easily be shown that the homogeneous equation $\mathcal{Q}g = 0$ has only the trivial solution for $p = 2$ and $s = -1/2$. Consequently, due to the general theory of pseudodifferential equations on manifolds without boundary, the same is valid for all $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbf{R}$, which completes the proof. □

Theorem 2.10. *Let S be Lipschitz. Then the operator*

$$\mathcal{Q} : [H_2^{-1/2}(S)]^3 \longrightarrow [H_2^{-1/2}(S)]^3 \tag{2.38}$$

is invertible.

Proof. It immediately follows from Theorems 2.3 and 2.4. □

As in the case of the operator $\mathcal{P}$, when dealing with the operator $\mathcal{Q}$, in the sequel, we shall assume that it is the mapping defined either by (2.37) for smooth S, or by (2.38) for Lipschitz S.

Theorem 2.11. *Let S be smooth. If $u \in [W^1_{p,\,\mathrm{loc}}(\Omega^+)]^3$, $1 < p < +\infty$, solves the homogeneous equation* (2.22), *then it is uniquely representable in the form*

$$u(x) = V(\mathcal{Q}^{-1}h)(x), \quad x \in \Omega^+, \tag{2.39}$$

where $\mathcal{Q}^{-1}$ is the inverse to the operator $\mathcal{Q}$ and

$$h := \{Tu\}^+ + i\,\omega\, c\,\{u\}^+ \in [B^{-1/p}_{p,p}(S)]^3. \tag{2.40}$$

In the case of a Lipschitz S, the same assertion holds with $p = 2$.

Proof. It is word for word the proof of Theorem 2.7. □

Corollary 2.12. *The problem* $(\mathrm{I})^+$ *is uniquely solvable and its solution is representable in the form* (2.39).

Remark 2.13. Note that if we apply the approach based on the single layer representation of solutions in the case of the exterior problem $(\mathrm{I})^-$, then it leads to the integral equation which is not unconditionally solvable for arbitrary values of the oscillation parameter ω. The case is that if a single layer potential vanishes in the exterior domain Ω^-, then the corresponding density is not zero, in general, since the homogeneous interior Dirichlet problem for the steady state oscillation equation may possess a nontrivial solution for the so-called exceptional values of the oscillation parameter ω. As we have seen, the representation (2.30) reduces the exterior problem $(\mathrm{I})^-$ to a pseudodifferential equation which is unconditionally solvable for arbitrary values of the oscillation parameter ω. Similar approaches have been applied to the Helmholtz equation by several authors (see, e.g., [**5, 27**] and the references therein).

3. Basic uniqueness, existence and regularity results.

3.1. Uniqueness results. Let us return to the mixed impedance problem formulated in Section 2.1 and prove the following uniqueness result.

Theorem 3.1. *Let S be Lipschitz. The homogeneous problems* $(\mathrm{MIP})^\pm$ *have only the trivial solution for $p = 2$.*

Proof. The proof is quite similar to the proof of Theorem 2.4. Indeed, let $u \in [W_2^1(\Omega^+)]^3$ be a solution to the homogeneous problem $(\mathrm{MIP})^+$. Since $A(\partial)u \in [L_2(\Omega^+)]^3$ we can apply Green's formula (2.11) with $v = u$, which leads to the relation

$$\int_{\Omega^+} [E(u,\overline{u}) - \varrho\,\omega^2\,|u|^2]dx + i\,\omega\,c\int_{S_I} |\{u\}^+|^2\,dS = 0.$$

Separating the imaginary part we get $r_{S_I}\{u\}^+ = 0$ and, consequently, $r_{S_I}\{Tu\}^+ = 0$ due to the homogeneous impedance condition (2.3). Thus the Cauchy data for u vanish on $S_I \subset S$. Then by (2.14) we conclude that the left hand side expression is real analytic in $\mathbf{R}^3 \setminus \overline{S_I}$ since the integration surfaces in the layer potentials coincide with S_D. Therefore, since the left hand side expression vanishes in Ω^-, it vanishes in Ω^+ as well (see the remark concerning the function $\widetilde{u}$ defined by (2.16)). Whence $u = 0$ in Ω^+ follows.

Now, let $u \in [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$ be a solution to the homogeneous problem $(\mathrm{MIP})^-$. We apply the notation introduced in the proof of Theorem 2.4 to obtain

$$\int_{\Omega_R^-} [E(u, \overline{u}) - \varrho\, \omega^2\, |u|^2]\, dx - i\, \omega\, c \times \int_{S_I} |\{u\}^-|^2\, dS - \int_{\Sigma_R} T(\partial, n) u \cdot u\, d\Sigma_R = 0.$$

With the help of word for word arguments we easily derive the equalities (one needs only substitute S_I for S in the formulae appearing in the proof of Theorem 2.4):

$$\begin{aligned} r_{S_I} \{u\}^- &= 0 \quad \text{on } S_I, \\ u_\infty^{(l)}(\widehat{x}) &= 0, \quad l = 1, 2, \quad \text{for all } \widehat{x} \in \Sigma_1. \end{aligned} \tag{3.1}$$

Therefore $u = u^{(1)} + u^{(2)} = 0$ follows from the equalities (3.1) since they yield $u^{(l)}(x) = 0$, $l = 1, 2$, in Ω^- due to the well known Rellich-Vekua theorem for metaharmonic functions. □

The uniqueness result for $p \neq 2$ will be shown later.

3.2. Existence results. First we consider the exterior problem in the case of C^∞-smooth boundary. Let h_0 be some fixed extension of the vector function h from S_I onto the whole of S preserving the function space, i.e., $h_0 \in [B^{-1/p}_{p,p}(S)]^3$ and $r_{S_I}\, h_0 = h$ on S_I. Then an arbitrary extension of the vector function h is representable as $h_0 + \varphi$, where $\varphi \in [\widetilde{B}^{-1/p}_{p,p}(S_D)]^3$.

