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AUTOCONVOLUTION EQUATIONS OF THE THIRD KIND WITH ABEL INTEGRAL

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ABSTRACT. In this paper a class of autoconvolution equations of the third kind with additional fractional integral is investigated. Two general existence theorems are proved, and a new type of solutions is shown for an exceptional equation of this class.

1. Introduction. In the paper [**2**] Berg and the author firstly investigate the general autoconvolution equation of the third kind with coefficient $k(x) \sim Ax$ as $x \to 0$ and without a free term. These investigations are being continued in recent papers by the author jointly with Hofmann and Janno [**5, 7**–**11**]. In particular, in [**9**] the case $k(x) \sim Ax^\alpha$, $\alpha > 0$ and in [**11**] the cases $k(x) \sim Ax$ and $k(x) \sim Ax^{1/2}$ with a free term $p(x) \sim -\gamma^2$,$\gamma > 0$ as $x \to 0$ have been dealt with.

In the present paper the more general equation with an additional fractional integral

(1.1)
$$k(x)y(x) = \int_0^x y(\xi)y(x-\xi)\,d\xi + \frac{\nu}{B(\alpha,\alpha)}\int_0^x y(\xi)(x-\xi)^{\alpha-1}d\xi + p(x)$$

where $\nu \in \mathbf{R}$, $k(x) \sim Ax^\alpha$ ($\alpha > 0$) and $p(x) \sim -\gamma^2 x^{2\alpha-1}$ ($\gamma > 0$) or $p(x) = o(x^{2\alpha-1})$ as $x \to 0$ is treated. For $\nu = 0$ with $\gamma = 0$ this equation has been considered in [**9**], for $\nu = 0$, $\alpha = 1/2$ with $\gamma > 0$ in [**11**]. Again using Janno's theorem [**6**] in the iteration method with weighted norms in C space, we prove the existence of a one-parametric family of (real) solutions and an additional solitary solution in the main case $\gamma > 0$. These solutions also hold for $\gamma = 0$ with the exception of

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one case where the existence proof for of a family of solutions requires a different approach. We show this existence for some typical equations by transforming these equations to related autoconvolution equations to which a slight generalization of Janno's method can be applied.

The plan of the paper is as follows. In Section 2 the main case $\gamma > 0$ and the extensions to the case $\gamma = 0$ and in Section 3 the exceptional case of $\gamma = 0$ in equation (1.1) together with some related equations are treated. In Appendix 1 a linear auxiliary equation, and in Appendix 2 the extension of Janno's theorem in space C with application to a class of autoconvolution equations are stated. Further, a generalized Volterra function with logarithmic factor which occurs in the treatment of the exceptional case is briefly discussed in Appendix 3.

2. Main case. In dealing with equation (1.1) in the finite interval $(0, T)$, $T > 0$, we make the following general *assumptions* besides $\alpha > 0$, $\nu \in \mathbf{R}$:

$$\begin{gathered} k \in C[0,T] \quad \text{with} \quad k(x) > 0 \text{ in } (0,T] \text{ and} \\ k(x) = Ax^{\alpha} + B(x) \ (A > 0), \quad B(x) = o(x^{\alpha}) \quad \text{as } x \to 0 \end{gathered} \tag{2.1}$$

and

$$p \in C(0,T]$$

with

$$p(x) = -\gamma^2 x^{2\alpha-1} + q(x) \ (\gamma \geq 0), \ q(x) = o(x^{2\alpha-1}) \text{ as } x \to 0. \tag{2.2}$$

In the *main case* $\gamma > 0$ we are looking for solutions y of equation (1.1) with asymptotic behavior $y(x) \sim Ex^{\alpha-1}$ as $x \to 0$. Equation (1.1) yields the equation

$$B(\alpha,\alpha)E^2 + [\nu - A]E - \gamma^2 = 0$$

with the beta function $B(u, v)$ for $E \in \mathbf{R}$ which has the two solutions

$$E_1 = \frac{1}{2B(\alpha,\alpha)} \left\{ [A-\nu] + \sqrt{[A-\nu]^2 + 4\gamma^2 B(\alpha,\alpha)} \right\} > 0 \tag{2.3}$$

and

(2.4)
$$E_2 = \frac{1}{2B(\alpha,\alpha)}\left\{[A-\nu] - \sqrt{[A-\nu]^2 + 4\gamma^2 B(\alpha,\alpha)}\right\} < 0.$$

For $E = E_1$ we prove the existence of a family of solutions y to equation (1.1) of the form

$$y(x) = E_1 x^{\alpha-1} + x^\beta z(x), \quad z \in C[0,T], \tag{2.5}$$

with a *suitable parameter* $\beta > \alpha - 1$ defined below (cf. [**9**]) assuming

$$\begin{aligned} q(x) &= o(x^{\alpha+\beta}) \text{ as } x \to 0 \text{ with } q(x)/x^{\alpha+\beta+1} \in L^1(0,T), \\ B(x) &= o(x^{1+\beta}) \text{ as } x \to 0 \text{ with } B(x)/x^{2+\beta} \in L^1(0,T). \end{aligned} \tag{2.6}$$

Inserting the ansatz (2.5) into equation (1.1) gives the equation for z

$$z(x) = f_1(x) + G_0[z](x) + L_1[z,z](x)$$

where

$$f_1(x) = \frac{h_1(x)}{x^\beta k(x)}, \qquad h_1(x) = q(x) - E_1 x^{\alpha-1} B(x), \tag{2.7}$$

$$G_0[z](x) = \frac{1}{x^\beta k(x)}\left[\frac{\nu}{B(\alpha,\alpha)} + 2E_1\right]\int_0^x \xi^\beta (x-\xi)^{\alpha-1} z(\xi)\,d\xi,$$

$$L_1[z_1,z_2](x) = \frac{1}{x^\beta k(x)}\int_0^x \xi^\beta (x-\xi)^\beta z_1(\xi) z_2(x-\xi)\,d\xi. \tag{2.8}$$

We further split up

$$G_0[z](x) = \frac{\lambda_1}{x^{\alpha+\beta}}\int_0^x$$

where
(2.9)
$$\lambda_1 = \frac{1}{A}\left[\frac{\nu}{B(\alpha,\alpha)} + 2E_1\right] = \frac{1}{B(\alpha,\alpha)}\left[1 + \frac{1}{A}\sqrt{(A-\nu)^2 + 4\gamma^2 B(\alpha,\alpha)}\right],$$

$$G_1[z](x) = -\frac{\lambda_1 B(x)}{x^{\alpha+\beta}k(x)} \int_0^x \xi^\beta (x-\xi)^{\alpha-1} z(\xi)\, d\xi \tag{2.10}$$

and obtain the *equation for* z in the form

$$z(x) - \frac{\lambda_1}{x^{\alpha+\beta}} \int_0^x \xi^\beta (x-\xi)^{\alpha-1} z(\xi) d\xi = g_1(x) \tag{2.11}$$

where

$$g_1(x) \equiv g_1[z](x) = f_1(x) + G_1[z](x) + L_1[z,z](x). \tag{2.12}$$

From (2.1), (2.2) with (2.6), we have $f_1 \in C[0,T]$ with $f_1(0) = 0$ and $f_1(x)/x \in L^1(0,T)$; further for any $z \in C[0,T]$ the following holds

$$|G_1[z](x)|, |L_1[z,z](x)| \leq \text{Const } x^{1+\beta-\alpha} \cdot \|z\| \quad \text{or} \quad \cdot \|z\|^2,$$

respectively, so that $G_1[z], L_1[z,z] \in C[0,T]$ with $G_1[z](0) = L_1[z,z](0) = 0$ and $G_1[z](x)/x, L_1[z,z](x)/x \in L^1(0,T)$. Hence, we have $g_1 \in C[0,T]$ with $g_1(0) = 0$ and $g_1(x)/x \in L^1(0,T)$.

