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NORM ESTIMATES FOR A PARTICULAR WEIGHTED INTEGRAL OPERATOR

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ABSTRACT. We provide norm estimates for a particular integral operator on Hardy and Bergman spaces of analytic functions on the unit disc.

1. Introduction. Let λ_i, $x_i \in [-1,1]$, $i = 1,2$, be such that $|x_i \pm \lambda_i| \leq 1$, $i = 1,2$. The linear segment $S_z = [x_1+\lambda_1 z, x_2+\lambda_2 z]$ is a subset of $\mathbf{D}$ for any $z \in \mathbf{D}$. We consider the function $r_z(t) = [S_z]t+(x_1+\lambda_1 z)$, $0 < t < 1$, $z \in \mathbf{D}$, where we define $[S_z] = (x_2+\lambda_2 z) - (x_1+\lambda_1 z)$. Let A, B, be linear complex functions with real coefficients. We assume that $[A(r_z(t))z + B(r_z(t))]^{-1}$ is bounded as a complex function of the variable t, for any $z \in \mathbf{D}$.

In this article we consider the integral operator

$$I(f)(z) = \frac{1}{[S_z]} \int_{S_z} \frac{f(\zeta)}{A(\zeta)z + B(\zeta)}\, d\zeta, \tag{1}$$

where f is an analytic function on the unit disc. For particular choices of the linear segment S_z and the functions A, B, we obtain certain well-known operators, like the Cesàro integral operator ([**6, 10, 11, 12**])

$$\mathcal{C}(f)(z) = \frac{1}{z} \int_0^z \frac{f(\zeta)}{1-\zeta}\, d\zeta,$$

which is of the form (1) for $A(\zeta) = 0$, $B(\zeta) = 1-\zeta$, $x_1 = x_2 = 0$, $\lambda_1 = 0$ and $\lambda_2 = 1$, or the Hilbert integral operator (as defined in [**2, 3**]),

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$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(\zeta)}{1-\zeta z}\, d\zeta,$$

for $A(\zeta) = -\zeta$, $B(\zeta) = 1$, $x_1 = 0$, $x_2 = 1$ and $\lambda_1 = \lambda_2 = 0$.

We study the operator I on Hardy space H^p, $p > 1$, of analytic function f on the unit disc for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < +\infty,$$

as well as the Bergman spaces A^p, $p > 2$, of analytic function f on the unit disc for which

$$\|f\|_{A^p}^p = \int_{\mathbf{D}} |f(z)|^p dm(z) < +\infty,$$

where $dm(z) = (1/\pi)\, dx\, dy$ is the normalized Lebesgue measure on the unit disc.

We easily verify that $r_z(0) = x_1 + \lambda_1 z$ and $r_z(1) = x_2 + \lambda_2 z$. Using a standard estimate for the growth rate of functions in Hardy spaces (see [**5**]), we find

$$\begin{aligned} |I(f)(z)| &= \left| \frac{1}{[S_z]} \int_{S_z} \frac{f(\zeta)}{A(\zeta)z + B(\zeta)}\, d\zeta \right| \\ &= \left| \int_0^1 \frac{f(r_z(t))}{A(r_z(t))z + B(r_z(t))}\, dt \right| \\ &\le \int_0^1 \frac{|f(r_z(t))|}{|A(r_z(t))z + B(r_z(t))|}\, dt \\ &\le \left\| \frac{1}{A(r_z(.))z + B(r_z(.))} \right\|_\infty \\ &\quad \times \int_0^1 \frac{1}{(1-|r_z(t)|)^{1/p}}\, dt \|f\|_{H^p}. \end{aligned}$$

The function $[A(r_z(t))z + B(r_z(t))]^{-1}$ is by assumption bounded as a function of variable t, for any $z \in \mathbf{D}$. Moreover, for any $z \in \mathbf{D}$, and $0 < t < 1$,

$$|r_z(t)| \le \min\{[S_{\pm 1}]t + (x_1 - \lambda_1), [S_{\pm 1}]t + (x_1 + \lambda_1)\};$$

thus,

$$1 - |r_z(t)| \geq \max\{1 - (x_1 - \lambda_1) - [S_{\pm 1}]t, 1 - (x_1 + \lambda_1) - [S_{\pm 1}]t\},$$

which implies that, for $p > 1$,

$$\int_0^1 \frac{1}{(1 - |r_z(t)|)^{1/p}}\, dt \leq \frac{1}{[1 - (x_1 \pm \lambda_1)]^{1/p}} \times \int_0^1 \frac{1}{\left[1 - \frac{S_{\pm 1}}{1-(x_1 \pm \lambda_1)} t\right]^{1/p}}\, dt < \infty.$$

