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STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACES WITH APPLICATIONS IN ELECTROMAGNETICS

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ABSTRACT. In this work we present some results on deterministic and stochastic integrodifferential equations in Hilbert spaces, motivated from and applied to problems arising from the mathematical modeling of electromagnetics fields in complex random media. We examine the mild, strong and classical well posedness for the Cauchy problem of the integrodifferential equation which describes Maxwell's equations complemented with the general (and therefore nonlocal in time) linear constitutive relations describing such media, with either additive or multiplicative noise.

1. Introduction. The propagation of electromagnetic waves in bianisotropic (general linear) media is the subject of many studies, and numerous references are available in the literature. Bianisotropic media find a wide range of applications from medicine to thin film technology. The mathematical modeling of such media is done through the modification of the constitutive relations for the well known Maxwell's equations in a region $\Omega \subset \mathbf{R}^3$, $t > 0$:

$$\frac{\partial D}{\partial t} - \operatorname{curl} H = -J_e, \qquad \frac{\partial B}{\partial t} + \operatorname{curl} E = -J_m \tag{1}$$

where E is the electric field, H is the magnetic field, D is the electric displacement, B is the magnetic induction and J_e, J_m are the densities of the electric and magnetic current, respectively. The complete constitutive relations for bianisotropic media are nonlocal in time and have the form:

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$$D = \varepsilon E + c_e \star E + c_m \star H, \qquad B = \mu H + \sigma_e \star E + \sigma_m \star H,$$

where by $\star$ we denote the convolution $\alpha \star U = \int_0^t \alpha(t-s,x)U(s,x)\,ds$.

Maxwell's system (1) under these constitutive relations is called the full nonlocal problem for bianisotropic media. A time-domain analysis for chiral media under the full constitutive relations can be found in [**44**]. Related work can be found in [**4, 7, 19, 28**], while a time domain analysis using approximating constitutive laws can be found in [**1, 2, 11, 13, 36, 43**]. In this paper the full nonlocal problem is studied using semigroup theory for an integrodifferential equation of Volterra type. Mild, strong and classical well posedness for the Cauchy problem of this integrodifferential equation, under sufficient conditions with respect to space regularity or time regularity assumptions, is treated.

In a number of applications it is of interest to study phenomena where the densities of the electric and magnetic currents J_e and J_m are assumed to be stochastic. These can be modeled as random fields, i.e., as random variables indexed by spatial and time coordinates. We will consider Gaussian random fields[1], which may be modeled as an infinite dimensional Wiener process. Therefore, the evolution of the electromagnetic fields in the medium will be given by a stochastic integrodifferential equation of Volterra type. One of the goals of the present work is to prove various types of well posedeness of these stochastic problems.

Deterministic problems for Volterra type integrodifferential equations in an abstract setting have been studied extensively by different methods since the early 1970s (see, e.g., [**42**] and references therein). One of the approaches employed, e.g., [**17, 25, 45**], uses semigroup methods, where the integral term is treated as a perturbation to the main term (involving an unbounded operator, which is the generator of a C_0-semigroup), whereas a closely related approach employs semigroup theory in the setting of product spaces, e.g., [**9, 10, 30**]. Yet another approach is based on the concept of resolvent families, e.g., [**16, 22, 23**]. This approach leads to more compact expressions for the solution; however, in many applications, a representation involving semigroups is preferable since for certain operators (e.g., the Maxwell operator) the corresponding semigroup is better studied and less abstract than the related resolvent family. Further, the semigroup approach is better suited for numerical implementation as well as for the determination of a local approximating scheme to these equation (see Remark 4 in subsection 2.3).

We have chosen to present in some length in subsection 2.2 certain results on the well posedeness–under different solution notions (mild, strong, classical)–of abstract Cauchy problems for Volterra type integrodifferential equations, although some of them may be deduced from existing results in different formulations. Apart from the sake of completeness, and the uniformity in presentation with the corresponding stochastic problem (subsection 3.2), the most important reason for doing so is the following: since there is very little work on the rigorous mathematical modeling of dispersive electromagnetic materials using Volterra type integrodifferential equations, we need to present the relevant abstract results in an escalating manner regarding the solution type (mild-strong-classical) and the required regularity of the data. Such a presentation allows easy access to the various conditions for well-poseness, in a form which makes it easy to check their validity for specific models. Another reason for choosing to present some of the results in abstract form and not only for the specific electromagnetic models is because we believe that some of these results may be extendable to other more general electromagnetic models than the one studied here as well as to other applications, e.g., viscoelasticity. On the other hand, some of the abstract results, as far as we know, have not appeared in the literature so far and we find it interesting to present them. For example Theorems 1, 2, 3 and 4 of subsection 2.2 which provide conditions for weak, strong and classical wellposedness by exchanging temporal with spatial regularity of the kernels is, to the best of our knowledge, new.

Contrary to what applies for the deterministic case, the problem is not equally well studied in the case of stochastic integrodifferential equations. One should mention the influential works of Bharucha-Kannan [**29**], Clément-Da Prato [**12**], Govindan [**20**] and the interesting recent works of Keck and McKibben [**33–35**], as well as the work of Karczewska [**32**] and Bonaccorsi-Fantozzi [**5, 6**]. These works are on the abstract problem; to the best of our knowledge stochastic integrodifferential equations have not as yet been used for the modeling on random electromagnetic media. Again we choose to present the theory in abstract form. One of the reasons for doing so is because the approach we employ to the abstract problems (namely, working directly in terms of predictable rather than of adapted solutions, as well as providing detailed information on the coefficients of the problem for weak and strong well posedness) is, as far as we know, novel.

2. The deterministic model.

2.1. Motivation. In this section, we follow [**44**] for the formulation of the deterministic full nonlocal problem in order to obtain a Cauchy problem for an integrodifferential equation of Volterra type which will be studied for weak, strong and classical well posedness.

We assume that Maxwell's equations (1) hold in Ω, for $t > 0$, where Ω is a bounded and a simply connected domain of $\mathbf{R}^3$ with smooth boundary $\partial\Omega$. Maxwell's equations (1), supplemented with the initial data $E(0,x) = E_0$, $H(0,x) = H_0$, $x \in \Omega$ and the boundary condition of a perfect conductor

$$E \times n = 0, \quad \text{in } \partial\Omega,$$

where n is the unit outward normal vector to $\partial\Omega$, under the complete constitutive relations

$$D = \varepsilon E + \int_0^t c_e(t-s,x)E(s,x)\,ds + \int_0^t c_m(t-s,x)H(s,x)\,ds,$$
$$B = \mu H + \int_0^t \sigma_e(t-s,x)E(s,x)\,ds + \int_0^t \sigma_m(t-s,x)H(s,x)\,ds,$$

lead to the following initial-boundary value problem for E, H:

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\varepsilon E + c_e \star E + c_m \star H) - \operatorname{curl} H &= -J_e && \text{in } \Omega, && t > 0,\\ \frac{\partial}{\partial t}(\mu H + \sigma_e \star E + \sigma_m \star H) + \operatorname{curl} E &= -J_m && \text{in } \Omega, && t > 0,\\ E \times n &= 0 && \text{in } \partial\Omega, && t > 0,\\ E(\cdot,0) = E_0,\ H(\cdot,0) &= H_0 && \text{in } \Omega. && \end{aligned}\right\} \tag{2}$$

We use the space $\mathbf{H} = L^2(\Omega)^3 \times L^2(\Omega)^3$, which is a Hilbert space when equipped with the inner product

$$\left(\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \cdot \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} \right)_{\mathbf{H}} = \int_\Omega (\varepsilon \phi_1 \cdot \overline{\phi_2} + \mu \psi_1 \cdot \overline{\psi_2})\,dx,$$

and the dense subspaces[2]

$$H(\mathrm{curl};\Omega) = \{U \in L^2(\Omega)^3 : \mathrm{curl}\, U \in L^2(\Omega)^3\},$$
$$H_0(\mathrm{curl};\Omega) = \{U \in H(\mathrm{curl};\Omega) : U \times n = 0 \quad \text{in } \partial\Omega\}.$$

We also define the matrices

$$A = \begin{pmatrix} \varepsilon I_3 & \mathbf{0} \\ \mathbf{0} & \mu I_3 \end{pmatrix}, \qquad K = \begin{pmatrix} c_e & c_m \\ \sigma_e & \sigma_m \end{pmatrix},$$

where I_3 is the 3×3 unit matrix and $\mathbf{0}$ is the zero matrix. Using the six vector notation,

$$\mathscr{E} = \begin{pmatrix} E \\ H \end{pmatrix}, \qquad \mathscr{E}_0 = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix},$$
$$M = \begin{pmatrix} \mathbf{0} & \mathrm{curl} \\ -\mathrm{curl} & \mathbf{0} \end{pmatrix}, \quad F = \begin{pmatrix} -J_e \\ -J_m \end{pmatrix},$$

system (2) takes the form of a Cauchy problem for a Sobolev type equation in $\mathbf{H}$:

$$\frac{d}{dt}(A\mathscr{E} + K \star \mathscr{E}) = M\mathscr{E} + F, \quad \mathscr{E}(0) = \mathscr{E}_0. \tag{3}$$

2.2. The abstract Cauchy problem. Suppose that $\mathscr{E} \in C([0,T];\mathbf{H})$ and that the matrix $K(t)$, $t \geq 0$, is appropriately smooth on t, with $K(0) = \mathbf{0}$ (see subsection 2.3 for details). Then we can formally justify for the convolution term:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int_0^t K(t-s)\mathscr{E}(s)\, ds\right) &= \int_0^t \frac{\partial}{\partial t} K(t-s)\mathscr{E}(s)\, ds + K(0)\mathscr{E}(t) \\ &= \int_0^t \frac{\partial}{\partial t} K(t-s)\mathscr{E}(s)\, ds. \end{aligned}$$

If we consider that A is invertible, then multiplying by A^{-1}, Problem (3) takes the form of a Cauchy problem for an integrodifferential equation of Volterra type in $\mathbf{H}$:

$$\frac{d}{dt}\mathscr{E} = \mathscr{M}\mathscr{E} + \mathscr{K} \star \mathscr{E} + \mathscr{F}, \quad \mathscr{E}(0) = \mathscr{E}_0, \tag{4}$$

where $\mathscr{M} = A^{-1}M$, $\mathscr{K} = -A^{-1}(\partial/\partial t)K$, $\mathscr{F} = A^{-1}F$ and $D(\mathscr{M}) = D(M)$.