From (2.9) we see that $\lambda_1 > 1/B(\alpha,\alpha)$. Defining now $\beta > \alpha - 1$ as the real root of the equation

$$\lambda_1 B(\alpha, \beta+1) = 1, \tag{2.13}$$

equation (2.11) has the general solution with arbitrary parameter K (cf. [**9**, Theorem 1])

$$z(x) = K + g_1(x) + \frac{\lambda_1}{x} \int_0^x r_1\left(\frac{\xi}{x}\right) g_1(\xi)\, d\xi, \quad K \in \mathbf{R}, \tag{2.14}$$

where the nonnegative resolvent function $r_1(u), 0 \leq u \leq 1$ is continuous in (0,1) and satisfies the estimation

$$r_1(u) \leq C\left[\frac{1}{u} + (1-u)^{\alpha-1}\right], \quad 0 < u < 1, \tag{2.15}$$

with a positive constant C. From (2.14) and (2.12) we finally obtain the *family of integral equations* for $z \in C[0,1]$

$$z(x) = f(x) + G[z](x) + L[z,z](x) \tag{2.16}$$

where

$$f(x) = K + f_1(x) + \frac{\lambda_1}{x}\int_0^x r_1\left(\frac{\xi}{x}\right) f_1(\xi)\, d\xi, \quad K \in \mathbf{R}, \tag{2.17}$$

$$G[z](x) = G_1[z](x) + \frac{\lambda_1}{x}\int_0^x r_1\left(\frac{\xi}{x}\right) G_1[z](\xi)\, d\xi, \tag{2.18}$$

$$L[z_1, z_2](x) = L_1[z_1, z_2](x) + \frac{\lambda_1}{x}\int_0^x r_1\left(\frac{\xi}{x}\right) L_1[z_1, z_2](\xi)\, d\xi. \tag{2.19}$$

We have $f \in C[0,T]$ with $f(0) = K$ and $G[z], L[z_1, z_2]$ are a bounded linear and a bounded bilinear operator on $C[0,T]$, respectively, with $G[z](0) = L[z,z](0) = 0$ for any $z \in C[0,T]$.

Equation (2.16) has the form of equation (A.7) in Appendix 2 and we apply Janno's Theorem A.1 to it. For this aim we estimate the operators G and L in the exponentially weighted norms $\|z\|_\sigma = \max_{0\le x\le T} |e^{-\sigma x} z(x)|$. We start with the operators G_1 in (2.10) and L_1 in (2.8). In view of (2.1), (2.6) we have

$$|e^{-\sigma x} G_1[z](x)| \le C_1 \frac{1}{x^{2\alpha-1}} \int_0^x e^{-\sigma(x-\xi)} \xi^\beta (x-\xi)^{\alpha-1} d\xi \cdot \|z\|_\sigma$$

and by [**9**, (A.3)]

$$\int_0^x e^{-\sigma(x-\xi)} \xi^\beta (x-\xi)^{\alpha-1} d\xi = \int_0^x e^{-\sigma\xi} \xi^{\alpha-1} (x-\xi)^\beta d\xi \le D\frac{x^\beta}{\sigma^\alpha}$$

so that

$$\|G_1[z]\|_\sigma \le C_1 D T^{\beta+1-2\alpha} \frac{\|z\|_\sigma}{\sigma^\alpha} \quad \text{if} \quad \beta \ge 2\alpha - 1. \tag{2.20}$$

For $\alpha - 1 < \beta < 2\alpha - 1$ we estimate

$$\int_0^x e^{-\sigma\xi} \xi^{\alpha-1}(x-\xi)^\beta d\xi \le \left(\int_0^x \xi^{\alpha-1}(x-\xi)^\beta d\xi\right)^\omega \left(D\frac{x^\beta}{\sigma^\alpha}\right)^{1-\omega}$$
$$= D_1 x^{\alpha\omega+\beta} \frac{1}{\sigma^{\alpha(1-\omega)}}, \quad 0 < \omega < 1.$$

Choosing $\omega = 2 - (1+\beta)/\alpha$ this gives

$$\int_0^x e^{-\sigma\xi}\xi^{\alpha-1}(x-\xi)^\beta d\xi \le D_1 \frac{x^{2\alpha-1}}{\sigma^{1+\beta-\alpha}}$$

so that

$$\|G_1[z]\|_\sigma \le C_1 D_1 \frac{\|z\|_\sigma}{\sigma^{1+\beta-\alpha}} \quad \text{if} \quad \alpha - 1 < \beta < 2\alpha - 1. \tag{2.21}$$

Analogously, due to (2.1) we obtain

$$|e^{-\sigma x} L_1[z_1, z_2](x)| \le C_2 \frac{1}{x^{\alpha+\beta}} \int_0^x e^{-\sigma\xi}\xi^\beta (x-\xi)^\beta d\xi \cdot \|z_1\|\,\|z_2\|_\sigma$$

with

$$\int_0^x e^{-\sigma\xi}\xi^\beta(x-\xi)^\beta d\xi \le D_2 x^{\beta+(\beta+1)\omega}\frac{1}{\sigma^{(1+\beta)(1-\omega)}}, \quad 0 < \omega < 1,$$

and choosing $\omega = \alpha/(\beta+1)$ we have the estimation

$$\|L_1[z_1, z_2]\|_\sigma \le C_2 D_2 \frac{\|z_1\|\|z_2\|_\sigma}{\sigma^{1+\beta-\alpha}}. \tag{2.22}$$

Moreover, the following holds

$$\begin{aligned} |e^{-\sigma x} L_1[z_1, z_2](x)| &\le C_2 \frac{1}{x^{\alpha+\beta}} \int_0^x \xi^\beta (x-\xi)^\beta d\xi \cdot \|z_1\|_\sigma \|z_2\|_\sigma \\ &\le C_0 x^{\beta+1-\alpha}\|z_1\|_\sigma\|z_2\|_\sigma \end{aligned}$$

so that

$$\|L_1[z_1, z_2]\|_\sigma \le C_0 T^{\beta+1-\alpha}\|z_1\|_\sigma \|z_2\|_\sigma. \tag{2.23}$$