In accordance with Theorem 2.7, we look for a solution to $(\mathrm{MIP})^-$ in the form

$$u(x) = W(\mathcal{P}^{-1}[h_0 + \varphi])(x) + i\, \varkappa\, V(\mathcal{P}^{-1}[h_0 + \varphi])(x), \quad x \in \Omega^-, \tag{3.2}$$

where h_0 is the above mentioned fixed extension and $\varphi \in [\widetilde{B}^{-1/p}_{p,p}(S_D)]^3$ is an unknown vector function. We see that $u \in [W^1_{p,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK\Omega^-)$

and the impedance boundary condition (2.3) on S_I is automatically satisfied since $\{Tu\}^- + i\,\omega\, c\,\{u\}^- = h_0 + \varphi$. The Dirichlet condition (2.2) leads to the pseudodifferential equation on S_D with respect to the unknown vector function φ

$$r_{S_D}\,[-2^{-1}\,I_3 + \mathcal{K}^* + i\,\omega\,c\,\mathcal{H}]\,\mathcal{P}^{-1}\,(h_0 + \varphi) = f \quad \text{on } S_D,$$

which can be rewritten as

$$r_{S_D}\,\mathcal{N}\,\varphi = F \quad \text{on } S_D, \tag{3.3}$$

where

$$\mathcal{N} := [2^{-1}\,I_3 - \mathcal{K}^* - i\,\omega\,c\,\mathcal{H}]\,\mathcal{P}^{-1}, \tag{3.4}$$

$$F := -f - r_{S_D}\,\mathcal{N}\,h_0 \in [B^{1-1/p}_{p,p}(S_D)]^3. \tag{3.5}$$

Due to Theorems 2.3 and 2.5 we have the following mapping property

$$\mathcal{N} : [H^s_p(S)]^3 \longrightarrow [H^{s+1}_p(S)]^3 \qquad \Big[[B^s_{p,t}(S)]^3 \longrightarrow [B^{s+1}_{p,t}(S)]^3\Big],$$
$$s \in \mathbf{R}, \quad 1 < p < +\infty, \quad 1 \le t \le +\infty.$$

Applying Theorems 2.3, 2.5 and A.1 (see the Appendix) we can establish the following property of the operator $r_{S_D}\,\mathcal{N}$.

Theorem 3.2. *Let* $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le t \le \infty$, *and let*

$$\frac{1}{p} - \frac{3}{2} < s < \frac{1}{p} - \frac{1}{2}. \tag{3.6}$$

Then the operators

$$r_{S_D}\,\mathcal{N} : [\widetilde{H}^s_p(S_D)]^3 \longrightarrow [H^{s+1}_p(S_D)]^3 \tag{3.7}$$

$$: [\widetilde{B}^s_{p,t}(S_D)]^3 \longrightarrow [B^{s+1}_{p,t}(S_D)]^3 \tag{3.8}$$

are invertible.

Proof. We prove the theorem in three steps.

Step 1. First we show that the principal homogeneous symbol matrix $\sigma(\mathcal{N})(\xi,x)$ of the operator $\mathcal{N}$ is positive definite for all $\xi\in\mathbf{R}^2$, $|\xi|=1$ and $x\in S$. Note that due to the relations (2.13) the principal homogeneous symbol matrices of the oscillation operators $\mathcal{H}$, $\mathcal{K}$, $\mathcal{K}^*$, $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{N}$ coincide with the principal homogeneous symbol matrices of the corresponding operators $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{K}_0^*$, $\mathcal{L}_0$, $\mathcal{P}_0$ and $\mathcal{N}_0$, constructed with the help of the fundamental matrix of statics $\Gamma(\cdot)$. Therefore, taking into consideration only the principal singular parts of the operators in (3.4) and (2.31) we get

$$\sigma(\mathcal{N})=\sigma(2^{-1}I_3-\mathcal{K}_0^*)\,[\sigma(\mathcal{P}_0)]^{-1}=\sigma(2^{-1}I_3-\mathcal{K}_0^*)\,[\sigma(\mathcal{L}_0)]^{-1}. \tag{3.9}$$

It is well known that the principal homogeneous symbol matrices of the operators $\mathcal{H}_0$, $2^{-1}I_3\pm\mathcal{K}_0$, $2^{-1}I_3\pm\mathcal{K}_0^*$ and $\mathcal{L}_0$ are elliptic, i.e., the matrices $\sigma(\mathcal{H}_0)$, $\sigma(2^{-1}I_3\pm\mathcal{K}_0)$, $\sigma(2^{-1}I_3\pm\mathcal{K}_0^*)$ and $\sigma(\mathcal{L}_0)$ are non-degenerate for all $\xi\in\mathbf{R}^2$, $|\xi|=1$ and $x\in S$. Moreover, the operator $\mathcal{H}_0:[B^s_{p,t}(S)]^3\to[B^{s+1}_{p,t}(S)]^3$ is invertible and the principal homogeneous symbol matrices $\sigma(-\mathcal{H}_0)$ and $\sigma(\mathcal{L}_0)$, as well as the principal homogeneous symbol matrices of the Steklov-Poincaré operators $\sigma([-2^{-1}I_3\pm\mathcal{K}_0]\,\mathcal{H}_0^{-1})$, are positive definite (for details see, e.g., [**10, 18, 22, 28**]). Further, due to the last equality in (2.21) we have

$$\sigma(\mathcal{H}_0)\,\sigma(\mathcal{L}_0)=\sigma(2^{-1}I_3+\mathcal{K}_0^*)\,\sigma(-2^{-1}I_3+\mathcal{K}_0^*),$$

whence

$$\begin{aligned}\sigma(2^{-1}I_3-\mathcal{K}_0^*)\,[\sigma(\mathcal{L}_0)]^{-1}&=[\sigma(-2^{-1}I_3-\mathcal{K}_0^*)]^{-1}\,\sigma(\mathcal{H}_0)\\&=\{[\sigma(\mathcal{H}_0)]^{-1}\,\sigma(-2^{-1}I_3-\mathcal{K}_0^*)\}^{-1}.\end{aligned}$$