Assumption 1. *Assume that*

1. *The operator* $\mathscr{M} : D(\mathscr{M}) \to \mathbf{H}$ *is the infinitesimal generator of a* C_0*-group of unitary operators* $T(t)$, $t \in \mathbf{R}$, *in* $\mathbf{H}$, *i.e.,* $\|T(t)\|_{\mathcal{L}}(\mathbf{H}) = 1$, *for every* $t \in \mathbf{R}$.

2. *The family* $\{\mathscr{K}(t)\}_{t\geq 0}$ *is a family of bounded operators in* $\mathbf{H}$, *which satisfies*:

$$\sup_{t\in[0,T]} \|\mathscr{K}(t)\|_{\mathcal{L}(\mathbf{H})} \leq M_{\mathcal{K}}, \quad \text{for some } M_{\mathcal{K}} > 0.$$

3. $\mathscr{F} \in L^1([0,T];\mathbf{H})$.

4. $\mathscr{E}_0 \in \mathbf{H}$.

In the sequel, unless otherwise stated, we assume that Assumption 1 is satisfied. We will give now the definitions of mild, strong and classical solutions of Problem (4).

Definition 1. A function $\mathscr{E} \in C([0,T];\mathbf{H})$ is called a mild solution of Problem (4), if:

$$\mathscr{E}(t) = T(t)\mathscr{E}_0 + \int_0^t T(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)dr\,ds + \int_0^t T(t-s)\mathscr{F}(s)\,ds, \quad t \in [0,T].$$

Definition 2. A function $\mathscr{E} \in C([0,T];\mathbf{H})$ is called a weak solution of Problem (4) if:

1. The function $\langle \mathscr{E}(t), \zeta\rangle$ is absolutely continuous on $[0,T]$, for every $\zeta \in D(\mathscr{M}^*) = D(\mathscr{M})$.

2. For every $\zeta \in D(\mathscr{M})$ the following holds

$$\frac{d}{dt}\langle \mathscr{E}(t), \zeta\rangle = -\langle \mathscr{E}(t), \mathscr{M}\zeta\rangle + \Big\langle \int_0^t \mathscr{K}(t-s)\mathscr{E}(s)\,ds, \zeta\Big\rangle + \langle F(t), \zeta\rangle,$$

almost everywhere on $[0, T]$.

3. $\mathscr{E}(0) = \mathscr{E}_0$.

Definition 3. An $\mathbf{H}$-valued function $\mathscr{E}$ is called a strong solution of Problem (4) if:

1. $\mathscr{E}(t) \in D(\mathscr{M})$, almost everywhere on $[0, T]$.
2. $\int_0^T [\|\mathscr{M}\mathscr{E}(s)\|_{\mathbf{H}} + \| \int_0^s \mathscr{K}(s-r)\mathscr{E}(r)\, dr\|_{\mathbf{H}}]\, ds < \infty$.
3. $\mathscr{E}(t)$ satisfies equations (4) almost everywhere on $[0, T]$.

Definition 4. An $\mathbf{H}$-valued function $\mathscr{E}$ is called a classical solution of Problem (4) if:

1. $\mathscr{E}(t) \in D(\mathscr{M})$, $t \in [0, T]$.
2. $\mathscr{M}\mathscr{E}(t)$ and $\int_0^t \mathscr{K}(t-s)\mathscr{E}(s)\, ds$ are continuous in $[0, T]$.
3. $\mathscr{E}(t)$ satisfies equations (4) for all $t \in [0, T]$.

Remark 1. Since $\mathscr{M}$ is closed and densely defined, one can easily prove that a mild solution is a weak solution and conversely. Further, as we can conclude by [**40**, 107–109] and the Theorems 2,3 below, the Definitions 3 and 4 are equivalent with the expressions: $\mathscr{E}(t)$ is differentiable almost everywhere on $[0, T]$ with $\mathscr{E}'(t) \in L^1([0, T]; \mathbf{H})$ and satisfies equations (4) almost everywhere on $[0, T]$, and $\mathscr{E}(t)$ is continuously differentiable on $[0, T]$ and satisfies equations (4) for all $t \in [0, T]$, respectively.

We have the following result:

Theorem 1. *Under Assumption* 1*, Problem* (4) *is weakly well-posed.*

Proof. In the Banach space $C([0, T]; \mathbf{H})$, for some $b > 0$, we consider the norm $\|\mathscr{E}\|_b = \sup_{t \in [0,T]} e^{-bt} \|\mathscr{E}(t)\|_H$ which is clearly equivalent to the usual norm of $C([0, T]; \mathbf{H})$. We define the map Φ on the Banach space $\mathbf{V} = (C([0,T];\mathbf{H}), \|\cdot\|_b)$:

$$\Phi(\mathscr{E}(t)) = T(t)\mathscr{E}_0 + \int_0^t T(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)dr\,ds + \int_0^t T(t-s)\mathscr{F}(s)\,ds,$$

with $s \le t \in [0,T]$. It is not hard to establish that Φ maps $\mathbf{V}$ into $\mathbf{V}$, and for appropriate choice of b, Φ is a contraction. □

Under some extra assumptions, it can be proved that this unique weak solution is also a strong solution.

Assumption 2. *Suppose that*

1. $\mathscr{K}(t)y \in D(\mathscr{M})$ *for every* $y \in D(\mathscr{M})$*, almost everywhere on* $[0,T]$ *and there is an* $M_{\mathcal{MK}} > 0$ *such that* $\|\mathscr{M}\mathscr{K}(t)y\|_{\mathbf{H}} \le M_{\mathcal{MK}}\|y\|_{\mathbf{H}_{\mathscr{M}}}$, $t \in [0,T]$.
2. $\mathscr{F}(t) \in D(\mathscr{M})$ *almost everywhere on* $[0,T]$ *and* $\mathscr{M}\mathscr{F} \in L^1([0,T];\mathbf{H})$.
3. $\mathscr{E}_0 \in D(\mathscr{M})$.

In view of the above assumptions we have the following

Theorem 2. *Under Assumption* 2*, Problem* (4) *is strongly well-posed.*

Proof. We consider the Banach space $\mathbf{L}^1 = L^1([0,T];\mathbf{H}_{\mathscr{M}})$, with $\mathbf{H}_{\mathscr{M}} = (D(\mathscr{M}), \|\cdot\|_{\mathscr{M}})$ and its usual norm $\|\mathscr{E}\|_{\mathbf{L}^1} = \int_0^T \|\mathscr{E}(t)\|_{H_{\mathscr{M}}}\,dt$, and we denote by W the space $L^1([0,T];\mathbf{H}_{\mathscr{M}})$ equipped with the norm $\|\mathscr{E}\|_b = \int_0^T e^{-bt}\|\mathscr{E}(t)\|_{H_{\mathscr{M}}}\,dt$, which is equivalent to the usual norm of $\mathbf{L}^1$. For $s \le t \in [0,T]$, we define now the map Φ on $\mathbf{L}^1$.

Based on the equivalence of the norms we can show that Φ maps $\mathbf{L}^1$ into $\mathbf{L}^1$. It is not hard to establish now that Φ is a contraction on W:

$$\|\Phi(\mathscr{E}_1)(t) - \Phi(\mathscr{E}_2)(t)\|_b \le \frac{1}{b^2}M_1\|\mathscr{E}_1(t) - \mathscr{E}_2(t)\|_b,$$

where $M_1 = M_{\mathcal{K}} + M_{\mathcal{MK}}$, for $b > 0$ sufficiently large. It is clear that the resulting fixed point satisfies Properties 1 and 2 of Definition 3. Since $\mathscr{E}(t)$ is absolutely continuous on $[0,T]$ and $\mathscr{E}(0) = \mathscr{E}_0$, Property 3 of Definition 3 is also confirmed. So, Problem (4), under the extra Assumptions 2.1–2.3, has a unique strong solution which is b-exponentially bounded. By the form of the solution we see that Problem (4) is strongly well posed. ◻

Furthermore, we can have:

Assumption 3. *Assume that*

1. *Assumption* 2 *holds and in particular* 2.1–2.2 *hold for every* $t \in [0,T]$.