Further, we estimate $G[z]$ in (2.18) and $L[z_1, z_2]$ in (2.19). In view of (2.15) we have

$$|e^{-\sigma x} G[z](x)| \le \left(1 + \frac{\lambda_1 C}{\alpha}\right)\|G_1[z]\|_\sigma + \lambda_1 C \int_0^x \frac{1}{\xi} e^{-\sigma x}|G_1[z](\xi)|\, d\xi.$$

But the integral on the right-hand side can be estimated due to (2.10) with (2.1), (2.6) by

$$
\begin{aligned}
C_3 \int_0^x \frac{1}{\xi^{2\alpha}} e^{-\sigma(x-\xi)} \int_0^\xi \eta^\beta (\xi-\eta)^{\alpha-1} d\eta \, d\xi \cdot \|z\|_\sigma \\
&\le C_3 B(\beta+1,\alpha) \int_0^x e^{-\sigma(x-\xi)} \xi^{\beta-\alpha} d\xi \cdot \|z\|_\sigma \\
&= \frac{C_3}{\lambda_1} \int_0^x e^{-\sigma\xi} (x-\xi)^{\beta-\alpha} d\xi \cdot \|z\|_\sigma
\end{aligned}
$$

and by [9, (A.2)]

(2.24)

$$
\int_0^x e^{-\sigma\xi}(x-\xi)^{\beta-\alpha} d\xi \le \begin{cases} \frac{x^{\beta-\alpha}}{\sigma} & \text{if } \beta \ge \alpha \\ (\Gamma(1+\beta-\alpha))/\sigma^{1+\beta-\alpha} & \text{if } -1 < \beta-\alpha < 0. \end{cases}
$$

Therefore, we get the estimation

$$
\begin{aligned}
\|G[z]\|_\sigma \le \left(1 + \frac{\lambda_1 C}{\alpha}\right) \|G_1[z]\|_\sigma \\
+ C C_3 \cdot \begin{cases} \frac{T^{\beta-\alpha}}{\sigma} & \text{if } \beta \ge \alpha \\ (\Gamma(1+\beta-\alpha))/\sigma^{1+\beta-\alpha} & \text{if } -1 < \beta-\alpha < 0 \end{cases}
\end{aligned}
$$

which, together with (2.20) and (2.21) proves assumption (A.8) of Theorem A.1.

Finally, analogously, in view of (2.15) we have

$$
\begin{aligned}
|e^{-\sigma x} L[z_1,z_2](x)| \le \left(1 + \frac{\lambda_1 C}{\alpha}\right) \|L_1[z_1,z_2]\|_\sigma \\
+ \lambda_1 C \int_0^x \frac{1}{\xi} e^{-\sigma x} |L_1[z_1,z_2](\xi)| \, d\xi
\end{aligned}
$$

and since $e^{-\sigma x} = e^{-\sigma(x-\xi)} e^{-\sigma(\xi-\eta)} e^{-\sigma\eta}$ from (2.8) with (2.1) it follows

$$
\begin{aligned}
\int_0^x \frac{1}{\xi} e^{-\sigma x} |L_1[z_1,z_2](\xi)| \, d\xi \\
&\le C_4 \int_0^x \frac{e^{-\sigma(x-\xi)}}{\xi^{1+\alpha+\beta}} \int_0^\xi \eta^\beta (\xi-\eta)^\beta d\eta \, d\xi \cdot \|z_1\|_\sigma \|z_2\|_\sigma \\
&\le C_5 \int_0^x e^{-\sigma(x-\xi)} \xi^{\beta-\alpha} d\xi \cdot \|z_1\|_\sigma \|z_2\|_\sigma
\end{aligned}
$$

where the last integral can be estimated by (2.24) again. This, together with (2.22) and (2.23), proves assumptions (A.9) and (A.10) of Theorem A.1. The theorem then shows the existence of a solution $z = z_K$ with $z(0) = K$ to equation (2.16) for any $K \in \mathbf{R}$.

Theorem 2.1. *Let k, p fulfill assumptions* (2.1) *and* (2.2) *with $\gamma > 0$ and* (2.6) *where $\beta > \alpha - 1$ is the real root of equation* (2.13) *with λ_1 defined in* (2.9)*. Then equation* (1.1) *in $(0, T)$, $T > 0$ has a one-parametric family of solutions y_K, $K \in \mathbf{R}$ of form* (2.5) *with $E_1 > 0$ defined in* (2.3) *and $z_K \in C[0, T]$ with $z_K(0) = K$. These solutions are the unique ones of equation* (1.1) *in the class of functions of form* (2.5) *with positive coefficient $E = E_1$.*

Corollary 2.1. *The existence assertion of Theorem* 2.1 *remains valid in the limit case $\gamma = 0$ if $\nu \neq A$. For $\nu < A$ we again have $E_1 > 0$ and $\lambda_1 = (2A - \nu)/AB(\alpha, \alpha) > 1/(B(\alpha, \alpha))$; for $\nu > A$, $E_1 = 0$ holds but again $\lambda_1 = \nu/(AB(\alpha, \alpha)) > 1/(B(\alpha, \alpha))$ implying $\beta > \alpha - 1$ in both cases.*

Remark 2.1. The case $\nu = 0$ with $\gamma = 0$ where $\beta = \alpha$ has been dealt with in [**9**, Theorem 3], the case $\nu = 0$ with $\gamma > 0$ for $\alpha = 1/2$ in [**11**, Theorem 3.1]. The above estimations of $G[z]$ and $L[z_1, z_2]$ for $\beta = \alpha$ improve the estimations in [**9**].

For $E = E_2$ we show the existence of a solitary solution y to equation (1.1) of the form

$$y(x) = E_2 x^{\alpha-1} + x^{\delta}\zeta(x), \quad \zeta \in C[0, T], \tag{2.25}$$

where the parameter $\delta > \alpha - 1$ is *prescribed* such that

$$\begin{aligned} q(x) &= dx^{\alpha+\delta} + e(x), \quad d \in \mathbf{R}, \qquad e(x) = o(x^{\alpha+\delta}), \\ B(x) &= bx^{1+\delta} + c(x), \quad b \in \mathbf{R}, \qquad c(x) = o(x^{1+\delta}) \end{aligned} \tag{2.26}$$

instead of (2.6).