Since $S_{\pm 1} \leq 1 - (x_1 \pm \lambda_1)$, the last integral is finite and thus the operator is well defined on Hardy spaces H^p, for $p > 1$. A similar argument using an analogous growth rate estimate for Bergman spaces (see [**13**]) proves that the operator I is well defined on Bergman spaces A^p, for $p > 2$, as well.

In order to state our results we need some definitions. Let $A(\zeta) = A_1\zeta + A_2$, $B(\zeta) = B_1\zeta + B_2$, where $\zeta \in \mathbf{D}$, and A_i, $B_i \in \mathbf{R}$, $i = 1, 2$. Moreover, for any $t \in (0, 1)$,

$$\begin{aligned}
a_1(t) &= \lambda_1\lambda_2 A_1,\\
a_2(t) &= A_2(\lambda_2 - \lambda_1)t + \lambda_1 A(x_2) + x_1\lambda_2 A_1 + \lambda_1\lambda_2 B_1,\\
a_3(t) &= [(x_2 - x_1)A_2 + (\lambda_2 - \lambda_1)B_2]t + x_1 A(x_2) + x_1\lambda_2 B_1 + \lambda_1 B(x_2),\\
a_4(t) &= (x_2 - x_1)B_2 t + x_1 B(x_2),\\
b_1(t) &= -(\lambda_2 - \lambda_1)A_1 t + \lambda_2 A_1,\\
b_2(t) &= -[(x_2 - x_1)A_1 + (\lambda_2 - \lambda_1)B_1]t + A(x_2) + \lambda_2 B_1,\\
b_3(t) &= -(x_2 - x_1)B_1 t + B(x_2).
\end{aligned}$$

We prove the following

Theorem 1. *We assume that for any $t \in (0, 1)$,*

- $|a_4(t) - a_3(t)| \leq |b_3(t) - b_2(t)|$,
- $|a_4(t) + a_3(t)| \leq |b_2(t) + b_3(t)|$,
- $|b_2(t)| < |b_3(t)|$,
- $b_1(t) = a_2(t) = 0$.

Then,

(1) *If* $a_3(t)+a_4(t) = b_2(t)+b_3(t)$, *for any* $t \in (0,1)$, *then the operator* I *is bounded on Hardy spaces* H^p, $p > 1$. *Moreover, for its norm we have*

$$\|I\|_{H^p \to H^p} \leq \int_0^1 \frac{Q(t)^{2/p-1}}{[a_3(t)b_3(t) - b_2(t)a_4(t)]^{1/p}}\, dt,$$

where

$$Q(t) = \begin{cases} \sup_{z\in \mathbf{D}} |b_2(t)z + b_3(t)| & 1 < p < 2, \\ \inf_{z\in \mathbf{D}} |b_2(t)z + b_3(t)| & p \geq 2. \end{cases}$$

(2) *The operator* I *is bounded on Bergman spaces* A^p, $p > 2$. *Moreover, for its norm we have*

$$\|I\|_{A^p \to A^p} \leq \int_0^1 \frac{Q(t)^{4/p-1}}{[a_3(t)b_3(t) - b_2(t)a_4(t)]^{2/p}}\, dt,$$

where

$$Q(t) = \begin{cases} \sup_{z\in \mathbf{D}} |b_2(t)z + b_3(t)| & 2 < p < 4, \\ \inf_{z\in \mathbf{D}} |b_2(t)z + b_3(t)| & p \geq 4. \end{cases}$$

In the next section we prove necessary preliminary lemmas. In Section 3 we find the expression of operator I in terms of weighted composition operators, from which we can estimate the Hardy and Bergman space norms and we prove Theorem 1.

The idea of representing an integral operator in terms of weighted composition operators has appeared in previous articles ([**2, 3, 6, 9, 10, 11**]) and was shown to be a fertile way of proving norm estimates on Hardy, Bergman or Dirichlet spaces. Those specific cases for Hardy and Bergman spaces will now be presented in Section 4 as corollaries of Theorem 1. The aim of the present article is to present a unified approach to the methodology that appeared in those articles and generalize the context in which this methodology is applied.