From the first equality in (2.21) it follows that

$$[\sigma(\mathcal{H}_0)]^{-1}\,\sigma(-2^{-1}I_3-\mathcal{K}_0^*)=\sigma(-2^{-1}I_3-\mathcal{K}_0)\,[\sigma(\mathcal{H}_0)]^{-1}.$$

Therefore we finally get from (3.9) that

$$\sigma(\mathcal{N})=\{\sigma(-2^{-1}I_3-\mathcal{K}_0)\,[\sigma(\mathcal{H}_0)]^{-1}\}^{-1},$$

i.e., $\sigma(\mathcal{N})$ is the inverse of the principal homogeneous symbol matrix of the Steklov-Poincaré operator, which is positive definite. Consequently, $\sigma(\mathcal{N})$ is positive definite.

Step 2. Here we show that the claim of the theorem holds for $p = t = 2$ and $s = -1/2$. Note that in this case both operators (3.7) and (3.8) coincide. Since the chosen values of the parameters p, t and s satisfy the inequalities (3.6), by Theorem A.1 the corresponding operator

$$r_{S_D}\mathcal{N} : [\widetilde{H}_2^{-1/2}(S_D)]^3 \longrightarrow [H_2^{1/2}(S_D)]^3 \tag{3.10}$$

is Fredholm with zero index.

Let us show that the null space of the operator (3.10) is trivial. To this end let $\varphi \in [\widetilde{H}_2^{-1/p}(S_D)]^3$ be a solution to the homogeneous equation $r_{S_D}\mathcal{N}\varphi = 0$ on S_D. Construct the vector

$$u(x) = W(\mathcal{P}^{-1}\varphi)(x) + i\varkappa V(\mathcal{P}^{-1}\varphi)(x), \quad x \in \Omega^-.$$

It can easily be seen that $u \in [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$ and solves the homogeneous mixed impedance problem, since $\{Tu\}^- + i\,\omega\,c\,\{u\}^- = \varphi = 0$ on S_I and $\{u\}^- = -\mathcal{N}\varphi = 0$ on S_D. Therefore by Theorem 3.1 we conclude $u = 0$ in Ω^-, whence $\varphi = 0$ follows. Thus the kernel of the operator (3.10) is trivial and consequently it is invertible.

Step 3. Now we treat the general case and assume that the parameters p and s satisfy the inequality (3.6), and $1 \le t \le +\infty$. Then by the results of Step 1 and Theorem A.1 the operators (3.7) and (3.8) are Fredholm with zero index. Moreover, by the final part of Theorem A.1 and in view of the results obtained in Step 2, we conclude that these operators have trivial null spaces. Therefore they are invertible. □

In accordance with Theorem 3.2 the nonhomogeneous equation (3.3) is uniquely solvable for the arbitrary right hand side. Note that the solution φ depends on the vector function h_0, which extends the given impedance boundary vector function h from S_I onto S i.e., φ depends on the extension operator. However, with the help of the injectivity property of the operators (3.7) and (3.8), it can easily be shown that the sum $h_0 + \varphi$ does not depend on the extension operator.

Now we are in position to formulate the basic existence result for the exterior mixed impedance problem.

Theorem 3.3. *Let* $4/3 < p < 4$ *and*

$$f \in [B^{1-1/p}_{p,p}(S_D)]^3, \quad h \in [B^{-1/p}_{p,p}(S_I)]^3. \tag{3.11}$$

Then Problem (MIP)$^-$ *has a unique solution* $u \in [W^1_{p,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$ *which is representable in the form of* (3.2) *where* $h_0 \in [B^{-1/p}_{p,p}(S)]^3$ *is some fixed extension of the vector function* $h \in [B^{-1/p}_{p,p}(S_I)]^3$ *from* S_I *onto* S *preserving the functional space and* $\varphi \in [\widetilde{B}^{-1/p}_{p,p}(S_D)]^3$ *is defined by the uniquely solvable pseudodifferential equation* (3.3) *with the right hand side given by* (3.5).

Proof. It is evident that the vector function (3.2), with densities assumed as above, solves the (MIP)$^-$ (see (2.1)–(2.4)). Therefore it remains to prove the uniqueness of solutions for $p \neq 2$.

Let $u \in [W^1_{p,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$ be a solution to the homogeneous mixed boundary value problem. Due to Theorem 2.7, u is representable in the form

$$u(x) = W(\mathcal{P}^{-1}\,\psi)(x) + i\,\varkappa\,V(\mathcal{P}^{-1}\,\psi)(x), \quad x \in \Omega^-,$$

where

$$\psi := \{Tu\}^- + i\,\omega\,c\,\{u\}^- \in [\widetilde{B}^{-1/p}_{p,p}(S_D)]^3$$

in view of the homogeneous impedance condition on S_I. Then the homogeneous Dirichlet condition on S_D gives

$$r_{S_D}\,\{u\}^- = -r_{S_D}\,\mathcal{N}\,\psi = 0.$$

Whence $\psi = 0$ follows by Theorem 3.2 if condition (3.6) is fulfilled with $s = -1/p$, i.e., if $4/3 < p < 4$. Therefore $u = 0$ in Ω^-. □

3.3. Regularity results. Here we present almost best regularity results for solutions to exterior mixed impedance problems.