Inserting the ansatz (2.25) into equation (1.1), analogously as above, we get the *equation for ζ*

$$\zeta(x) - \frac{\lambda_2}{x^{\alpha+\delta}} \int_0^x \xi^{\delta}(x - \xi)^{\alpha-1}\zeta(\xi)d\xi = g_2(x) \tag{2.27}$$

where

$$\lambda_2 = \frac{1}{A}\left[\frac{\nu}{B(\alpha,\alpha)} + 2E_2\right] = \frac{1}{B(\alpha,\alpha)}\left[1 - \frac{1}{A}\sqrt{(A-\nu)^2 + 4\gamma^2 B(\alpha,\alpha)}\right] \tag{2.28}$$

with $\lambda_2 < 1/B(\alpha,\alpha)$ and

$$g_2(x) \equiv g_2[\zeta](x) = f_2(x) + G_2[\zeta](x) + L_2[\zeta,\zeta](x) \tag{2.29}$$

with

$$f_2(x) = \frac{h_2(x)}{x^\delta k(x)}, \qquad h_2(x) = q(x) - E_2 B(x) x^{\alpha-1}, \tag{2.30}$$

$$G_2[\zeta](x) = -\frac{\lambda_2 B(x)}{x^{\alpha+\delta}k(x)} \int_0^x \xi^\delta (x-\xi)^{\alpha-1}\zeta(\xi)\,d\xi, \tag{2.31}$$

$$L_2[\zeta_1,\zeta_2](x) = \frac{1}{x^\delta k(x)} \int_0^x \xi^\delta (x-\xi)^\delta \zeta_1(\xi)\zeta_2(x-\xi)\,d\xi. \tag{2.32}$$

It holds $f_2 \in C[0,T]$ with $f_2(0) = (d - E_2 b)/A$. Further, G_2 is a bounded linear and L_2 a bounded bilinear operator in $C[0,T]$ and $C[0,T] \times C[0,T]$, respectively, with

$$|G_2[\zeta](x)| \le \text{Const}\, x^{\delta+1-\alpha}\|\zeta\| \quad \text{for any } \zeta \in C[0,T],$$
$$|L_2[\zeta_1,\zeta_2](x)| \le \text{Const}\, x^{\delta+1-\alpha}\|\zeta_1\|\|\zeta_2\| \quad \text{for any } \zeta_1,\zeta_2 \in C[0,T]$$

implying $G_2[\zeta](0) = L_2[\zeta,\zeta](0) = 0$ for any $\zeta \in C[0,T]$. For the solution ζ of equation (2.27) we then have

$$\zeta(0) = \frac{f_2(0)}{1 - \lambda_2 B(\delta+1,\alpha)} = \frac{d - E_2 b}{A} \cdot \frac{1}{1 - \lambda_2 B(\delta+1,\alpha)}. \tag{2.33}$$

Further, as above in (2.20)–(2.23) for G_1 and L_1, from (2.31) and (2.32) we obtain the estimations for G_2 and L_2 in weighted norms

$$\|G_2[\zeta]\|_\sigma \le \text{Const} \begin{cases} \frac{\|\zeta\|_\sigma}{\sigma^\alpha} & \text{if } \delta \ge 2\alpha - 1 \\ \frac{\|\zeta\|_\sigma}{\sigma^{1+\delta-\alpha}} & \text{if } \alpha - 1 < \delta < 2\alpha - 1 \end{cases} \tag{2.34}$$

and

(2.35)
$$\|L_2[\zeta_1,\zeta_2]\|_\sigma \leq \text{Const} \begin{cases} \|\zeta_1\|\,\|\zeta_2\|_\sigma/\sigma^{1+\delta-\alpha} \\ \|\zeta_1\|_\sigma\,\|\zeta_2\|_\sigma. \end{cases}$$

We now have to distinguish the three cases: $0 < \lambda_2 < 1/(B(\alpha,\alpha))$, $\lambda_2 = 0$ and $\lambda_2 < 0$. In the *first case* $0 < \lambda_2 < 1/(B(\alpha,\alpha))$ equation (2.27) has the form of equation (A.2) with the unique solution

(2.36)
$$\zeta(x) = f(x) + G[\zeta](x) + L[\zeta,\zeta](x)$$

where

(2.37)
$$f(x) = f_2(x) + \frac{\lambda_2}{x}\int_0^x r_2\left(\frac{\xi}{x}\right) f_2(\xi)\,d\xi,$$

(2.38)
$$G[\zeta](x) = G_2[\zeta](x) + \frac{\lambda_2}{x}\int_0^x r_2\left(\frac{\xi}{x}\right) G_2[\zeta](\xi)\,d\xi,$$

(2.39)
$$L[\zeta_1,\zeta_2](x) = L_2[\zeta_1,\zeta_2](x) + \frac{\lambda_2}{x}\int_0^x r_2\left(\frac{\xi}{x}\right) L_2[\zeta_1,\zeta_2](\xi)\,d\xi$$

and the nonnegative resolvent function $r_2(u)$ is continuous in $(0,1)$ and satisfies the estimation

(2.40)
$$r_2(u) \leq C\left[\frac{1}{u^{p_0}} + (1-u)^{\alpha-1}\right], \quad 0 < u < 1,$$

with some $p_0 \in (-\delta, \alpha-\delta)$. Like $f_2 \in C[0,T]$ it holds $f \in C[0,T]$ since $p_0 < \alpha - \delta < 1$.

We estimate $G[\zeta]$ in (2.38) and $L[\zeta_1,\zeta_2]$ in (2.39). As above we have

$$|e^{\sigma x} G[\zeta](x)| \leq \left(1 + \frac{\lambda_2 C}{\alpha}\right)\|G_2[\zeta]\|_\sigma + \lambda_2 C x^{p_0-1}\int_0^x \xi^{-p_0} e^{-\sigma x}|G_2[\zeta](\xi)|\,d\xi$$

and the last integral can be estimated in view of (2.31) with (2.1), (2.26) by

$$(2.41)\quad C_6 x^{p_0-1}\int_0^x e^{-\sigma\xi}(x-\xi)^{1-p_0+\delta-\alpha}d\xi\cdot\|\zeta\|_\sigma \leq C_6\int_0^x e^{-\sigma\xi}(x-\xi)^{\delta-\alpha}d\xi\cdot\|\zeta\|_\sigma$$

since $(x-\xi)^{1-p_0} \leq x^{1-p_0}$ due to $p_0 < 1$. But the integral in (2.41) fulfills the estimation (2.24) with δ instead of β and, observing also (2.34), we see that $G[\zeta]$ satisfies the assumption (A.8) of Theorem A.1 again.

Analogously as above, we also have

$$|e^{-\sigma x}L[\zeta_1,\zeta_2](x)| \leq \left(1+\frac{\lambda_2 C}{\alpha}\right)\|L_2[\zeta_1,\zeta_2]\|_\sigma + \lambda_2 C x^{p_0-1}\int_0^x \xi^{-p_0}e^{-\sigma x}|L_2[\zeta_1,\zeta_2](\xi)|\,d\xi$$

and from (2.32)

$$\int_0^x \xi^{-p_0}e^{-\sigma x}|L_2[\zeta_1,\zeta_2](\xi)|\,d\xi \leq C_7 x^{1-p_0}\int_0^x e^{-\sigma\xi}(x-\xi)^{\delta-\alpha}d\xi\cdot\|\zeta_1\|_\sigma\|\zeta_2\|_\sigma$$

with the same integral as in (2.41). This together with (2.35) proves the assumptions (A.9), (A.10) of Theorem A.1.

In the *second case* $\lambda_2 = 0$ we have the solution $\zeta(x) = g_2(x)$ of equation (2.27) leading to the equation

$$\zeta(x) = f_2(x) + G_2[\zeta](x) + L_2[\zeta,\zeta](x)$$

for which by (2.34) and (2.35) the assumptions of Theorem A.1 are fulfilled.