2. Preliminaries. Let a, b, c, $d \in \mathbf{R}$ and

$$\omega(z) = cz + d, \quad \phi(z) = \frac{az+b}{cz+d}, \quad z \in \mathbf{D}.$$

We show

Lemma 2. *Let ϕ defined as above be a non constant self map of the unit disc with 1 as a fixed point. The weighted composition operator*

$$T(f)(z) = \frac{1}{\omega(z)} f \circ \phi(z),$$

is bounded on H^p, $1 < p < +\infty$, and for its norm we have

$$\|T(f)\|_{H^p} \leq \frac{C^{2/p-1}}{(ad-bc)^{1/p}} \|f\|_{H^p},$$

where $C = \sup_{z\in\mathbf{D}} |\omega(z)|$, $1 < p < 2$, and $C = \inf_{z\in\mathbf{D}} |\omega(z)|$, $p \geq 2$.

Proof. We will transfer T to an operator $\widetilde{T}$ acting on Hardy spaces of the right half plane, which are isometric to Hardy spaces of the disc, and thereby estimate the norm.

The Hardy space $H^p(\Pi)$ of the right half plane $\Pi = \{z : \Re(z) > 0\}$, consists of analytic functions $f : \Pi \to \mathbf{C}$, such that

$$\|f\|^p_{H^p(\Pi)} = \sup_{0<x<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy < \infty.$$

These are Banach spaces for $1 \leq p < \infty$.

Let $\mu(z) = (1+z)/(1-z)$ be the conformal map of $\mathbf{D}$ onto Π with inverse $\mu^{-1}(z) = (z-1)/(z+1)$, and let

$$V(f)(z) = \frac{(4\pi)^{1/p}}{(1-z)^{2/p}} f(\mu(z)), \quad f \in H^p(\Pi).$$

It can be checked that this map is a linear isometry from $H^p(\Pi)$ onto H^p with inverse given by

$$V^{-1}(g)(z) = \frac{1}{\pi^{1/p}(1+z)^{2/p}} g(\mu^{-1}(z)), \quad g \in H^p.$$

Let $\widetilde{T} : H^p(\Pi) \to H^p(\Pi)$ be the operator defined by $\widetilde{T} = V^{-1}TV$, and suppose $f \in H^p(\Pi)$. A calculation shows that $\widetilde{T}$ is a weighted composition operator given by

$$\widetilde{T}(f)(z) = \frac{\left(c\mu^{-1}(z) + d\right)^{(2/p)-1}}{(a-c)^{2/p}} f(\Phi(z)),$$

where

$$\Phi(z) = \mu \circ \phi \circ \mu^{-1}(z) = \frac{(a+b+c+d)z + b + d - a - c}{(c+d-a-b)z + a + d - b - c}.$$

Since $\phi(1) = 1$, we get $a + b = c + d$; thus,

$$\Phi(z) = \frac{a+b}{a-c} z + \frac{b-c}{a-c}.$$

Notice that $a \neq c$ since in the opposite case the assumption $\phi(1) = 1$ would imply $b = d$, thus $ad - bc = 0$, which is a contradiction since ϕ is a non constant map. Moreover, since Φ is by definition a well defined self map of the half plane, we get $(a+b)/(a-c) > 0$, and $(b-c)/(a-c) > 0$. For $1 < p < 2$, we compute $2/p - 1 > 0$, and

$$|\widetilde{T}(f)(z)| \leq \frac{\sup_{z \in \mathbf{D}} |cz + d|^{2/p-1}}{(a-c)^{2/p}} |f(\Phi(z))|.$$

In a similar way, for $p \geq 2$, we compute $2/p - 1 \leq 0$, and

$$|\widetilde{T}(f)(z)| \leq \frac{\inf_{z \in \mathbf{D}} |cz + d|^{2/p-1}}{(a-c)^{2/p}} |f(\Phi(z))|.$$