2. *The family of operators,* $\{\mathscr{K}(t)\}_{t\geq 0}$, *is continuous on* $[0,T]$ *in* $\mathcal{L}(\mathbf{H})$.

3. $\mathscr{F}$ *is continuous on* $[0,T]$.

Theorem 3. *Under Assumption* 3, *Problem* (4) *is classically well-posed.*

Proof. We now consider the Banach space $C([0,T];\mathbf{H}_{\mathscr{M}})$ equipped with the norm $\|\mathscr{E}\|_b = \sup_{t\in[0,T]} e^{-bt}\|\mathscr{E}(t)\|_{H_{\mathscr{M}}}$, for some $b>0$, which is equivalent to the usual norm of $C([0,T];\mathbf{H}_{\mathscr{M}})$, and we define the map Φ, as before, on the Banach space $U = (C([0,T];\mathbf{H}_{\mathscr{M}}), \|\cdot\|_b)$. It is not hard to show that the function $G(t) = \int_0^t \mathscr{K}(t-s)\mathscr{E}(s)\,ds \in C([0,T];\mathbf{H})$. Furthermore, since $G(t) \in D(\mathscr{M})$ for every $t \in [0,T]$ and $\mathscr{M}G \in L^1([0,T];\mathbf{H})$ the functions $u(t) = \int_0^t T(t-s)G(s)\,ds$ and $\mathscr{M}u(t)$ are continuous on $[0,T]$, so $u(t) \in C([0,T];\mathbf{H}_{\mathscr{M}})$.

By a similar argument we find that the functions $v(t)=\int_0^t T(t-s) \times \mathscr{F}(s)\,ds$ and $\mathscr{M}v(t)$ are continuous on $[0,T]$, hence $v(t) \in C([0,T];\mathbf{H}_{\mathscr{M}})$. So, Φ maps U into U. Furthermore, we may show that for $b>0$ large enough, Φ is a contraction on U, and thus has a unique fixed point in U which satisfies the assertions of Definition 4. Following the proof of [**40**, Theorem 2.4, page 107], it can be shown that this solution is continuously differentiable. So, Problem (4) is classically well posed and the unique classical solution is b-exponentially bounded. ◻

We saw that in order to obtain a classical solution for Problem (4), we must have $G, F \in C([0,T];\mathbf{H})\cap L^1([0,T];\mathbf{H}_{\mathscr{M}})$. An alternative assumption concerning G and $\mathscr{F}$ can be the following: $G, F \in W^{1,1}([0,T];\mathbf{H})$, see e.g., [**18, 23**]. More precisely, replacing space regularity by time regularity assumptions, we have the following:

Assumption 4. *Suppose that*

1. *For the family of bounded operators* $\{\mathscr{K}(t)\}_{t\geq 0}$, *Assumption* 1.2 *holds.*

2. *For any* $y \in \mathbf{H}_{\mathscr{M}}$, *the map:* $t \to \mathscr{K}(t)y \in W^{1,1}([0,T];\mathbf{H})$ *and*

$$\left\| \frac{\partial}{\partial t}\mathscr{K}(t)y\|_{\mathbf{H}} \leq b(t)\|y \right\|_{\mathbf{H}_{\mathscr{M}}}, \quad b \in L^1([0,T];\mathbf{R}).$$

Theorem 4. *Under Assumption* 4, *Problem* (4) *has a unique classical solution if* $\mathscr{E}_0 \in D(\mathscr{M})$ *and* $\mathscr{F} \in C([0,T];\mathbf{H}) \cap L^1([0,T];\mathbf{H}_{\mathscr{M}})$ *or* $\mathscr{F} \in W^{1,1}([0,T];\mathbf{H})$.

Proof. We will prove that the map Φ of Theorem 3, under the assumptions of Theorem 4, is a contraction on the space $U = (C([0,T];\mathbf{H}_{\mathscr{M}}), \|\cdot\|_b)$.

For the continuous function $G(t) = \int_0^t \mathscr{K}(t-s)\mathscr{E}(s)\,ds$, since $\mathscr{E} \in U$, we observe that

$$G'(t) = \mathscr{K}(0)\mathscr{E}(t) + \int_0^t \frac{\partial}{\partial t}\mathscr{K}(t-s)\mathscr{E}(s)\,ds \in L^1([0,T];\mathbf{H}).$$

So the function $u(t) = \int_0^t T(t-s)G(s)\,ds$ is differentiable and its derivative

$$u'(t) = T(t)G(0) + \int_0^t T(t-s)G'(s)\,ds = \int_0^t T(t-s)G'(s)\,ds,$$

is continuous on $[0,T]$. So, we obtain that $u(t) \in D(\mathscr{M})$ and $\mathscr{M}u(t)$ is continuous on $[0,T]$, with $\mathscr{M}u(t) = u'(t) - G(t)$. Moreover, since Assumption 1.2 holds, for any $y \in \mathbf{H}_{\mathscr{M}}$ we have:

$$\|\mathscr{K}(t)y\|_{\mathbf{H}} \leq M_{\mathcal{K}}\|y\|_{\mathbf{H}} \leq M_{\mathcal{K}}\|y\|_{\mathbf{H}_{\mathscr{M}}},$$

so

$$\|\mathscr{K}(t)\mathscr{E}(t)\|_{\mathbf{H}} \le M_{\mathcal{K}}\|\mathscr{E}(t)\|_{\mathbf{H}_{\mathscr{M}}}, \quad t \in [0,T]. \tag{5}$$

Hence, by (5) and our assumptions, we take:

$$\begin{aligned}\|\mathscr{M}u(t)\|_{\mathbf{H}} &\le \|u'(t)\|_{\mathbf{H}} + \|G(t)\|_{\mathbf{H}} \\ &\le \left\| \int_0^t T(t-s)G'(s)\,ds \right\|_{\mathbf{H}} + \|G(t)\|_{\mathbf{H}} \\ &\le T\Big(\|\mathscr{K}(0)\|_{\mathcal{L}(\mathbf{H})} + \int_0^t b(s)\,ds + M_{\mathcal{K}}\Big) \sup_{t\in[0,T]} \|\mathscr{E}(t)\|_{\mathbf{H}_{\mathscr{M}}}.\end{aligned}$$

Finally we obtain that, for fixed T, there is a constant N so that

$$\|u(t)\|_{\mathbf{H}_{\mathscr{M}}} \le N \sup_{t\in[0,T]} \|\mathscr{E}, (t)\|_{\mathbf{H}_{\mathscr{M}}} \tag{6}$$

with $N > \overline{T}(\|\mathscr{K}(0)\|_{\mathcal{L}(\mathbf{H})} + \int_0^T b(s)\,ds + M_{\mathcal{K}})$, $\overline{T} = \max\{T, T^2\}$.

By a similar argument as before, since $\mathscr{F} \in W^{1,1}([0,T];\mathbf{H})$, we see that for the continuous function $v(t) = \int_0^t T(t-s)\mathscr{F}(s)\,ds$ hold: $v(t) \in D(\mathscr{M})$ and $\mathscr{M}v(t)$ is continuous on $[0,T]$. So Φ maps U into U.

Now, using (6), we can check that Φ is a contraction on the space U. Indeed, we have

$$\begin{aligned} e^{-bt}\left\| \int_0^t T(t-s) \int_0^s \mathscr{K}(s-r)(\mathscr{E}_1(r) - \mathscr{E}_2(r))\,dr\,ds \right\|_{H_{\mathscr{M}}} \\ \le e^{-bt} N \sup_{t\in[0,T]} \|(\mathscr{E}_1(r) - \mathscr{E}_2(r))\|_{\mathbf{H}_{\mathscr{M}}}, \end{aligned}$$

so it is clear that, for $b > 0$ sufficiently large, the map Φ is a contraction on U. It is easy to check that the unique fixed point $\mathscr{E}(t)$, $t \in [0,T]$, satisfies the assertions of Definition 4 and consequently equations (4). So, Problem (4), under the assumptions of Theorem 4, is classically well posed. □

More compact expressions of the solution of Volterra type problems of forms more general than Problem (4), can be found in [**16, 22, 23**], in terms of the theory of resolvent operators. According to this theory, the unique classical solution of Problem (4) is given by

$$\mathscr{E}(t) = R(t)\mathscr{E}_0 + \int_0^t R(t-s)\mathscr{F}(s)\,ds, \quad t \in [0,T],$$

where $\{R(t)\}_{t\geq 0}$ is the resolvent operator family which is admitted for Problem (4). In particular, since, as we have already seen, for any $y \in \mathbf{H}_{\mathscr{M}}$, we have

$$\|\mathscr{K}(t)y\|_{\mathbf{H}} \leq M_{\mathcal{K}}\|y\|_{\mathbf{H}} \leq M_{\mathcal{K}}\|y\|_{\mathbf{H}_{\mathscr{M}}},$$

it is clear that $\mathscr{K}(t)$ is continuous as an operator from $\mathbf{H}_{\mathscr{M}}$ to $\mathbf{H}$ for any $t \geq 0$. We see that Hypothesis (H2) of [**16**] is fulfilled, so we have the following result[3], along the lines of [**16**]:

Theorem 4′. *Under the assumptions of Theorem* 4*, Problem* (4) *admits a resolvent operator* $\{R(t)\}_{t\geq 0}$ *and has a unique solution given by the form*

$$\mathscr{E}(t) = R(t)\mathscr{E}_0 + \int_0^t R(t-s)\mathscr{F}(s)\,ds, \quad t \in [0,T].$$

The above theorems may be generalized for the case where the source terms have a nonlinear dependance on the field $\mathscr{E}$, under suitable Lipschitz conditions. For the sake of brevity, we refrain from stating and proving the related theorems for the deterministic case here. However, in the stochastic case, we assume the possible nonlinear dependence of the sources on the field and provide general results for the nonlinear case.