Theorem 3.4. *Let the conditions* (3.11) *and the inequalities*
(3.12)
$$4/3 < p < 4, \quad 1 < t < \infty,\; 1 \le q \le \infty,\; 1/t - 1/2 < s < 1/t + 1/2,$$

be fulfilled, and let $u \in [W^1_{p,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-)$ *be the unique solution to the mixed problem* (MIP)$^-$. *In addition to* (3.11),

(i) *if*
$$f \in [B^s_{t,t}(S_D)]^3, \qquad h \in [B^{s-1}_{t,t}(S_I)]^3,$$

then

$$u \in [H^{s+\frac{1}{t}}_{t,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-);$$

(ii) *if*

$$f \in [B^s_{t,q}(S_D)]^3, \qquad h \in [B^{s-1}_{t,q}(S_I)]^3,$$

then

$$u \in [B^{s+\frac{1}{t}}_{t,q,\,\mathrm{loc}}(\Omega^-)]^3 \cap SK(\Omega^-); \tag{3.13}$$

(iii) *if*

$$f \in [C^\alpha(S_D)]^3, \qquad h \in [B^{\alpha-1}_{\infty,\infty}(S_I)]^3, \quad \alpha > 0, \tag{3.14}$$

then

$$u \in [C^\beta(\overline{\Omega^-})]^3 \cap SK(\Omega^-) \quad \text{with any } \beta \in (0,\gamma),\ \gamma := \min\{\alpha,\ 1/2\}.$$

Proof. Parts (i) and (ii) can be shown by the word for word arguments applied in the proof of Theorem 3.3. To prove (iii) we need the following chain embeddings (see, e.g., [**33**])

$$C^\alpha(\mathcal{S}) = B^\alpha_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha-\varepsilon}_{\infty,1}(\mathcal{S}) \subset B^{\alpha-\varepsilon}_{\infty,q}(\mathcal{S}) \subset B^{\alpha-\varepsilon}_{t,q}(\mathcal{S}) \subset C^{\alpha-\varepsilon-k/t}(\mathcal{S}), \tag{3.15}$$

where ε is an arbitrary small positive number, $\mathcal{S} \subset \mathbf{R}^3$ is a compact k-dimensional ($k = 2,3$) smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < t < \infty$, $\alpha - \varepsilon - k/t > 0$, α and $\alpha - \varepsilon - k/t$ are not integers. From (3.14) and the embeddings (3.15) the condition (3.13) follows with any $s \le \alpha - \varepsilon$.

Bearing in mind (3.12) and taking t sufficiently large and ε sufficiently small, we may put $s = \alpha - \varepsilon$ if

$$1/t - 1/2 < \alpha - \varepsilon < 1/t + 1/2, \tag{3.16}$$

and $s \in (1/t - 1/2, 1/t + 1/2)$ if

$$1/t + 1/2 < \alpha - \varepsilon. \tag{3.17}$$

By (3.13) the solution u belongs then to $[B^{s+(1/t)}_{t,q,\mathrm{loc}}(\Omega^-)]^3$ with $s+1/t=\alpha-\varepsilon+1/t$ if (3.16) holds, and with $s+1/t\in(2/t-1/2,2/t+1/2)$ if (3.17) holds. In the last case we can take $s+1/t=2/t+1/2-\varepsilon$. Therefore, we have either $u\in[B^{\alpha-\varepsilon+(1/t)}_{t,q,\mathrm{loc}}(\Omega^-)]^3$, or $u\in[B^{1/2+(2/t)-\varepsilon}_{t,q,\mathrm{loc}}(\Omega^-)]^3$ in accordance with the inequalities (3.16) and (3.17). The last embedding in (3.15) (with $k=3$) yields that either $u\in[C^{\alpha-\varepsilon-(2/t)}(\overline{\Omega^-})]^3$, or $u\in[C^{1/2-\varepsilon-1/t}(\overline{\Omega^-})]^3$ which lead to the inclusion

$$u\in[C^{\gamma-\varepsilon-2/t}(\overline{\Omega^-})]^3, \tag{3.18}$$

where $\gamma:=\min\{\alpha,1/2\}$. Since t is sufficiently large and ε is sufficiently small, the embedding (3.18) completes the proof. □

3.4. Interior mixed impedance problem. We can develop the same approach in the case of the interior problem $(\mathrm{MIP})^+$ on the basis of the representation formula (2.39)–(2.40), see Theorem 2.11. Indeed, let $h_0\in[B^{-1/p}_{p,p}(S)]^3$ and $\varphi\in[\widetilde{B}^{-1/p}_{p,p}(S_D)]^3$ be defined as in Section 3.2. We look for a solution to the problem $(\mathrm{MIP})^+$ in the form

$$u(x)=V(\mathcal{Q}^{-1}[h_0+\varphi])(x),\quad x\in\Omega^+. \tag{3.19}$$

We see that $u\in[W^1_p(\Omega^+)]^3$ and the interior impedance boundary condition (2.3) on S_I is automatically satisfied since $\{Tu\}^+ + i\,\omega\,c\,\{u\}^+ = h_0+\varphi$ due to Theorem 2.11. The interior Dirichlet condition (2.2) leads to the pseudodifferential equation on S_D with respect to the unknown vector function φ

$$r_{S_D}\,\mathcal{H}\,\mathcal{Q}^{-1}\,(h_0+\varphi)=f\quad \text{on } S_D,$$

which can be rewritten as

$$r_{S_D}\,\mathcal{R}\,\varphi=F\quad\text{on } S_D, \tag{3.20}$$

where

$$\mathcal{R}:=\mathcal{H}\,\mathcal{Q}^{-1}, \tag{3.21}$$

$$F:=f-r_{S_D}\,\mathcal{R}\,h_0\in[B^{1-1/p}_{p,p}(S_D)]^3. \tag{3.22}$$

Due to Theorems 2.3 and 2.5 we have the following mapping property

$$\mathcal{R} : [H_p^s(S)]^3 \longrightarrow [H_p^{s+1}(S)]^3 \qquad \Big[[B_{p,t}^s(S)]^3 \longrightarrow [B_{p,t}^{s+1}(S)]^3\Big], \quad s \in \mathbf{R}, \quad 1 < p < +\infty, \quad 1 \leq t \leq +\infty. \tag{3.23}$$

Applying Theorems 2.3, 2.5 and A.1 (see the Appendix) we can establish the following property of the operator $r_{S_D}\,\mathcal{Q}$.