In the *third case* $\lambda_2 < 0$ equation (2.27) has been studied in [**11**, Appendix 2]. From [**11**, Corollary A4] we obtain the unique solution of this equation in the form (2.36) with the resolvent function $\tilde{r}_2$ which is continuous in (0,1) and satisfies the estimation

$$|\tilde{r}_2(u)| \leq C[u^\delta + (1-u)^{\alpha-1}], \quad \delta > \alpha - 1 > -1$$

corresponding to $p_0 = -\delta$ in (2.40). Hence the assumption of Theorem A.1 are fulfilled for the corresponding equation (2.36). Therefore, from Theorem A.1 we obtain

Theorem 2.2. *Let k, p satisfy the assumptions* (2.1), (2.2) *with* $\gamma > 0$ *and* (2.26) *where $\delta > \alpha - 1$ is prescribed. Then equation* (1.1) *in $(0, T)$, $T > 0$ has a unique solution y of form* (2.25) *with $E_2 < 0$ defined in* (2.4) *and $\zeta \in C[0, T]$ with value $\zeta(0)$ given by* (2.33).

Corollary 2.2. *The assertion of Theorem* 2.2 *remains valid in the limit case $\gamma = 0$ where $E_2 = 0$ for $\nu \leq A$ and $E_2 = (A - \nu)/B(\alpha, \alpha) < 0$ for $\nu > A$. In the case $\gamma = \nu = 0$ where $\lambda_2 = 0$ the assumption* (2.26) *on B can be weakened to $B(x) = o(x^\alpha)$ as $x \to 0$.*

Remark 2.2. The case $\nu = 0$ with $\gamma = 0$ where $\beta = \alpha$ has been dealt with in [**9**, Theorem 4], the case $\nu = 0$ with $\gamma > 0$ for $\alpha = 1/2$ in [**11**, Theorem 3.1].

As an *example* to both theorems we consider the modified Bernstein-Doetsch equation (cf. [**11**])

$$\begin{aligned} Ax^\alpha y(x) &= \int_0^x y(\xi) y(x-\xi)\, d\xi \\ &\quad + \frac{A}{B(\alpha,\alpha)} \int_0^x y(\xi)(x-\xi)^{\alpha-1}\, d\xi - \frac{1}{\Gamma(2\alpha)} x^{2\alpha-1}. \end{aligned} \tag{2.42}$$

It has a family of solutions of the form

$$y_K(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} + x^\beta z_K(x)$$

with $z_K(0) = K \in \mathbf{R}$ and $\beta > \alpha - 1$ the real root of the equation

$$B(\alpha, \beta+1) = \frac{A\Gamma^2(\alpha)}{A\Gamma(2\alpha) + 2\Gamma(\alpha)},$$

in particular the solution $y_1(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}$ for $K = 0$, and the solitary solution $y_2(x) = -1/\Gamma(\alpha)x^{\alpha-1}$.

For $\alpha = 1$ with $A = 2$ we have $\beta = 1$ and the functions $z_K(x) = 2[\exp((K/2)x) - 1]/2$, for $\alpha = 1$ with $A = 1$ it is $\beta = 2$ and we get the functions $z_K(x) = 2[\cosh\sqrt{K}x - 1]/x^2$, $K > 0$ and $z_K(x) = 2[\cos\sqrt{-K}x - 1]/x^2$, $K < 0$.

6. Exceptional case. It remains to prove the existence of a family of solutions to equation (1.1) in the *exceptional case* $\gamma = 0$, $\nu = A$. Putting $A = 1$ without loss of generality, we consider the basic equation for this case

$$x^\alpha y(x) = \int_0^x y(\xi)y(x-\xi)\,d\xi + \frac{1}{B(\alpha,\alpha)}\int_0^x y(\xi)(x-\xi)^{\alpha-1}d\xi. \tag{3.1}$$

Looking for solutions $y(x) = o(x^{\alpha-1})$ as $x \to 0$ we make the ansatz

$$y(x) = \int_0^\infty \frac{x^{t+\alpha-1}}{\Gamma(t+\alpha)}\varphi(t)\,dt, \quad x > 0, \tag{3.2}$$

where the function $\varphi \in C[0,\infty)$ satisfies an estimation

$$|\varphi(t)| \le C_0 e^{\rho t}, \quad t \ge 0 \ (\rho \in \mathbf{R}) \tag{3.3}$$

with a positive constant C_0.

From the integrals

$$\int_0^x y(\xi)y(x-\xi)\,d\xi = \int_0^\infty \frac{x^{t+2\alpha-1}}{\Gamma(t+2\alpha)}\left[\int_0^t \varphi(s)\varphi(t-s)\,ds\right]dt,$$
$$\int_0^x y(\xi)(x-\xi)^{\alpha-1}d\xi = \Gamma(\alpha)\int_0^\infty \frac{x^{t+2\alpha-1}}{\Gamma(t+2\alpha)}\varphi(t)\,dt$$

we see that the function (3.2) is a solution to equation (3.1) if φ is a solution to the equation

$$\lambda(t)\varphi(t) = \int_0^t \varphi(s)\varphi(t-s)\,ds, \quad t > 0, \tag{3.4}$$

where

$$\lambda(t) = \frac{\Gamma(t+2\alpha)}{\Gamma(t+\alpha)} - \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}. \tag{3.5}$$

The function $\lambda \in C[0,\infty)$ satisfies $\lambda(0) = 0$, $\lambda(t) > 0$ for $t > 0$ and $\lambda(t) \sim t^\alpha$ as $t \to \infty$. It has the asymptotic expansion

$$\lambda(t) = A_0 t + B_0 t^2 + O(t^3) \quad \text{as } t \to 0,$$

where

$$\begin{aligned} A_0 &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \left[\Psi(2\alpha) - \Psi(\alpha)\right], \\ B_0 &= \frac{1}{2}\frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \left[(\Psi(2\alpha) - \Psi(\alpha))^2 + \Psi'(2\alpha) - \Psi'(\alpha)\right] \end{aligned} \tag{3.6}$$

with Gauss psi-function $\Psi = \Gamma'/\Gamma$. Therefore, by Theorem A.2 in Appendix 2 we get a solution φ_0 to equation (3.4) of the form

$$\varphi_0(t) = A_0 - 2B_0 t \ln t + t\chi_0(t)$$

with $\chi_0 \in C[0,\infty)$ satisfying $\chi_0(0) = 0$ and the general solution $\varphi = e^{ct}\varphi_0$, $c \in \mathbf{R}$ of the form

$$\varphi(t) = A_0 - 2B_0 t \ln t + t\chi(t) \tag{3.7}$$

where $\chi \in C[0,\infty)$ fulfills $\chi(0) = K$, $K = A_0 c \in \mathbf{R}$. Further, the solutions φ obey inequalities of type (3.3). Hence the corresponding functions y in (3.2) are solutions to equation (3.1).