Remark 2. We note that the assumptions of Theorems 4 and 4′ are the same. In many applications we prefer the expression of the solution of Problem (4) using semigroup theory instead of using the theory of resolvent operators, even if in the latter case the solution has a simpler form. The resolvent operator $\{R(t)\}_{t\geq 0}$ is an abstract mathematical object; on the contrary, the unitary group $(T(t))_{t\in\mathbf{R}}$, generated by Maxwell's operator, is quite well studied in the literature. Generally, in [**24**], several results concerning the expansion of a solution, which is expressed using semigroup theory, are proved. These results may lead to a numerical scheme.

2.3. Applications in electromagnetics. The abstract results are applied to provide well posedness results for Problem (3) as follows:

Assumption 5. *Suppose that*

1. ε, ε^{-1} *and* μ, μ^{-1} *are positive and bounded scalar functions of* $x \in \Omega$.

2. *For the* $\mathbf{R}^{3\times 3}$*-valued functions* $k(\cdot, x) = c_e, c_m, \sigma_e, \sigma_m : [0,T] \to L^\infty(\Omega)^9$, $x \in \Omega$, *we assume that*: $k \in W^{1,1}([0,T]; L^\infty(\Omega)^9)$, *with* $k(0,x) = \mathbf{O}_{3\times 3}$ *and that*

$$\sup_{t\in[0,T]} \left\| \frac{\partial}{\partial t} k(t) \right\|_{L^\infty(\Omega)^3} < \infty.$$

3. $M : D(M) = H_0(\mathrm{curl}; \Omega) \times H(\mathrm{curl}; \Omega) \to L^2(\Omega)^3 \times L^2(\Omega)^3$.

4. $F \in L^1([0,T]; L^2(\Omega)^3 \times L^2(\Omega)^3)$.

Since by Assumption 5.1 the matrix A is invertible, multiplying by A^{-1}, Problem (3) takes the form of a Cauchy problem for an integrodifferential equation of Volterra type in $L^2(\Omega)^3 \times L^2(\Omega)^3$:

$$\frac{d}{dt}\mathscr{E} = \mathscr{M}\mathscr{E} + \mathscr{K} \star \mathscr{E} + \mathscr{F}, \qquad \mathscr{E}(0) = \mathscr{E}_0, \tag{7}$$

where $\mathscr{M} = A^{-1}M$, $\mathscr{K} = -A^{-1}(\partial/\partial t)K$, $\mathscr{F} = A^{-1}F$ and $D(\mathscr{M}) = H_0(\mathrm{curl}; \Omega) \times H(\mathrm{curl}; \Omega)$.

It is well known (see, e.g., [**44**]) that the operator $i\mathscr{M}$ is self-adjoint. This means that the densely defined operator $\mathscr{M}$ is skew-adjoint, therefore by Stone's theorem (see, e.g., [**18**]) it generates a unitary group $(T(t))_{t\in\mathbf{R}}$ on $L^2(\Omega)^3 \times L^2(\Omega)^3$.

Theorem 5. *Under Assumption* 5, *there is a unique continuous* (*with respect to* t) *function* $\mathscr{E}(t) \in L^2(\Omega)^3 \times L^2(\Omega)^3$, *for* $t \in [0,T]$, *which* *satisfies equation* (3) *in the weak sense, and Problem* (3) *is weakly well posed in* $L^2(\Omega)^3 \times L^2(\Omega)^3$.

Assumption 6. *Assume that Assumption* 5 *holds and additionally that*

1. *for every* $y \in H_0(\text{curl};\Omega) \times H(\text{curl};\Omega)$, $\mathscr{K}(t)y \in H_0(\text{curl};\Omega) \times H(\text{curl};\Omega)$ *almost everywhere on* $[0,T]$, *and there is an* $M_{\mathcal{MK}} > 0$ *such that* $\|\mathscr{M}\mathscr{K}(t)y\|_{L^2(\Omega)^3\times L^2(\Omega)^3} \le M_{\mathcal{MK}}\|y\|_{H_0(\text{curl};\Omega)\times H(\text{curl};\Omega)}$.

2. $\mathscr{F}(t) \in H_0(\text{curl};\Omega) \times H(\text{curl};\Omega)$ *almost everywhere on* $[0,T]$ *and* $\mathscr{M}\mathscr{F} \in L^1([0,T];L^2(\Omega)^3 \times L^2(\Omega)^3)$.

3. $\mathscr{E}_0 \in H_0(\text{curl};\Omega) \times H(\text{curl};\Omega)$.

Theorem 6. *Under Assumption* 6 *there is a unique function* $\mathscr{E}(t) \in H_0(\text{curl};\Omega) \times H(\text{curl};\Omega)$ *which is differentiable almost everywhere on* $[0,T]$ *with* $E'(t) \in L^1([0,T];L^2(\Omega)^3 \times L^2(\Omega)^3)$ *and satisfies equations* (3) *almost everywhere on* $[0,T]$. *Problem* (3) *is strongly well-posed in* $L^2(\Omega)^3 \times L^2(\Omega)^3$.

Assumption 7. *Assume that*

1. *Assumption* 6 *holds, and in particular that* 6.1–6.2 *hold, for every* $t \in [0,T]$.

2. *The family of operators* $\mathscr{K}(t) \in \mathcal{L}(L^2(\Omega)^3 \times L^2(\Omega)^3)$, *is continuous on* $[0,T]$.

3. $\mathscr{F}$ *is continuous on* $[0,T]$.

Theorem 7. *Under Assumption* 7 *there is a unique function* $\mathscr{E}(t) \in H_0(\text{curl};\Omega) \times H(\text{curl};\Omega)$ *which is continuously differentiable on* $[0,T]$ *and satisfies equations* (3) *for all* $t \in [0,T]$. *Problem* (3) *is classically well-posed in* $L^2(\Omega)^3 \times L^2(\Omega)^3$.

We saw in Theorem 3 in subsection 2.2 that, in order to obtain a classical solution for Problem (4), we must have $G, \mathscr{F} \in C([0,T];\mathbf{H}) \cap L^1([0,T];\mathbf{H}_{\mathscr{M}})$. In Theorem 4 we considered an alternative assumption concerning G and $\mathscr{F}$, i.e., $G, \mathscr{F} \in W^{1,1}([0,T];\mathbf{H})$. It is clear that if $K \in W^{2,1}([0,T];L^\infty(\Omega)^3)$, then $G \in W^{1,1}([0,T];L^2(\Omega)^3 \times L^2(\Omega)^3)$. More precisely, replacing space regularity by time regularity assumptions, we have the following theorem:

Theorem 7′. *Suppose that for the family of bounded operators* $\{\mathscr{K}(t)\}_{t\geq 0}$*, Assumption* 5.2 *holds. In addition we assume that*:

For any $y \in H_0(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega)$*, the map*: $t \to \mathscr{K}(t)y \in W^{1,1}([0,T];L^2(\Omega)^3 \times L^2(\Omega)^3)$ *and*

$$\left\| \frac{\partial}{\partial t}\mathscr{K}(t)y \right\|_{L^2(\Omega)^3\times L^2(\Omega)^3} \leq b(t)\|y\|_{H_0(\mathrm{curl};\Omega)\times H(\mathrm{curl};\Omega)}, \quad b \in L^1([0,T];\mathbf{R}).$$

Then Problem (3) *has a unique classical solution if* $\mathscr{E}_0 \in H_0(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega)$ *and*

$$\mathscr{F} \in C([0,T];\; L^2(\Omega)^3 \times L^2(\Omega)^3) \cap L^1([0,T];\; H_0(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega))$$

or

$$\mathscr{F} \in W^{1,1}([0,T];L^2(\Omega)^3 \times L^2(\Omega)^3).$$

Remark 3. In Assumption 5.2 we assumed that $k(0,x) = \mathbf{O}_{3\times 3}$; hence, $K(0) = \mathbf{O}_{6\times 6}$. This is not really necessary. Considering that $K(0) \neq \mathbf{O}_{6\times 6}$, we can replace the operator $\mathscr{M}$ in the equation (4) by the operator $\mathscr{M} - A^{-1}K(0)$. Since $-A^{-1}K(0)$ is a multiplicative (hence bounded) operator one can check by [**40**, page 76] that the operator $\mathscr{M} - A^{-1}K(0) : D(\mathscr{M}) \to \mathbf{H}$ is the infinitesimal generator of a C_0-semigroup of operators $S(t)$, $t \geq 0$, in $\mathbf{H}$, satisfying $\|S(t)\|_{\mathcal{L}(\mathbf{H})} \leq e^{T\|A^{-1}K(0)\|_{\mathcal{L}(\mathbf{H})}}$, for $t \in [0,T]$.