Theorem 3.5. *Let $s \in \mathbf{R}$, $1 < p < \infty$, $1 \leq t \leq \infty$, and let*

$$\frac{1}{p} - \frac{3}{2} < s < \frac{1}{p} - \frac{1}{2}.$$

Then the operators

$$r_{S_D}\,\mathcal{R} : [\widetilde{H}_p^s(S_D)]^3 \longrightarrow [H_p^{s+1}(S_D)]^3 \tag{3.24}$$

$$: [\widetilde{B}_{p,t}^s(S_D)]^3 \to [B_{p,t}^{s+1}(S_D)]^3 \tag{3.25}$$

are invertible.

Proof. It is word for word of the proof of Theorem 3.2. One needs only to show that the principal homogeneous symbol matrix $\sigma(\mathcal{R})(\xi, x)$ of the operator $\mathcal{R}$ is positive definite for all $\xi \in \mathbf{R}^2$, $|\xi| = 1$ and $x \in S$. This follows from the equality

$$\begin{aligned}\sigma(\mathcal{R}) &= \sigma(\mathcal{H}_0)\,[\sigma(\mathcal{Q}_0)]^{-1}\\ &= \sigma(\mathcal{H}_0)\,[\sigma(-2^{-1}I_3 + \mathcal{K}_0)]^{-1}\\ &= \{\sigma(-2^{-1}I_3 + \mathcal{K}_0)\,[\sigma(\mathcal{H}_0)]^{-1}\}^{-1}\end{aligned}$$

and from the positive definiteness of the principal homogeneous symbol matrices of the Steklov-Poincaré operators $\sigma([\,-2^{-1}I_3 \pm \mathcal{K}_0]\,\mathcal{H}_0^{-1})$. Now, the proof follows from Theorems 3.1 and A.1. □

In accordance with Theorem 3.5 the nonhomogeneous equation (3.20) is uniquely solvable for arbitrary right hand side. As above let us remark that the solution φ depends on the vector function h_0, which extends the given impedance boundary vector function h from S_I onto

S, i.e., φ depends on the extension operator. However, with the help of the injectivity property of the operators (3.24) and (3.25), it can easily be shown that the sum $h_0 + \varphi$ does not depend on the extension operator.

Now we can formulate the basic existence result for the interior mixed impedance problem.

Theorem 3.6. *Let* $4/3 < p < 4$ *and*

$$f \in [B^{1-1/p}_{p,p}(S_D)]^3, \qquad h \in [B^{-1/p}_{p,p}(S_I)]^3.$$

Then Problem $(\mathrm{MIP})^+$ *has a unique solution* $u \in [W^1_p(\Omega^+)]^3$ *which is representable in the form of* (3.19) *where* $h_0 \in [B^{-1/p}_{p,p}(S)]^3$ *is some fixed extension of the vector function* $h \in [B^{-1/p}_{p,p}(S_I)]^3$ *from* S_I *onto* S *preserving the functional space and* $\varphi \in [\widetilde{B}^{-1/p}_{p,p}(S_D)]^3$ *is defined by the uniquely solvable pseudodifferential equation* (3.20) *with the right hand side given by* (3.21).

Proof. It is word for word of the proof of Theorem 2.3. □

Note that the counterpart of the smoothness Theorem 3.4 holds in the case of the problem $(\mathrm{MIP})^+$ as well.

3.5. The Lipschitz case. All the results obtained in subsections 3.1 and 3.2 for Bessel potential spaces are valid also in the case of Lipschitz boundaries provided $p = 2$ and $s = -1/2$. To prove this let us show that $\mathcal{N}$ can be represented as (see (3.4))

$$\mathcal{N} = \mathcal{N}_1 + \widetilde{\mathcal{N}}, \tag{3.26}$$

where

$$\begin{aligned} \mathcal{N}_1 &:= \mathcal{P}_1\,\mathcal{P}_0^{-1}, \\ \mathcal{P}_1 &:= 2^{-1}\,I_3 - \mathcal{K}_0^* - i\,\varkappa\,\mathcal{H}_0, \\ \mathcal{P}_0 &:= \mathcal{L}_0 + i\,\varkappa\,[2^{-1}\,I_3 + \mathcal{K}_0], \end{aligned}$$

and

$$\widetilde{\mathcal{N}} := \mathcal{N} - \mathcal{N}_1 : [H^{1/2}_2(S)]^3 \longrightarrow [H^{-1/2}_2(S)]^3 \tag{3.27}$$

is a compact operator (for details see, e.g., [**1, 2**]). As before, the subscript "0" denotes that the corresponding operators are constructed with the help of the fundamental solution of statics $\Gamma(\cdot)$.

First we formulate some technical lemmata.

Lemma 3.7. *Let S be a Lipschitz boundary. The operators*

$$\mathcal{P}_0 = \mathcal{L}_0 + i\,\varkappa\,[2^{-1}\,I_3 + \mathcal{K}_0] : [H_2^{1/2}(S)]^3 \longrightarrow [H_2^{-1/2}(S)]^3,$$
$$\mathcal{P}_1 = 2^{-1}\,I_3 - \mathcal{K}_0^* - i\,\varkappa\,\mathcal{H}_0 \quad : [H_2^{1/2}(S)]^3 \longrightarrow [H_2^{1/2}(S)]^3,$$

are invertible.

Proof. Since, due to Theorem 2.3, these operators are Fredholm with zero index, we need only to prove their injectivity, which follows from Green's formulae for the equations of statics and the corresponding uniqueness results for the appropriate BVPs.