From (3.7) we obtain for these solutions the representation

$$y_K(x) = A_0\nu(x,\alpha-1) - 2B_0\omega(x,1,\alpha-1) + \mu(x,1,\alpha-1)z(x) \tag{3.8}$$

with $z \in C[0,\infty)$ satisfying $z(0) = K$, $K \in \mathbf{R}$ and the Volterra functions (cf. [**4**, Section 18.3] and Appendix 3)

$$\begin{aligned} \nu(x,\alpha-1) &= \int_0^\infty \frac{x^{t+\alpha-1}dt}{\Gamma(t+\alpha)}, \\ \mu(x,1,\alpha-1) &= \int_0^\infty \frac{x^{t+\alpha-1}t\,dt}{\Gamma(t+\alpha)}, \\ \omega(x,1,\alpha-1) &= \int_0^\infty \frac{x^{t+\alpha-1}t\ln t\,dt}{\Gamma(t+\alpha)}. \end{aligned}$$

Theorem 3.1. *The integral equation* (3.1) *has* (*besides* $y \equiv 0$) *the one-parametric family of solutions* y_K, $K \in \mathbf{R}$ *of form* (3.8) *given by formula* (3.2) *where* $\varphi = e^{ct}\varphi_0$, $c \in \mathbf{R}$ *are the solutions of equation* (3.4) *of form* (3.7) *with coefficients* (3.6).

From (3.8) by formulas [**4**, Section 18.3, (9)] and (A.22) it follows

Corollary 3.1. *The functions* y_K *have the asymptotic expansion as* $x \to 0$:

$$y_K(x) = \frac{A_0}{\Gamma(\alpha)} x^{\alpha-1}\left(\ln\frac{1}{x}\right)^{-1} + \frac{2B_0}{\Gamma(\alpha)} x^{\alpha-1}\left(\ln\frac{1}{x}\right)^{-2} \ln\ln\frac{1}{x} + K_0 x^{\alpha-1}\left(\ln\frac{1}{x}\right)^{-2} + o\left(x^{\alpha-1}\left(\ln\frac{1}{x}\right)^{-2}\right),$$

where $K_0 \in \mathbf{R}$ *is given by*

$$K_0 = \frac{1}{\Gamma(\alpha)}[K - A_0\Psi(\alpha) - 2B_0\Psi(2)], \quad K \in \mathbf{R}$$

with $\Psi(2) = 1 - \widetilde{C}$, $\widetilde{C}$ *Euler's constant.*

From Remark A.2 in Appendix 2 a further corollary to Theorem 3.1 holds.

Corollary 3.2. *Equation* (3.1) *with free term* $p(x)$ *of the form*

$$p(x) = \int_0^\infty \frac{x^{t+2\alpha-1}}{\Gamma(t+2\alpha)} f(t)\, dt,$$

where $f \in C[0,\infty)$ *satisfies* $f(t) = o(t^2)$ *as* $t \to 0$ *with* $f(t)/t^3 \in L^1(0,T)$ *for any* $T > 0$ *and an inequality of type* (3.3), *has a family of solutions* y_K *as in Theorem* 3.1.

In the case of the more general equation than (3.1)

$$(3.9)\quad k(x)y(x) = \int_0^x y(\xi)y(x-\xi)\, d\xi + \frac{1}{B(\alpha,\alpha)}\int_0^x y(\xi)(x-\xi)^{\alpha-1} d\xi$$

where

$$k(x) = x^\alpha + \sum_{n=1}^{N} b_n x^{\alpha+\beta_n}, \quad b_n \in \mathbf{R},\ 0 < \beta_1 < \cdots < \beta_N$$

by the ansatz (3.2) we get the equation for φ

$$\lambda(t)\varphi(t) = \int_0^t \varphi(s)\varphi(t-s)\, ds - \sum_{n=1}^{N} b_n \psi_n(t), \quad t > 0 \tag{3.10}$$

where λ is given by (3.5) and

$$\psi_n(t) = \begin{cases} 0 & \text{if } 0 < t < \beta_n \\ \frac{\Gamma(t+2\alpha)}{\Gamma(t+\alpha-\beta_n)}\varphi(t-\beta_n) & \text{if } t > \beta_n. \end{cases}$$

Equation (3.10) can be solved successively in the intervals (β_{n-1}, β_n), $n = 1, 2, \dots$.

Remark 3.1. We *conjecture* that under suitable assumptions on $k(x) \sim x^\alpha$ and $p(x) = o(x^{2\alpha-1})$ as $x \to 0$ equation (3.9) with additional free term $p(x)$ has a family of solutions $y_K, K \in \mathbf{R}$ of form (3.8).

The ansatz (3.2) can be used for other types of equations to reduce them to autoconvolution equations of the third kind. We give some examples. The *integral-functional equation*

$$x^\alpha \sum_{n=0}^{N} a_n y(c_n x) = \int_0^x y(\xi) y(x-\xi)\, d\xi, \tag{3.11}$$

where $c_0 > c_1 > \cdots > c_N > 0$ with $c_0 \geq 1$ and $a_0 = 1$, $a_n \in \mathbf{R}$ $(n = 1, \dots, N)$ leads to equation (3.4) with

$$\lambda(t) = \frac{\Gamma(t+2\alpha)}{\Gamma(t+\alpha)} \sum_{n=0}^{N} a_n c_n^{t+l-1} \sim t^\alpha c_0^{t+\alpha-1} \text{ as } t \to \infty$$

which by the conditions

$$\sum_{n=0}^{N} a_n c_n^{\alpha-1} = 0, \qquad \sum_{n=0}^{N} a_n c_n^{t+\alpha-1} > 0 \quad \text{for } t > 0,$$

fulfills the assumptions of Theorem A.2. Also related Stieltjes integrals can be brought into the left-hand side of equation (3.11).

The *integral-differential equation*

$$(3.12)\quad x^\alpha y(x) + \sum_{n=1}^{N} c_n x^{\gamma_n} \frac{d}{dx}[x^{\alpha+1-\gamma_n} y] + \sum_{m=1}^{M} d_m x^{-\delta_m} \int_0^x \xi^{\delta_m-\beta_m}(x-\xi)^{\alpha-1+\beta_m} y(\xi)\, d\xi = \int_0^x y(\xi)y(x-\xi)\, d\xi$$

($\gamma_m, \delta_m, \beta_m \in \mathbf{R}$ with $\alpha+\beta_m > 0$, $\delta_m > \beta_m - 1$; $c_n, d_m \in \mathbf{R}$) yields equation (3.4) with

$$\lambda(t) = \frac{\Gamma(t+2\alpha)}{\Gamma(t+\alpha)}\Bigg[1 + \sum_{n=1}^{N} c_n(t+2\alpha-\gamma_n) + \sum_{m=1}^{M} d_m B(\alpha+\beta_m, t+\alpha+\delta_m-\beta_m)\Bigg] \sim t^{\alpha+1}\sum_{n=1}^{N} c_n \quad \text{as } t\to\infty \quad \text{if } \sum_{n=1}^{N} c_n > 0$$

or

$$\lambda(t) \sim t^\alpha\left(1 - \sum_{n=1}^{N} c_n\gamma_n\right) \quad \text{as } t\to\infty \text{ if } \sum_{n=1}^{N} c_n = 0, \sum_{n=1}^{N} c_n\gamma_n < 1.$$

If $c_n, d_m \in \mathbf{R}$ satisfy the relations

$$1 + \sum_{n=1}^{N} c_n(2\alpha-\gamma_n) + \sum_{m=1}^{M} d_m B(\alpha+\beta_m, \alpha+\delta_m-\beta_m) = 0,$$

$$1 + \sum_{n=1}^{N} c_n(t+2\alpha-\gamma_n) + \sum_{m=1}^{M} d_m B(\alpha+\beta_m, t+\alpha+\delta_m-\beta_m) > 0 \quad \text{for } t > 0,$$

and

$$\sum_{n=1}^{N} c_n > 0 \quad \text{or} \quad \sum_{n=1}^{N} c_n = 0, \sum_{n=1}^{N} c_n \gamma_n < 1,$$

then Theorem A.2 proves the existence of a family of solutions to equation (3.12).