Remark 4. A possible use of integral representation for the solution of the electromagnetic problem is the following: Though mathematical treatment of the integrodifferential equation which modifies Maxwell's equations under the complete non local constitutive relations for chiral media is feasible, in a number of important applications this may be cumbersome to handle. Thus, local approximations to the full constitutive relations have been proposed, that are capable of keeping the general features of chiral media, without the mathematical complications introduced by the non locality of the integral terms. In practice, a very common approximation scheme to the full constitutive relations for chiral media, is the Drude-Born-Fedorov (DBF) approximation (see e.g., [**43**]) which leads to the constitutive relations:

$$D = \varepsilon(E + \beta \operatorname{curl} E), \qquad B = \mu(H + \beta \operatorname{curl} H).$$

A local approximation to the integral terms usually leads to a Cauchy problem for an equation of pseudoparabolic type, where well posedness results are established (see [**36**]). The error of the solution of this "approximating" problem can be expressed if we consider the solution of the "full" problem in the implicit form, therefore in terms of the unitary group generated by Maxwell's operator.

3. The stochastic models.

3.1. Motivation. Models based on stochastic integrodifferential equations in the form of (4) allow us to describe phenomena (clearly not covered by deterministic models) arising from various forms of uncertainty in space and time. This uncertainty may be related to stochastic densities of electric and magnetic currents J_e and J_m, respectively, which may depend nonlinearly on the electromagnetic field. If we assume that the evolution of electromagnetic fields in a bianisotropic medium takes place in an environment which is disturbed by some electromagnetic noise, an extra term containing the stochastic effects which may be modeled by functionals of a Wiener process must then be added in equation (4). This noise may either be of the additive or the multiplicative type.

While the literature on deterministic integrodifferential equations is extended (see e.g., the references in Section 2 and references therein), there is still relatively little work done on stochastic integrodifferential equations of type (4) (see e.g., [**20, 29, 33, 34, 35**]). These studies concern only adapted processes. We follow the approach of [**14, 21, 27**] to stochastic differential equations in Hilbert space concerning predictable processes, which we modify accordingly for the case of stochastic integrodifferential equations.

3.2. The abstract Cauchy problem. Let U be a real separable and infinite dimensional Hilbert space and consider the real and separable Hilbert space $\mathbf{H} = L^2(V)^3 \times L^2(V)^3$, where V is a bounded and simply connected domain of $\mathbf{R}^3$ with smooth boundary ∂V, the probability space $(\Omega, \mathscr{F}, P)$ with a normal filtration $\mathscr{F}_t$, $t \geq 0$, and the predictable $\boldsymbol{\sigma}$-field P_T in the space $\Omega_T = [0,T] \times \Omega$. Consider also the measurable spaces $(U, \mathscr{B}(U))$, $(\mathbf{H}, \mathscr{B}(\mathbf{H}))$, $(\Omega_T \times \mathbf{H}, \mathscr{P}_T \times \mathscr{B}(\mathbf{H}))$ (as usual $\mathscr{B}$ is the Borrel $\boldsymbol{\sigma}$-field) and $(L_2^0, \mathscr{B}(L_2^0))$, where by L_2^0 we denote the space of all Hilbert-Schmidt operators in $L_2(U_0, H)$ with $U_0 = Q^{1/2}(U)$, and $Q \in \mathcal{L}(U)$ is a nonnegative, nuclear operator ($\mathrm{Tr}\,[Q] < \infty$). For the necessary notions and results concerning probability theory and stochastic analysis we refer to [**31, 37**].

A non linear stochastic model, with multiplicative noise, for Problem (4), is described as the Cauchy problem for a stochastic integrodifferential equation of the form:

$$d\mathscr{E}_t = \left[\mathscr{M}\mathscr{E}_t + \int_0^t \mathscr{K}(t-s)\mathscr{E}_s\,ds + F(t,\ \mathscr{E}_t) \right] dt + B(\mathscr{E}_t)\,dW_t, \qquad t \geq 0, \quad \mathscr{E}_0 = \xi, \tag{8}$$

where W_t, $t \geq 0$, is a U-valued Q-Wiener process in the probability space $(\Omega, \mathscr{F}, P)$.

Assumption 8. *We assume the following*:

1. *The operator* $\mathscr{M}$ *and the family of bounded operators* $\{\mathscr{K}(t)\}_{t\geq 0}$, *satisfy the Assumptions* 1.1–2 *of Section* 2.

2. *For the operator* $B : \mathbf{H} \to L_2^0$ *the following hold*:

(a) $E\left[\int_0^T \|B(\mathscr{E}_s)\|_{L_2^0}^2\,ds\right] < \infty$.

(b) *There exists a* $C_B > 0$, *such that*: $\|B(x) - B(y)\|_{L_2^0} \leq C_B\|x-y\|_{\mathbf{H}}$ *where* $x, y \in \mathbf{H}$.

3. *The function* $F : \Omega_T \times \mathbf{H} \to \mathbf{H}$ *with* $(t,\omega,x) \to F(t,\omega,x)$ *is measurable from* $(\Omega_T \times \mathbf{H}, \mathbf{P}_T \times \mathscr{B}(\mathbf{H}))$ *to* $(\mathbf{H}, \mathscr{B}(H))$ *and there exist* $C,\ C_F > 0$, *such that*:

(a) $\|F(t,\omega,x)\|_{\mathbf{H}} \leq C\|x\|_{\mathbf{H}}$ *where* $x \in \mathbf{H}$, $t \in [0,T]$, $\omega \in \Omega$,

(b) $\|F(t,\omega,x) - F(t,\omega,y)\|_{\mathbf{H}} \leq C_F\|x-y\|_{\mathbf{H}}$ *where* $x, y \in \mathbf{H}$, $t \in [0,T]$, $\omega \in \Omega$.

4. ξ *is an* $\mathbf{H}$*-valued,* $\mathscr{F}_0$*-measurable, square integrable random variable, i.e.,* $E[\|\xi\|_{\mathbf{H}}^2] < \infty$.

We will employ the space of all continuous (in mean square) and square integrable predictable processes

$$\mathcal{C}([0,T];\mathbf{H}) = \{Y \in C([0,T];L^2(\Omega,\mathbf{H}) : Y \text{ is predictable}\}.$$

This space equipped with the norm

$$\|Y\|_{\mathcal{C}} = \sup_{t\in[0,T]} \left(E\left[\|Y_t\|^2_{\mathbf{H}}\right]\right)^{1/2}$$

is a Banach space. We note that an adapted stochastic process which is continuous itself is predictable. We will give now the definitions of mild and strong solutions for the stochastic Problem (8).

Definition 5. A stochastic process $E_t \in \mathcal{C}([0,T];\mathbf{H})$ is called a mild solution of problem (8) if:

$$\begin{aligned}\mathscr{E}_t = T(t)\xi &+ \int_0^t T(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}_r\,dr\,ds \\ &+ \int_0^t T(t-s)F(s,\mathscr{E}_s)\,ds \\ &+ \int_0^t T(t-s)B(\mathscr{E}_s)\,dW_s, \quad t\in[0,T], \text{ P-as.}\end{aligned}$$

Definition 6. An $\mathbf{H}$-valued predictable process $\mathscr{E}_t$, $t \in [0,T]$, is called a weak solution of problem (8), if:

1. $\int_0^T \|\mathscr{E}_s\|_{\mathbf{H}}\,ds < \infty$, P-a.s.
2. For every $\zeta \in D(\mathscr{M}^*) = D(\mathscr{M})$ there holds

$$\begin{aligned}\langle \mathscr{E}_t,\zeta\rangle = \langle \xi\ ,\ \zeta\rangle + \int_0^t \Bigg[&- \langle \mathscr{E}_s\ ,\ \mathscr{M}\zeta\rangle + \langle F(s,\mathscr{E}_s),\zeta\rangle \\ &+ \left\langle \int_0^s \mathscr{K}(s-r)\mathscr{E}_r\,dr\ ,\ \zeta\right\rangle\Bigg]\,ds \\ &+ \int_0^t \langle B(\mathscr{E}_s)\,dW_s,\zeta\rangle, \quad t\in[0,T], \text{ P-as.}\end{aligned}$$

Definition 7. An $\mathbf{H}$-valued predictable process $\mathscr{E}_t$, $t \in [0,T]$, is called a strong solution of problem (8), if:

1. $\mathscr{E}_t \in D(\mathscr{M})$, P-a.s., almost everywhere on $[0,T]$.
2. $\int_0^T [\|\mathscr{M}\mathscr{E}_s\|_{\mathbf{H}} + \| \int_0^s \mathscr{K}(s-r)\mathscr{E}_r \, dr\|_{\mathbf{H}}] \, ds < \infty$, P-a.s.
3. $\mathscr{E}_t = \xi + \int_0^t \mathscr{M}\mathscr{E}_s \, ds + \int_0^t \int_0^s \mathscr{K}(s-r)\mathscr{E}_r \, dr \, ds + \int_0^t \mathscr{F}(s, \mathscr{E}_s) \, ds + \int_0^t B(\mathscr{E}_s) \, dW_s$, $t \in [0,T]$ P-as.