First, let us show that the kernel of the operator $\mathcal{P}_0$ is trivial. Let $\psi \in [H_2^{1/2}(S)]^3$ be a solution to the homogeneous equation $\mathcal{P}_0\,\psi = 0$ on S and construct the linear combination of the layer potentials of statics

$$u(x) = W_0(\psi)(x) + i\,\varkappa\,V_0(\psi)(x). \tag{3.28}$$

Evidently, the vector u belongs to the class $[W_2^1(\Omega^+)]^3 \cap [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^3$, solves the homogeneous equilibrium equations of statics $A(\partial)\,u(x) = 0$ in $\Omega^\pm$, and decays at infinity as $\mathcal{O}(|x|^{-1})$. The first order partial derivatives of u decay as $\mathcal{O}(|x|^{-2})$. Moreover, $\mathcal{P}_0\,\psi = \{Tu\}^- = 0$. With the help of Green's formulae ([**7, 27**])

$$\int_{\Omega^\pm} E(u, \overline{u})\,dx = \pm\,\langle \{Tu\}^\pm,\, \overline{\{u\}^\pm}\rangle_S \tag{3.29}$$

where $\langle\cdot,\cdot\rangle_S$ denotes the duality brackets between the adjoint spaces $[H_2^{-1/2}(S)]^3$ and $[H_2^{1/2}(S)]^3$, we get $E(u, \overline{u}) = 0$. Therefore u is a rigid displacement vector (see the remark after formula (2.10)) and since u decays at infinity we easily derive that $u(x) = W_0(\psi) + i\,\varkappa\,V_0(\psi) = 0$ in Ω^-. Due to the properties of the layer potentials this relation yields $\{Tu\}^+ + i\,\varkappa\,\{u\}^+ = 0$ on S. Applying again Green's formula (3.29) we get $u = 0$ in Ω^+ and consequently, $\psi = 0$ on S. Thus the operator $\mathcal{P}_0$ has the trivial kernel.

Now, let $\psi \in [H_2^{1/2}(S)]^3$ be a solution to the homogeneous equation $\mathcal{P}_1\,\psi = 0$ on S and construct again the vector function (3.28). As above, we see that the vector u belongs to the class $[W_2^1(\Omega^+)]^3 \cap [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^3$, solves the homogeneous equilibrium equations of statics $A(\partial)\,u(x) = 0$ in $\Omega^\pm$, and decays at infinity as $\mathcal{O}(|x|^{-1})$. The first order partial derivatives of u decay as $\mathcal{O}(|x|^{-2})$. Moreover, $\mathcal{P}_1\,\psi = -\{u\}^- = 0$. By the uniqueness theorem for the exterior Dirichlet problem of statics, which can be proved by Green's formula (3.29), we deduce $u = 0$ in Ω^-. This in turn implies $u = 0$ in Ω^+ and finally we get $\psi = 0$ on S. Thus the operator $\mathcal{P}_1$ has the trivial kernel. □

Corollary 3.8. *Let S be a Lipschitz boundary. The operator*

$$\mathcal{N}_1 : [H_2^{-1/2}(S)]^3 \longrightarrow [H_2^{1/2}(S)]^3$$

is invertible. Consequently, for arbitrary $g \in [H_2^{-1/2}(S)]^3$

$$||g||_{[H_2^{-1/2}(S)]^3} \le c\,||\mathcal{N}_1\,g||_{[H_2^{1/2}(S)]^3},$$

where c is some positive constant independent of g.

Next we prove the following coercivity property of the operator $\mathcal{N}_1$.

Lemma 3.9. *Let S be a Lipschitz boundary. The sesquilinear form $\langle g, \overline{\mathcal{N}_1\,g}\rangle_S$ is nonnegative, and there exists a positive constant C such that*

$$\langle g, \overline{\mathcal{N}_1\,g}\rangle_S \ge C\,||g||^2_{[H_2^{-1/2}(S)]^3}$$

for all $g \in [H_2^{-1/2}(S)]^3$.

Proof. Let $g \in [H_2^{-1/2}(S)]^3$, and consider the linear combination of the layer potentials

$$u(x) = -W_0(\mathcal{P}_0^{-1}g)(x) - i\,\varkappa\,V_0(\mathcal{P}_0^{-1}g)(x). \tag{3.30}$$

Due to Lemma 3.7 and the mapping properties of the layer potentials, it is evident that the vector u belongs to the class $[W^1_2(\Omega^+)]^3 \cap [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^3$, solves the homogeneous equilibrium equations of statics $A(\partial)\,u(x) = 0$ in $\Omega^{\pm}$, and decays at infinity as $\mathcal{O}(|x|^{-1})$. The first order partial derivatives of u decay as $\mathcal{O}(|x|^{-2})$. Actually, the vector (3.30) belongs to the Beppo-Levi type space (see [**8**, Ch. XI])

$$BL(\Omega^-) := \{v \in [W^1_{2,loc}(\Omega^-)]^3 : \\ (1+|x|^2)^{-1/2}\, v_k \in L_2(\Omega^-),\ \partial_j v_k \in L_2(\Omega^-),\ k,j = 1,2,3\},$$

where the norm is defined as

$$(3.31)\quad ||v||^2_{BL(\Omega^-)} := ||(1+|x|^2)^{-1/2}\, v||^2_{L_2(\Omega^-)} + \sum_{k,j=1}^{3} ||\partial_j v_k||^2_{L_2(\Omega^-)}.$$

Moreover, $\{u\}^- = \mathcal{N}_1\, g$ and $\{Tu\}^- = -g$ on S. Therefore from Green's identity (3.29) we have the following equality

$$(3.32)\qquad \int_{\Omega^-} E(u,\overline{u})\,dx = -\langle \{Tu\}^-,\, \overline{\{u\}^-}\rangle_S = \langle g,\, \overline{\mathcal{N}_1\, g}\rangle_S.$$

It is known (see [**22**, Chapter 3]) that $E(u,\overline{u}) \geq c_1\, e_{kj}(u)\, e_{kj}(\overline{u})$, where $e_{kj}(u) = 2^{-1}(\partial_j u_k + \partial_k u_j)$ and $c_1 > 0$ is a constant depending only on the material parameters λ and μ. Therefore from (3.32) we have

$$(3.33)\qquad \langle g,\, \overline{\mathcal{N}_1\, g}\rangle_S \geq c_1 \int_{\Omega^-} \sum_{k,j=1}^{3} |\, e_{kj}(u)\,|^2\, dx.$$