Finally, by the slightly more general ansatz than (3.2)

$$y(x) = \int_0^\infty \frac{x^{t+\alpha-1}}{\Gamma(t+\beta)} \varphi(t)\, dt, \quad \beta \geq \alpha > 0,$$

the related *integral equation* to (3.1)

$$\begin{aligned} x^\beta y(x) &= \int_0^x \xi^{\beta-\alpha} y(\xi) y(x-\xi)\, d\xi \\ &\quad + \frac{1}{B(\alpha,\beta)} \int_0^x \xi^{\beta-\alpha} y(\xi)(x-\xi)^{\alpha-1} d\xi \end{aligned} \tag{3.13}$$

is reduced to the equation

$$\lambda(t)\varphi(t) = \int_0^t \mu(s)\varphi(s)\varphi(t-s)\, ds \tag{3.14}$$

with

$$\begin{aligned} \lambda(t) &= \frac{\Gamma(t+\alpha+\beta)}{\Gamma(t+\beta)} - \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sim t^\alpha \quad \text{as } t \to \infty, \\ \mu(t) &= \frac{\Gamma(t+\alpha)}{\Gamma(t+\beta)} \sim t^{\alpha-\beta} \quad \text{as} \quad t \to \infty. \end{aligned}$$

Equation (3.14) (after multiplication with the constant factor $\Gamma(\beta)/\Gamma(\alpha)$) also satisfies the assumptions of Theorem A.2.

APPENDIX

1. A linear auxiliary equation. We deal with the equation

$$x^\alpha w(x) = \lambda \int_0^x (x-\xi)^{\alpha-1} w(\xi)\, d\xi + h(x) \quad (\alpha > 0) \tag{A.1}$$

where $0 < \lambda < 1/B(\alpha,\alpha)$. In [**9**] this equation for $\lambda > 0$ has been considered in the class of solutions $w(x) = x^\beta z(x), z \in C[0,T], T > 0$

with $\beta > -1$ real root of $\lambda B(l, \beta + 1) = 1$ in which the homogeneous equation has the solution $w_0(x) = x^\beta$.

For $0 < \lambda < 1/B(\alpha, \alpha)$ we have $-1 < \beta < \alpha - 1$ and now we are looking for the uniquely determined solution to equation (A.1) of the form $w(x) = x^\delta \zeta(x)$, $\zeta \in C[0, T]$ with prescribed $\delta > \alpha - 1$. Then ζ satisfies the equation

$$\zeta(x) = \frac{\lambda}{x^{\alpha+\delta}} \int_0^x (x - \xi)^{\alpha-1} \xi^\delta \zeta(\xi)\, d\xi + g(x) \tag{A.2}$$

where $g(x) = h(x)/x^{\alpha+\delta}$ is assumed to be in $C[0, T]$. Then equation (A.2) has the unique solution in $C[0, T]$

$$\zeta(x) = g(x) + \frac{\lambda}{x} \int_0^x r\left(\frac{\xi}{x}\right) g(\xi)\, d\xi \tag{A.3}$$

where the nonnegative resolvent function $r(u), 0 \leq u \leq 1$ is continuous in $(0, 1)$ and fulfills the equation

$$r(u) = \lambda u^\delta \int_u^1 (v - u)^{\alpha-1} v^{-(\alpha+\delta)} r(v)\, dv + u^\delta (1 - u)^{\alpha-1}. \tag{A.4}$$

Substituting $t = \ln(1/u)$ and $\varphi(t) = r(u)$ from (A.4) we obtain the Laplace transform of φ

$$\Phi(p) = \frac{B(p + \delta, \alpha)}{1 - \lambda B(p + \delta, \alpha)}. \tag{A.5}$$

As in [**9**], from (A.4) and (A.5) we have the estimation

$$r(u) \leq C\left[\frac{1}{u^{p_0}} + (1 - u)^{\alpha-1}\right], \quad 0 < u < 1, \tag{A.6}$$

with some positive constant C where $p_0 \in (-\delta, \alpha - \delta)$ is the real root of the equation $\lambda B(p_0 + \delta, \alpha) = 1$.

2. Existence theorem for quadratic operator equation. We state a corollary to an existence theorem by Janno [**6**] for operator equations in spaces of continuous functions on **R**. We consider the operator equation

$$z(x) = f(x) + G[z](x) + L[z,z](x) \tag{A.7}$$

with $f \in C[0,T]$, a linear operator $G : C[0,T] \to C[0,T]$ and a bilinear operator $L : C[0,T] \times C[0,T] \to C[0,T]$, $0 < T < \infty$. The space $C[0,T]$ is equipped with the exponentially weighted norms

$$\|z\|_\sigma = \|e^{-\sigma x} z(x)\| = \max_{0 \le x \le T} |e^{-\sigma x} z(x)|, \quad \sigma \ge 0.$$

Then we have (cf. [**2**, Lemma 1])

Theorem A.1. *Let $T > 0$. Let the linear operator G and the bilinear operator L in $C[0,T]$ fulfill the inequalities*

$$\|G[z]\|_\sigma \le M_T(\sigma) \|z\|_\sigma, \quad \sigma \ge \sigma_0 > 0, \tag{A.8}$$

for any $z \in C[0,T]$ with a continuous function M_T satisfying $M_T(\sigma) \to 0$ as $\sigma \to \infty$,

$$\|L[z_1, z_2]\|_\sigma \le N_T \|z_1\|_\sigma \|z_2\|_\sigma, \quad \sigma \ge \sigma_0 > 0, \tag{A.9}$$

with a constant N_T and

$$\|L[z_1, z_2]\|_\sigma \le \begin{cases} \mu_{1,T}(\sigma) \|z_1\| \|z_2\|_\sigma \\ \mu_{2,T}(\sigma) \|z_1\|_\sigma \|z_2\| \end{cases}, \quad \sigma \ge \sigma_0 > 0, \tag{A.10}$$

with continuous functions $\mu_{k,T}$, $k = 1, 2$, satisfying $\mu_{k,T}(\sigma) \to 0$ as $\sigma \to \infty$ for any pair $z_1, z_2 \in C[0,T]$.

Then, for any $f \in C[0,T]$, equation (A.1) *has a uniquely determined solution $z \in C[0,T]$.*

From the proof of the general theorem of Janno in [**6**] the following corollary to Theorem A.1 easily follows.