We have the following result:

Theorem 8. *Under Assumption* 8, *Problem* (8) *is weakly well-posed.*

Proof. In the Banach space $\mathcal{C}([0,T];\mathbf{H})$, for some $b > 0$, we consider the norm

$$\|\mathscr{E}\|_b = \sup_{t \in [0,T]} e^{-bt} \left(E\left[\|\mathscr{E}_t\|_H^2 \right] \right)^{1/2}$$

which is clearly equivalent to the usual norm of $\mathcal{C}([0,T];\mathbf{H})$. We define the map Φ on the Banach space $\mathcal{V} = (\mathcal{C}([0,T];\mathbf{H}), \|\cdot\|_b)$:

$$\Phi(\mathscr{E}_t) = T(t)\xi + \int_0^t T(t-s) \int_0^s \mathscr{K}(s-r)\mathscr{E}_r \, dr \, ds \\ + \int_0^t T(t-s)F(s, \mathscr{E}_s) \, ds + \int_0^t T(t-s)B(\mathscr{E}_s) \, dW_s,$$

which we rewrite as $\Phi(\mathscr{E}_t) = T(t)\xi + \Phi_1(\mathscr{E}_t) + \Phi_2(\mathscr{E}_t) + \Phi_3(\mathscr{E}_t)$, with $s \le t \in [0,T]$.

By Assumption 8.2 (a) and Proposition 6.2 in [**14**], we conclude that the stochastic convolution $\Phi_3(\mathscr{E}_t)$ has a predictable version. Moreover, by Itô's formula and Lemma 7.2 in [**14**], we compute

$$\begin{aligned}\sup_{t\in[0,T]} E\bigg[\bigg\| \int_0^t T(t-s)B(\mathscr{E}_s)\,dW_s\|_{\mathbf{H}}^2\bigg] \\ &= \sup_{t\in[0,T]} E\bigg[\int_0^t \|T(t-s)B(\mathscr{E}_s)\|_{L_2^0}^2\,ds\bigg] \\ &\le E\bigg[\int_0^T \|T(t-s)B(\mathscr{E}_s)Q^{1/2}\|_{L_2(U,\mathbf{H})}^2\,ds\bigg] \\ &= E\bigg[\int_0^T \sum_{n=1}^{\infty} \|T(t-s)B(\mathscr{E}_s)Q^{1/2}e_n\|_{\mathbf{H}}^2\,ds\bigg] \\ &\le E\bigg[\int_0^T \|B(\mathscr{E}_s)\|_{L_2^0}^2\,ds\bigg] \\ &< \infty,\end{aligned}$$

where by $\{e_n\}_{n=1}^{\infty}$ we denote an orthogonal basis in U.

By the properties of the stochastic convolution, one can easily see that Φ_3 is well defined and maps $\mathcal{V}$ into $\mathcal{V}$. Following the proof of Theorem 1 in [**27**], we estimate

$$\begin{aligned}e^{-bt}\bigg(E\bigg[\bigg\| \int_0^t T(t-s)(B(\mathscr{E}_s^1)-B(\mathscr{E}_s^2))\,dWs\bigg\|_{\mathbf{H}}^2\bigg]\bigg)^{1/2} \\ &= e^{-bt}\bigg(E\bigg[\int_0^t \|T(t-s)(B(\mathscr{E}_s^1)-B(\mathscr{E}_s^2))\|_{L_2^0}^2\,ds\bigg]\bigg)^{1/2} \\ &\le \bigg(\int_0^t e^{-2bt}E\Big[\|(B(\mathscr{E}_s^1)-B(\mathscr{E}_s^2))\|_{L_2^0}^2\Big]\,ds\bigg)^{1/2} \\ &\le \bigg(\int_0^t e^{-2b(t-s)}\,ds\bigg)^{1/2} \\ &\quad \times C_B\Big\{\sup_{s\in[0,t]} e^{-bs}\big(E\big[\|\mathscr{E}_s^1-\mathscr{E}_s^2\|_{\mathbf{H}}^2\big]\big)^{1/2}\Big\} \\ &\le \bigg(\frac{1}{2b}\bigg)^{1/2} C_B\|\mathscr{E}^1-\mathscr{E}^2\|_b.\end{aligned}$$

So, for the map Φ_3 we find that

$$\|\Phi_3(\mathscr{E}_t^1-\mathscr{E}_t^2)\|_b \le \bigg(\frac{1}{2b}\bigg)^{1/2} C_B\|\mathscr{E}^1-\mathscr{E}^2\|_b. \tag{A}$$

As the composition of measurable functions is measurable, taking into account Assumption 8.3 and that

$$\begin{aligned}\sup_{t\in[0,T]} E\left[\left\| \int_0^t T(t-s)F(s,\mathscr{E}_s)\,ds\right\|_{\mathbf{H}}^2\right] &\le E\left[\left(\int_0^T \|F(s,\mathscr{E}_s)\|_{\mathbf{H}}\,ds\right)^2\right]\\ &\le TE\left[\int_0^T \|F(s,\mathscr{E}_s)\|_{\mathbf{H}}^2\,ds\right]\\ &\le TC^2E\left[\int_0^T \|\mathscr{E}_s\|_{\mathbf{H}}^2\,ds\right]\\ &\le T^2C^2\|\mathscr{E}\|_{\mathcal{C}} < \infty,\end{aligned}$$

one can obtain that $\Phi_2(\mathscr{E}_t)$ has a predictable version and is continuous in mean square. Thus, Φ_2 is well defined and maps $\mathcal{V}$ into $\mathcal{V}$. Furthermore, we estimate

$$\begin{aligned}&e^{-bt}\left(E\left[\left\| \int_0^t T(t-s)F(s,\mathscr{E}_s^1)-F(s,\mathscr{E}_s^2)\,ds\right\|_{\mathbf{H}}^2\right]\right)^{1/2}\\ &\qquad\le T^{1/2}e^{-bt}\left(E\left[\int_0^t \|F(s,\mathscr{E}_s^1)-F(s,\mathscr{E}_s^2)\|_{\mathbf{H}}^2\,ds\right]\right)^{1/2}\\ &\qquad\le T^{1/2}\left(\int_0^t e^{-2bt}E\left[\|F(s,\mathscr{E}_s^1)-F(s,\mathscr{E}_s^2)\|_{\mathbf{H}}^2\right]\,ds\right)^{1/2}\\ &\qquad\le T^{1/2}\left(\int_0^t e^{-2b(t-s)}\,ds\right)^{1/2}\\ &\qquad\quad\times C_F\left\{\sup_{s\in[0,t]} e^{-bs}\left(E\left[\|\mathscr{E}_s^1-\mathscr{E}_s^2\|_{\mathbf{H}}^2\right]\right)^{1/2}\right\}\\ &\qquad\le T^{1/2}\left(\frac{1}{2b}\right)^{1/2} C_F\|\mathscr{E}^1-\mathscr{E}^2\|_b.\end{aligned}$$

So, for the map Φ_2 we find that

$$\text{(B)}\qquad \|\Phi_2(\mathscr{E}_t^1-\mathscr{E}_t^2)\|_b \le T^{1/2}\left(\frac{1}{2b}\right)^{1/2} C_F\|\mathscr{E}^1-\mathscr{E}^2\|_b.$$

Similarly we see that the process $\Phi_1(E_t)$ satisfies:

$$\begin{aligned}\sup_{t\in[0,T]} E\Big[\| \int_0^t T(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)\,dr\,ds\|_{\mathbf{H}}^2\Big] \\ &\leq E\Big[\Big(\int_0^T \Big\|\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)\,dr\Big\|_{\mathbf{H}}\,ds\Big)^2\Big] \\ &\leq TE\Big[\int_0^T \|\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)\,dr\Big\|_{\mathbf{H}}^2\,ds\Big] \\ &\leq T^2E\Big[\sup_{s\in[0,t]}\Big\|\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)\,dr\Big\|_{\mathbf{H}}^2\Big] \\ &\leq T^2E\Big[\Big(\int_0^T \|\mathscr{K}(t-r)\mathscr{E}(r)\|_{\mathbf{H}}\,dr\Big)^2\Big] \\ &\leq T^3E\Big[\int_0^T \|\mathscr{K}(t-r)\mathscr{E}(r)\|_{\mathbf{H}}^2\,dr\Big] \\ &\leq T^3M_{\mathcal{K}}^2E\Big[\int_0^T \|\mathscr{E}(r)\|_{\mathbf{H}}^2\,dr\Big] \\ &\leq T^4M_{\mathcal{K}}^2\|\mathscr{E}\|_{\mathcal{C}} < \infty,\end{aligned}$$

so, arguing as for the map Φ_2, one can find that Φ_1 is well defined and maps $\mathcal{V}$ into $\mathcal{V}$. We also estimate