Further, due to Korn's inequality in unbounded domains (see [**29**], [**20**, Section 3, Theorem 3]) we have

$$(3.34)\qquad \int_{\Omega^-} \sum_{k,j=1}^{3} |e_{kj}(u)|^2\, dx \geq c_2 \int_{\Omega^-} \sum_{k,j=1}^{3} |\partial_j u_k|^2,$$

where c_2 is a positive constant independent of u. On the other hand, the right hand side expression in (3.34) is equivalent to the norm (3.31) in Beppo-Levi space $BL(\Omega^-)$. Therefore we get from (3.33)

$$(3.35)\qquad \langle g,\, \overline{\mathcal{N}_1\, g}\rangle_S \geq c_3\, ||u||^2_{BL(\Omega^-)}$$

with some positive constant c_3. By the well known trace theorem

$$||\{u\}^-||_{[H_2^{1/2}(S)]^2)} \leq c_4\, ||u||_{BL(\Omega^-)}. \tag{3.36}$$

Taking into account that $\{u\}^- = \mathcal{N}_1\, g$ from (3.35) and (3.36) we deduce

$$\langle g, \overline{\mathcal{N}_1\, g}\rangle_S \geq c_4\, ||\mathcal{N}_1\, g||^2_{[H_2^{1/2}(S)]^2)}.$$

Now Corollary 3.8 completes the proof. □

Corollary 3.10. *Let S be a Lipschitz boundary. The operator*

$$r_{S_D}\mathcal{N}_1 : [\widetilde{H}_2^{-1/2}(S_D)]^3 \longrightarrow [H_2^{1/2}(S_D)]^3$$

is invertible.

Proof. It follows immediately from Lemma 3.9. Indeed, we have the inequality

$$\langle g, \overline{\mathcal{N}_1\, g}\rangle_S = \langle g, \overline{r_{S_D}\mathcal{N}_1\, g}\rangle_{S_D} \geq C\, ||g||^2_{[H_2^{-1/2}(S)]^3} \tag{3.37}$$

for all $g \in [\widetilde{H}_2^{-1/2}(S_D)]^3$, where the symbol $\langle\cdot,\cdot\rangle_{S_D}$ denotes the duality brackets between the adjoint spaces $[\widetilde{H}_2^{-1/2}(S_D)]^3$ and $[H_2^{1/2}(S_D)]^3$. As is well known the coercivity property (3.37) implies the invertibility of the corresponding operator (see, e.g., [**25**, Chapter 2, Lemma 2.32]). □

Now we prove the basic invertibility result for the operator $r_{S_D}\mathcal{N}$.

Theorem 3.11. *Let S be a Lipschitz boundary. The operator*

$$r_{S_D}\mathcal{N} : [\widetilde{H}_2^{-1/2}(S_D)]^3 \longrightarrow [H_2^{1/2}(S_D)]^3 \tag{3.38}$$

is invertible.

Proof. In view of the representation (3.26), the compactness of the operator (3.27) and Corollary 3.10 we see that the operator (3.38) is Fredholm with zero index, since it is a compact perturbation of the invertible operator. So we need only to show the injectivity of the operator (3.38). We proceed as follows. Let $\psi \in [\widetilde{H}_2^{-1/2}(S_D)]^3$ be a solution of the homogeneous equation $r_{S_D}\mathcal{N}\psi = 0$ on S_D and construct the vector

$$u(x) = W(\mathcal{P}^{-1}\psi)(x) + i\varkappa V(\mathcal{P}^{-1}\psi)(x).$$

It can easily be seen that $u \in [W_2^1(\Omega^+)]^3 \cap [W_{2,\,\mathrm{loc}}^1(\Omega^-)]^3 \cap SK(\Omega^-)$ and solves the homogeneous mixed impedance problem $(\mathrm{MIP})^-$, since $\{Tu\}^- + i\omega c\{u\}^- = \psi = 0$ on S_I and $\{u\}^- = -\mathcal{N}\psi = 0$ on S_D. Therefore by Theorem 3.1 we conclude $u = 0$ in Ω^-, whence $\psi = 0$ follows. Thus the kernel of the operator (3.38) is trivial and consequently it is invertible. □

Finally we have the following existence result for the mixed impedance problem $(\mathrm{MIP})^-$ which directly follows from Theorem 3.11.

Theorem 3.12. *Let S be a Lipschitz boundary and*

$$f \in [H_2^{1/2}(S_D)]^3, \quad h \in [H_2^{-1/2}(S_I)]^3.$$

Then Problem $(\mathrm{MIP})^-$ has a unique solution $u \in [W_{2,\mathrm{loc}}^1(\Omega^-)]^3 \cap SK(\Omega^-)$ which is representable in the form of (3.2) where $h_0 \in [H_2^{-1/2}(S)]^3$ is some fixed extension of the vector function $h \in [H_2^{-1/2} \times (S_I)]^3$ from S_I onto S preserving the functional space and $\varphi \in [\widetilde{H}_2^{-1/2}(S_D)]^3$ is defined by the uniquely solvable boundary integral equation (3.3) where the right hand side is given by (3.5) with $p = 2$.

A similar existence result holds also for the mixed impedance problem $(\mathrm{MIP})^+$.

Theorem 3.13. *Let S be a Lipschitz boundary and*

$$f \in [H_2^{1/2}(S_D)]^3, \quad h \in [H_2^{-1/2}(S_I)]^3.$$

Then Problem $(\mathrm{MIP})^+$ has a unique solution $u \in [W_2^1(\Omega^+)]^3$ which is representable in the form of (3.19) where $h_0 \in [H_2^{-1/2}(S)]^3$ is some fixed extension of the vector function $h \in [H_2^{-1/2}(S_I)]^3$ from S_I onto

S preserving the functional space and $\varphi \in [\widetilde{H}_2^{-1/2}(S_D)]^3$ *is defined by the uniquely solvable boundary integral equation* (3.20) *where the right hand side is given by* (3.22) *with* $p = 2$.