Corollary A.1. *Let the assumptions on the operators G and L in Theorem* A.1 *be fulfilled for any $T > 0$ where the functions $M_T(\sigma)$ in*

(A.2) and $\mu_{k,T}(\sigma)$, $k = 1, 2$ in (A.4) can be chosen independently of T for $T > T_0$ with some $T_0 \in (0, \infty)$. Let further $f \in C[0, \infty)$ satisfy the inequality

$$|f(x)| \le C_0 e^{\sigma_1 x}, \quad x \ge 0, \qquad \text{with some } \sigma_1 \ge \sigma_0. \tag{A.11}$$

Then the uniquely determined solution $z \in C[0, \infty)$ of equation (A.7) obeys the estimation

$$|z(x)| \le C e^{\sigma_2 x}, \quad x \ge 0, \qquad \text{with some } \sigma_2 \ge \sigma_1. \tag{A.12}$$

We apply Corollary A.1 to an existence theorem for the equation

$$k(x)y(x) = \int_0^x a(\xi)y(\xi)y(x-\xi)d\xi, \ x > 0 \ , \tag{A.13}$$

extending Theorem 3 in [**2**].

Theorem A.2. *Let $k \in C[0, \infty)$ with $k(x) > 0$ for $0 < x < \infty$ and $k(0) = 0$ possess the expansion*

$$k(x) = Ax + Bx^2 + C(x) \quad (A > 0) \tag{A.14}$$

where $B \in \mathbf{R}$ and $C(x)/x^3 \in L^1(0, T)$ for any $T > 0$ and satisfy the inequality

$$\frac{1}{k(x)} \le D_1 \quad \text{for } x \ge 1 \tag{A.15}$$

with a positive constant D_1. Let further $a \in C[0, \infty)$ with $a(0) = 1$ possess the expansion

$$a(x) = 1 + ux + v(x) \tag{A.16}$$

where $u \in \mathbf{R}$ and $v(x)/x^2 \in L^1(0, T)$ for any $T > 0$ and satisfy the inequality

$$|a(x)| \le D_2 \quad \text{for } x \ge 0 \tag{A.17}$$

with a positive constant D_2.

Then equation (A.13) has a solution $y_0 \in C[0,\infty)$ *of the form*

$$y_0(x) = A + \mu x \ln x + x z_0(x), \quad \mu = Au - 2B, \tag{A.18}$$

with $z_0 \in C[0,\infty)$ *satisfying* $z_0(0) = 0$ *and obeying the inequality*

$$|z_0(x)| \le D_0 e^{\varepsilon x}, \quad x \ge 0, \tag{A.19}$$

with positive constants D_0, ε*. The solution* y_0 *is unique in the class of functions of type (A.18). The general solution* y *of equation* (A.13) *is given by* $y = e^{cx} y_0$, $c \in \mathbf{R}$.

The *proof* of Theorem A.2 follows repeating the proof of Theorem 3 in [**2**] and observing the assumptions (A.15) and (A.17).

Remark A.1. From the expansion (A.18) for y_0, the estimation (A.19) for z_0 and the representation $y = e^{cx} y_0$, $c \in \mathbf{R}$ the general inequality

$$|y(x)| \le D e^{\rho x}, \quad x \ge 0, \tag{A.20}$$

for any solution y of equation (A.13) obtained by Theorem A.2 follows, where D is a positive constant and the constant $\rho \in \mathbf{R}$ depends on the specific solution y.

Remark A.2. Theorem A.2 with the estimation (A.20) also holds for equation (A.13) with the additional free term $p(x) \in C[0,\infty)$ which satisfies $p(x)/x^3 \in L^1(0,T)$ for any $T > 0$ and an inequality of form (A.11).

For the *proof* compare Theorem 3 with Remark 3 in [**9**]. For further extension to more general free terms p we refer to Theorem 5 in [**7**] and Theorem 2.1 in [**11**].

3. A generalized Volterra function. In analogy to the Volterra functions μ [**4**, Section 18.3, (2)] we introduce the *generalized Volterra functions*

$$\omega(x,\beta,\delta;n) = \int_0^\infty \frac{x^{\delta+t} t^\beta (\ln t)^n \, dt}{\Gamma(\beta+1)\Gamma(\delta+t+1)} \quad (\beta > -1) \tag{A.21}$$

for $n = 0, 1, \ldots$, where $\omega(x,\beta,\delta;0) = \mu(x,\beta,\delta)$, and we put $\omega(x,\beta,\delta;1) = \omega(x,\beta,\delta)$. We are interested in the asymptotic expansion [**1, 3**] of the function $\omega(x,\beta,\delta;n)$ as $x \to 0$. Using as in [**4**, Section 18.3, (8)] the series

$$\frac{1}{\Gamma(\delta+t+1)} = \sum_{m=0}^{\infty} \mu(1,-m-1,\delta)\frac{(-t)^m}{m!}$$

where $\mu(1,-1,\delta) = 1/\Gamma(\delta+1)$ (see [**4**, Section 18.3, (5)]) and $x^t = \exp(-t\ln(1/x))$ in (A.21) and integrating term-by-term, we obtain the asymptotic expansion of the function ω in descending powers of $\ln(1/x)$ times powers of $\ln\ln(1/x)$:

$$\begin{aligned}\omega(x,\beta,\delta;n) \approx \frac{1}{\Gamma(\beta+1)} & x^{\delta}\left(\ln\frac{1}{x}\right)^{-\beta-1} \\ & + \sum_{m=0}^{\infty}\mu(1,-m-1,\delta)\frac{(-1)^m}{m!}\left(\ln\frac{1}{x}\right)^{-m} \\ & \cdot \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\Gamma^{(j)}(\beta+m+1)\left(\ln\ln\frac{1}{x}\right)^{n-j}.\end{aligned}$$

In particular, for $n = 1$ we have

$$\begin{aligned}\omega(x,\beta,\delta) \approx \frac{1}{\Gamma(\beta+1)} & x^{\delta}\left(\ln\frac{1}{x}\right)^{-\beta-1} \\ & + \sum_{m=0}^{\infty}\mu(1,-m-1,\delta)\frac{(-1)^{m+1}}{m!}\left(\ln\frac{1}{x}\right)^{-m} \\ & \cdot\left[\Gamma(\beta+m+1)\ln\ln\frac{1}{x} - \Gamma'(\beta+m+1)\right]\end{aligned}$$

which implies the *finite asymptotic expansion* for $\beta > -1$, $\delta > -1$:

$$\begin{aligned}\omega(x,\beta,\delta) = & -\frac{1}{\Gamma(\delta+1)}x^{\delta}\left(\ln\frac{1}{x}\right)^{-\beta-1}\left[\ln\ln\frac{1}{x} - \Psi(\beta+1)\right] \\ & + O\left(x^{\delta}\left(\ln\frac{1}{x}\right)^{-\beta-2}\ln\ln\frac{1}{x}\right) \quad \text{as } x \to 0\end{aligned} \tag{A.22}$$

with the Gauss psi-function $\Psi = \Gamma'/\Gamma$.

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