$$\begin{aligned}&e^{-bt}\Big(E\Big[\Big\|\int_0^t T(t-s)\int_0^s \mathscr{K}(s-r)(\mathscr{E}_r^1-\mathscr{E}_r^2)\,dr\,ds\Big\|_{\mathbf{H}}^2\Big]\Big)^{1/2} \\ &\leq T^{1/2}e^{-bt}\Big(E\Big[\int_0^t\Big\|\int_0^s \mathscr{K}(s-r)(\mathscr{E}_r^1-\mathscr{E}_r^2)\,dr\Big\|_{\mathbf{H}}^2\,ds\Big]\Big)^{1/2} \\ &\leq Te^{-bt}\Big(E\Big[\sup_{s\in[0,t]}\|\int_0^s \mathscr{K}(s-r)(\mathscr{E}_r^1-\mathscr{E}_r^2)\,dr\Big\|_{\mathbf{H}}^2\Big]\Big)^{1/2} \\ &\leq T^{3/2}e^{-bt}\Big(E\Big[\sup_{s\in[0,t]}\int_0^s \|\mathscr{K}(s-r)(\mathscr{E}_r^1-\mathscr{E}_r^2)\|_{\mathbf{H}}^2\,dr\Big]\Big)^{1/2} \\ &\leq T^{3/2}e^{-bt}\Big(E\Big[\int_0^t \|\mathscr{K}(t-r)(\mathscr{E}_r^1-\mathscr{E}_r^2)\Big\|_{\mathbf{H}}^2\,dr\Big]\Big)^{1/2} \\ &\leq T^{3/2}M_{\mathcal{K}}\Big(\int_0^t e^{-2b(t-s)}\,ds\Big)^{1/2}\{\sup_{r\in[0,t]} e^{-br}\left(E\left[\|\mathscr{E}_r^1-\mathscr{E}_r^2\|_{\mathbf{H}}^2\right]\right)^{1/2}\}\end{aligned}$$

$$\leq T^{3/2} M_{\mathcal{K}} \left(\frac{1}{2b}\right)^{1/2} \|\mathscr{E}^1 - \mathscr{E}^2\|_b.$$

So, for the map Φ_1 we find that

$$\text{(C)} \qquad \|\Phi_1(\mathscr{E}_t^1 - \mathscr{E}_t^2)\|_b \leq T^{3/2} M_{\mathcal{K}} \left(\frac{1}{2b}\right)^{1/2} \|\mathscr{E}^1 - \mathscr{E}^2\|_b.$$

Summing up the obtained estimates (A), (B) and (C), we take:

$$\|\Phi(\mathscr{E}_t^1 - \mathscr{E}_t^2)\|_b \leq \left(\frac{1}{2b}\right)^{1/2} (T^{3/2} M_{\mathcal{K}} + T^{1/2} C_F + C_B) \|\mathscr{E}^1 - \mathscr{E}^2\|_b.$$

Hence for $b > 0$ sufficiently large, the map Φ is a contraction on $\mathcal{V}$, thus has a unique fixed point in $\mathcal{V}$. Therefore, Problem (8), under the Assumptions 8.1–8.4 has a unique mild solution which is b-exponentially bounded. By a straightforward modification of Theorem 6.5 in [**14**] and Assumption 8.2 (a), this mild solution is also a weak solution. By the form of the solution we can check that Problem (4) is weakly well-posed. □

Remark 5. As far as the regularity of this weak solution is concerned, one can observe that, in our case, the hypothesis of Theorem 6.10 in [**14**] of the generation of a contraction semigroup by an operator A (i.e., $\langle Ax, x\rangle \leq 0$ for every $x \in \mathbf{H}$), is fulfilled (we have that $\langle Ax, x\rangle = 0$ for every $x \in \mathbf{H}$). Hence, we obtain that the stochastic convolution $\Phi_3(\mathscr{E}_t)$ and therefore also $\mathscr{E}_t$, $t \in [0, T]$, has a continuous modification. We note that the coefficients F, B are defined for every $x \in \mathbf{H}$, but there are cases that are defined only on a subspace of $\mathbf{H}$, (see [**26, 27**]). We also note that the above result stands even in the case of a cylindrical Wiener process ($\mathrm{Tr}\,[Q] = \infty$).

In the spirit of [**21, 27**], we can prove the existence and uniqueness of a strong solution.

Assumption 9. *Suppose that Assumptions* 8.1–8.4 *and* 2.1 *of Section* 2 *hold. Suppose also that*

1. $\xi \in D(\mathscr{M})$, $F(t,x) \in D(\mathscr{M})$ *and* $B(x)Q^{1/2}h \in D(\mathscr{M})$ *P-as for all* $t \in [0,T]$, $x \in \mathbf{H}$, $h \in U$.

2. $\|\mathscr{M}F(t,x)\|_{\mathbf{H}} \le g_1(t)\|x\|_{\mathbf{H}}$, $g_1 \in L^1([0,T];\mathbf{R})$, $x \in \mathbf{H}$.
3. $\|\mathscr{M}B(x)\|_{L_2^0} \le g_2(t)\|x\|_{\mathbf{H}}$, $g_2 \in L^2([0,T];\mathbf{R})$, $x \in \mathbf{H}$.

Theorem 9. *Under Assumption* 9, *Problem* (8) *is strongly well posed.*

Proof. In the Banach space $\mathcal{W} = L^1([0,T];L^2(\Omega;\mathbf{H}_{\mathscr{M}}))$, for some $b>0$, we consider the norm

$$\begin{aligned}\|\mathscr{E}\|_b &= \int_0^T e^{-bt}\|\mathscr{E}(t)\|_{L^2(\Omega;\mathbf{H}_{\mathscr{M}})}\,dt\\ &= \int_0^T e^{-bt}\left(E\left[\|\mathscr{E}_t\|^2_{\mathbf{H}_{\mathscr{M}}}\right]\right)^{1/2}\,dt, \quad t\in[0,T]\end{aligned}$$

which is clearly equivalent to the usual norm of $L^1([0,T];\ L^2(\Omega;\mathbf{H}_{\mathscr{M}}))$. By a combination of Theorem 2 of Section 2 and Proposition 2.3 in [**27**] (or Theorem 2.1 in [**21**]), one can show that the solution map defined by the mild solution of Theorem 8 has a unique fixed point in the space $(\mathcal{W},\|\cdot\|_b)$ which satisfies Properties 1 and 2 of Definition 7. Following [**27**, page 26], by Fubini's theorem, we can see that

$$\begin{aligned}&\int_0^t\int_0^s \mathscr{M}T(s-r)\bigg(\int_0^r \mathscr{K}(r-u)\mathscr{E}(u)\,du\bigg)\,dr\,ds\\ &\qquad = \int_0^t\int_r^t \mathscr{M}T(s-r)\bigg(\int_0^r \mathscr{K}(r-u)\mathscr{E}(u)\,du\bigg)\,ds\,dr\\ &\qquad = \int_0^t T(t-r)\int_0^r \mathscr{K}(r-u)\mathscr{E}(u)\,du\,dr\\ &\qquad\quad - \int_0^t\int_0^r \mathscr{K}(r-u)\mathscr{E}(u)\,du\,dr\\ &\qquad := I_1\end{aligned}$$

and that

$$\begin{aligned}\int_0^t \int_0^s \mathscr{M} T(s-r)\mathscr{F}(r,\mathscr{E}_r)\,dr\,ds &= \int_0^t \int_r^t \mathscr{M} T(s-r)\mathscr{F}(r,\mathscr{E}_r)\,ds\,dr \\ &= \int_0^t T(t-r)\mathscr{F}(r,\mathscr{E}_r)\,dr \\ &\quad - \int_0^t \mathscr{F}(r,\mathscr{E}_r)\,dr := I_2.\end{aligned}$$

By the stochastic Fubini's theorem, we can also have

$$\begin{aligned}\int_0^t \int_0^s \mathscr{M} T(s-r)B(\mathscr{E}_r)\,dW_r\,ds &= \int_0^t \int_r^t \mathscr{M} T(s-r)B(\mathscr{E}_r)\,ds\,dW_r \\ &= \int_0^t T(t-r)B(\mathscr{E}_r)\,dW_r \\ &\quad - \int_0^t B(\mathscr{E}_r)\,dW_r \\ &:= I_3.\end{aligned}$$

Hence, applying $\mathscr{M}$ in the fixed point equation and using the closedness of $\mathscr{M}$ and the above results we have

$$\begin{aligned}\int_0^t \mathscr{M}\mathscr{E}(s)\,ds &= \int_0^t \mathscr{M} T(s)\mathscr{E}_0\,ds \\ &\quad + \int_0^t \int_0^s \mathscr{M} T(s-r)\int_0^r \mathscr{K}(r-u)\mathscr{E}(u)\,du\,dr\,ds \\ &\quad + \int_0^t \int_0^s \mathscr{M} T(s-r)\mathscr{F}(r,\mathscr{E}_r)\,dr\,ds \\ &\quad + \int_0^t \int_0^s \mathscr{M} T(s-r)B(\mathscr{E}_r)\,dW_r\,ds \\ &= T(t)\mathscr{E}_0 - \mathscr{E}_0 + I_1 + I_2 + I_3 \\ &= \mathscr{E}(t) - \xi - \int_0^t \int_0^r \mathscr{K}(r-u)\mathscr{E}(u)\,du\,dr\end{aligned}$$

$$- \int_0^t \mathscr{F}(r, \mathscr{E}_r)\, dr$$
$$- \int_0^t B(\mathscr{E}_r)\, dW_r,$$

therefore, the unique fixed point satisfies Property 3 of Definition 7. Thus, we conclude that Problem (8) is strongly well posed. □

3.3. Applications in electromagnetics. Recently, a certain interest for the study of stochastic models in physics (hence in electromagnetic theory also) has developed. One can refer to the introduction in [**3**] for certain reasons that clarify the implication of stochastic terms in the equations of mathematical physics. For the use of stochastic models in electromagnetic theory, containing also engineering literature, one can refer to [**38, 41**] and references therein.