3.6. The two-dimensional case. Exactly the same results, with *verbatim* statements and proofs, hold in the two-dimensional case. For the sake of brevity we only mention here the definition of the 2×2 fundamental matrix, and the Sommerfeld-Kupradze radiation conditions for the two-dimensional case. The elastic oscillation 2×2 operator $A(\partial, \omega)$, its principal part $A(\partial)$ and the stress operator $T(\partial, n)$ (necessary for the formulation of the boundary conditions), are defined exactly as the corresponding operators of the three-dimensional case.

Denote by $G(x, \omega)$ and $G(x)$ the matrices of fundamental solutions of the operators $A(\partial, \omega)$ and its principal part $A(\partial)$. These matrices are defined (see, e.g., [**3, 21**], [**24**, Chapter IX, Section148]) as

$$G(x, \omega) = [G_{kj}(x, \omega)]_{2\times 2},$$

where

$$G_{kj}(x, \omega) = i \sum_{l=1}^{2} (\delta_{kj} \widetilde{\alpha}_l + \widetilde{\beta}_l \, \partial_k \, \partial_j) \, H_0^{(1)}(\kappa_l \, |x|), \quad x \in \mathbf{R}^2 \setminus \{0\},$$

$H_0^{(1)}(z)$ being the Hankel function of first kind and zero order, with

$$\widetilde{\alpha}_l = -\frac{\delta_{2l}}{4\,\mu}, \qquad \widetilde{\beta}_l = \frac{(-1)^{l+1}}{4\,\varrho\,\omega^2}, \quad l = 1, 2,$$

and

$$G(x) = [G_{kj}(x)]_{2\times 2},$$

where

$$G_{kj}(x) = \frac{1}{4\pi\,\mu\,(\lambda + 2\mu)} \left(\delta_{kj} (\lambda{+}3\mu) \, \ln|x| - (\lambda{+}\mu) \frac{x_j \, x_k}{|x|^2} \right), \; x \in \mathbf{R}^2 \setminus \{0\}.$$

The matrix $G(x)$ corresponds to the equilibrium equations of statics and represents the principal singular part of the matrix $G(x, \omega)$. It is evident that $G(x, \omega)$ satisfies the Sommerfeld-Kupradze radiation conditions at infinity. Note that for two-dimensional exterior problems, the vector u is said to satisfy the *Sommerfeld-Kupradze type radiation conditions at infinity* if and only if u is representable as a sum of two metaharmonic vectors (the so called longitudinal $u^{(1)} = u^{(p)}$ and transverse parts $u^{(2)} = u^{(s)}$ of u), i.e.,

$$u = u^{(1)} + u^{(2)} \text{ with } \Delta\, u^{(1)} + k_1^2\, u^{(1)} = 0, \quad \Delta\, u^{(2)} + k_2^2\, u^{(2)} = 0,$$
$$k_1 \equiv k_p = \omega\sqrt{\frac{\varrho}{\lambda + 2\,\mu}}, \qquad k_2 \equiv k_s = \omega\sqrt{\frac{\varrho}{\mu}},$$

and for sufficiently large $r = |x|$

$$\frac{\partial u^{(1)}(x)}{\partial r} - i\,k_1\,u^{(1)}(x) = \mathcal{O}(r^{-3/2}), \quad \frac{\partial u^{(2)}(x)}{\partial r} - i\,k_2\,u^{(2)}(x) = \mathcal{O}(r^{-3/2}).$$

APPENDIX

A. Some results from the theory of pseudodifferential equations on manifolds with boundary.

Here we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission and crack problems by the potential methods. They can be found in [**12, 15, 31**].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$, and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbf{R}$ on $\overline{\mathcal{M}}$. Denote by $\sigma(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbf{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\sigma(x, 0, \dots, 0, +1)]^{-1}[\sigma(x, 0, \dots, 0, -1)], \quad x \in \partial\overline{\mathcal{M}},$$

and introduce the notation

$$\delta_j(x) = \Re\left[(2\pi\, i)^{-1} \ln \lambda_j(x)\right], \quad j = 1, \dots, N.$$

Here the branch in the logarithmic function $\ln\zeta$ is chosen with regard to the inequality $-\pi < \arg\zeta \le \pi$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$, $j = \overline{1,N}$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. In the particular case, when $\sigma(x,\xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbf{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1,\dots,N$, since all the eigenvalues $\lambda_j(x)$ $(j = \overline{1,N})$ are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem.

Theorem A.1. *Let $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbf{R}$, that is, there is a positive constant c_0 such that*

$$\Re\,\sigma(x,\xi)\,\eta\cdot\eta \ge c_0\,|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbf{R}^n$ with $|\xi| = 1$, and $\eta \in \mathcal{C}^N$. Then the operators

(A.1)

$$\mathcal{A} : \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \longrightarrow \big[H^{s-\nu}_p(\mathcal{M})\big]^N \quad \Big[\big[\widetilde{B}^s_{p,t}(\mathcal{M})\big]^N \longrightarrow \big[B^{s-\nu}_{p,t}(\mathcal{M})\big]^N\Big],$$

are Fredholm with zero index if

$$\text{(A.2)} \qquad \frac{1}{p} - 1 + \sup_{\substack{x\in\partial\mathcal{M}\\ 1\le j\le N}} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{\substack{x\in\partial\mathcal{M}\\ 1\le j\le N}} \delta_j(x).$$

Moreover, the null-spaces and indices of the operators (A.1) *are the same* (*for all values of the parameter $t \in [1,+\infty]$*) *provided p and s satisfy the inequality* (A.2).

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