3.3.1. Multiplicative noise. A nonlinear stochastic model, with multiplicative noise, for system (3) is the following:

$$\begin{aligned} d[A\mathscr{E}_t + K \star \mathscr{E}_t] = [M\mathscr{E}_t + F(t,\ \mathscr{E}_t)]\, dt + B(\mathscr{E}_t)\, dW_t, \\ t \geq 0, \quad \mathscr{E}_0 = \xi, \end{aligned} \tag{9}$$

where W_t, $t \geq 0$ is an $L^2(V)^3 \times L^2(V)^3$-valued Q-Wiener process in the probability space $(\Omega, \mathscr{F}, P)$. We denote by L_2^0 the space of all Hilbert-Schmidt operators in $L_2(U_0, L^2(V)^3 \times L^2(V)^3)$ with $U_0 = Q^{1/2}(L^2(V)^3 \times L^2(V)^3)$, and $Q \in \mathcal{L}(L^2(V)^3 \times L^2(V)^3)$ is a nonnegative, nuclear operator ($\mathrm{Tr}\,[Q] < \infty$). Let us note that, more generally, instead of $U_0 = Q^{1/2}(L^2(V)^3 \times L^2(V)^3)$ we can consider $U_0 = Q^{1/2}(U)$, where U is a real separable and infinite dimensional Hilbert space.

Applying the abstract results for problem (8) we obtain the following:

Theorem 10. *Suppose that*

1. *The operator* $\mathscr{M}$ *and the family of bounded operators* $\{\mathscr{K}(t)\}_{t\geq 0}$, *satisfy the Assumptions* 5.1–5.2 *of the deterministic model.*

2. *For the operator* $B:\ L^2(V)^3 \times L^2(V)^3 \to L_2^0$ *the following hold*:

(a) $E\left[\int_0^T \|B(\mathscr{E}_s)\|_{L_2^0}^2\, ds\right] < \infty.$

(b) *There exists a* $C_B > 0$, *such that*: $\|B(x) - B(y)\|_{L_2^0} \leq C_B \|x - y\|_{L^2(V)^3 \times L^2(V)^3}$ *where* $x, y \in L^2(V)^3 \times L^2(V)^3$.

3. *The function* $F : \Omega_T \times L^2(V)^3 \times L^2(V)^3 \to L^2(V)^3 \times L^2(V)^3$ *with* $(t, \omega, x) \to F(t, \omega, x)$ *is measurable from* $(\Omega_T \times L^2(V)^3 \times L^2(V)^3, \mathbf{P}_T \times \mathscr{B}(L^2(V)^3 \times L^2(V)^3))$ *to* $(L^2(V)^3 \times L^2(V)^3, \mathscr{B}(L^2(V)^3 \times L^2(V)^3))$ *and for every* $x, y \in L^2(V)^3 \times L^2(V)^3$, $t \in [0, T]$, $\omega \in \Omega$, *there exist* $C, C_F > 0$, *such that*:

(a) $\|F(t, \omega, x)\|_{L^2(V)^3 \times L^2(V)^3} \leq C \|x\|_{L^2(V)^3 \times L^2(V)^3}$.

(b) $\|F(t, \omega, x) - F(t, \omega, y)\|_{L^2(V)^3 \times L^2(V)^3} \leq C_F \|x - y\|_{L^2(V)^3 \times L^2(V)^3}$.

4. ξ *is an* $L^2(V)^3 \times L^2(V)^3$-*valued*, $\mathscr{F}_0$-*measurable, square integrable random variable, i.e.*, $E[\|\xi\|^2_{L^2(V)^3 \times L^2(V)^3}] < \infty$.

Then there is a unique, continuous in mean square, $L^2(V)^3 \times L^2(V)^3$-*valued predictable process* $\mathscr{E}_t$, $t \in [0, T]$, *which satisfies equation* (9) *in the weak sense and Problem* (9) *is weakly well-posed.*

Remark 6. By Remark 5, since we have that $\langle \mathscr{M} x, x \rangle = 0$ for every $x \in H_0(\mathrm{curl}\,; \Omega) \times H(\mathrm{curl}\,; \Omega)$, this weak solution has a continuous modification.

We can also have the following:

Theorem 11. *Suppose that Assumptions* 1–4 *of Theorem* 10 *and* 6.1 *of subsection* 2.3 *hold. Suppose also that* $\xi, F(t, x), B(x)Q^{1/2}h \in H_0(\mathrm{curl}\,; \Omega) \times H(\mathrm{curl}\,; \Omega))$, *P-as for all* $t \in [0, T]$, $x, h \in L^2(V)^3 \times L^2(V)^3$ *and that*

1. $\|\mathscr{M} F(t, x)\|_{L^2(V)^3 \times L^2(V)^3} \leq g_1(t) \|x\|_{L^2(V)^3 \times L^2(V)^3}, g_1 \in L^1([0, T]; \mathbf{R})$, $x \in L^2(V)^3 \times L^2(V)^3$,

2. $\|\mathscr{M} B(x)\|_{L_2^0} \leq g_2(t) \|x\|_{L^2(V)^3 \times L^2(V)^3}, g_2 \in L^2([0, T]; \mathbf{R})$, $x \in L^2(V)^3 \times L^2(V)^3$.

Then Problem (8) *is strongly well posed in* $L^2(V)^3 \times L^2(V)^3$.

3.3.2. More general noises. Another nonlinear stochastic model, with a time dependent multiplicative noise, for Problem (4) is of the form:

$$d\mathscr{E}_t = \Big[\mathscr{M}\mathscr{E}_t + \int_0^t \mathscr{K}(t-s)\mathscr{E}_s\,ds + F(t,\ \mathscr{E}_t)\Big]\,dt + B(t,\mathscr{E}_t)\,dW_t, \quad t \geq 0, \quad \mathscr{E}_0 = \xi. \tag{10}$$

Assumption 10. *Let* $\mathbf{H} = L^2(V)^3 \times L^2(V)^3$. *Assume that, for the coefficients* F, B, *the following hold*:

1. $B(t,x) \in L_2^0$, $x \in \mathbf{H}$ *and the mapping* $B : \Omega_T \times \mathbf{H} \to L_2^0 : (t,\omega,x) \to B(t,\omega,x)$ *is measurable from* $(\Omega_T \times \mathbf{H}, \mathbf{P}_T \times \mathscr{B}(\mathbf{H}))$ *into* $(L_2^0, \mathscr{B}(L_2^0))$.

2. *The function* $F : \Omega_T \times \mathbf{H} \to \mathbf{H}$ *with* $(t,\omega,x) \to F(t,\omega,x)$ *is measurable from* $(\Omega_T \times \mathbf{H}, \mathbf{P}_T \times \mathscr{B}(\mathbf{H}))$ *into* $(\mathbf{H}, \mathscr{B}(\mathbf{H}))$.

3. *There is a* $C > 0$, *such that*:

(a) $\|F(t,\omega,x) - F(t,\omega,y)\|_{\mathbf{H}} + \|B(t,\omega,x) - B(t,\omega,y)\|_{L_2^0} \leq C\|x-y\|_{\mathbf{H}}$ *where* $x, y \in \mathbf{H}$, $t \in [0,T]$, $\omega \in \Omega$.

(b) $\|F(t,\omega,x)\|_{\mathbf{H}}^2 + \|B(t,\omega,x)\|_{L_2^0}^2 \leq C^2(1 + \|x\|^2)$, $x \in \mathbf{H}$, $t \in [0,T]$, $\omega \in \Omega$.

Mild, weak and strong solutions can be defined in a way similar to this in Problem (8). This case can be treated by a combination of Theorem 8 and Chapter 7 in [**14**] as far as the mild solvability is concerned and under some extra assumptions mild solvability stands even in the case of a cylindrical Wiener process ($\mathrm{Tr}\,[Q] = \infty$). Furthermore, one can find sufficient conditions for mild and strong well posedness for Problem (10) by a combination of Theorem 9 and Chapter 2 in [**21**] where the case of time varying systems is also covered.

A stochastic model, with a convolution type noise for Problem (4) is the following:

$$\mathscr{E}'_t = \mathscr{M}\mathscr{E}_t + \int_0^t \mathscr{K}(t-s)\mathscr{E}_s\,ds + \mathscr{F}(t,\mathscr{E}_t) + \int_0^t B(t-s)\mathscr{E}_s\,dW_s, \quad \mathscr{E}_0 = \xi. \tag{11}$$

More general forms of equations than (11) have been already studied in [**20, 33**–**35**] covering also the case that the coefficients F, B are not defined for every $x \in \mathbf{H}$, but only on a subspace of $\mathbf{H}$. Time varying systems can also be covered as we can see in the Remark 3.1 in [**20**].

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ENDNOTES

1. The Gaussian property is a reasonable assumption in view of arguments based on the independence of fluctuations and the central limit theorem.

2. For the properties of the function spaces introduced in electromagnetic theory, we refer to [**8, 15, 39**].

3. We use the numbering Theorem $4'$, to highlight the fact that this result, as well as Theorem 4, are related to Assumption 4. The same applies for Theorems 7 and $7'$ of subsection 2.3.

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