We get that, for every $1 \leq p < +\infty$,

$$|\widetilde{T}(f)(z)| \leq \frac{C^{2/p-1}}{(a-c)^{2/p}} |f(\Phi(z))|,$$

where $C = \sup_{z \in \mathbf{D}} |\omega(z)|$, for $1 < p < 2$, and $C = \inf_{z \in \mathbf{D}} |\omega(z)|$, for $p \geq 2$. An integration gives

$$\begin{aligned}
\|\widetilde{T}(f)\|_{H^p(\Pi^+)} &= \sup_{0<x<+\infty} \left(\int_{-\infty}^{+\infty} |\widetilde{T}(f)(z)|^p \, dy \right)^{1/p} \\
&\leq \frac{C^{2/p-1}}{(a-c)^{2/p}} \sup_{0<x<+\infty} \left(\int_{-\infty}^{+\infty} |f(\frac{a+b}{a-c}(x+iy) + \frac{b-c}{a-c})|^p \, dy \right)^{1/p} \\
&\leq \frac{C^{2/p-1}}{(a-c)^{2/p}} \left(\frac{a-c}{a+b} \right)^{1/p} \|f\|_{H^p(\Pi^+)} \\
&= \frac{C^{2/p-1}}{[(a+b)(a-c)]^{1/p}} \|f\|_{H^p(\Pi^+)} \\
&= \frac{C^{2/p-1}}{(ad-bc)^{1/p}} \|f\|_{H^p(\Pi^+)},
\end{aligned}$$

which is the desired result. □

Lemma 3. *Let ϕ defined as above be a non constant self map of the unit disc. Under the notations above, the operator*

$$T(f)(z) = \frac{1}{\omega(z)} f \circ \phi(z),$$

is bounded on A^p, $2 < p < +\infty$, and for its norm we have

$$\|T(f)\|_{A^p} \leq \frac{C^{4/p-1}}{(ad-bc)^{2/p}} \|f\|_{A^p},$$

where $C = \sup_{z \in \mathbf{D}} |\omega(z)|$, $2 < p < 4$, and $C = \inf_{z \in \mathbf{D}} |\omega(z)|$, $p \geq 4$.

Proof. The function ϕ is a non constant self map of the unit disc thus, $ad - bc \neq 0$. We easily verify that

$$\frac{1}{\omega^2(z)} = \frac{1}{ad-bc} \phi'(z).$$

Let $f \in A^p$. We compute

$$\begin{aligned}
\|T(f)\|^p_{A^p} &= \iint_D |\omega(z)|^{-p} |f(\phi(z))|^p \, dm(z) \\
&= \iint_D |\omega(z)|^{4-p} |\omega(z)|^{-4} |f(\phi(z))|^p \, dm(z) \\
&= \frac{1}{(ad-bc)^2} \iint_D |\omega(z)|^{4-p} |f(\phi(z))|^p |\phi'(z)|^2 \, dm(z) \\
&= \frac{1}{(ad-bc)^2} \iint_{\phi(\mathbf{D})} |\omega(\phi^{-1}(z))|^{4-p} |f(z)|^p \, dm(z) \\
&\leq \frac{C^{4-p}}{(ad-bc)^2} \iint_{\phi(\mathbf{D})} |f(z)|^p \, dm(z) \\
&\leq \frac{C^{4-p}}{(ad-bc)^2} \iint_{\mathbf{D}} |f(z)|^p \, dm(z),
\end{aligned}$$

and we obtain the desired result. □

3. Proof of the theorem. We consider the linear fractional transformation of the variable t,

$$\gamma(t,z) = \frac{R_1(z)t + R_2(z)}{R_3(z)t + R_4(z)},$$

where

$$\begin{aligned} R_1(z) &= [S_z](A_2 z + B_2),\\ R_2(z) &= (x_1 + \lambda_1 z)[(A(x_2) + A_1\lambda_2 z)z + B(x_2) + B_1\lambda_2 z],\\ R_3(z) &= -[S_z](A_1 z + B_1),\\ R_4(z) &= (A(x_2) + A_1\lambda_2 z)z + B(x_2) + B_1\lambda_2 z. \end{aligned}$$

We easily verify that $\gamma(0,z) = x_1 + \lambda_1 z$, and $\gamma(1,z) = x_2 + \lambda_2 z$.

To evaluate the integral (1) we apply the transformation $\zeta \to \gamma(t,z)$, and we compute

$$I(f)(z) = \frac{1}{[S_z]} \int_0^1 \frac{f(\gamma(t,z))}{A(\gamma(t,z))z + B(\gamma(t,z))} \frac{\partial \gamma(t,z)}{\partial t}\, dt.$$

We calculate

$$\begin{aligned} &\frac{1}{A(\gamma(t,z))z + B(\gamma(t,z))}\\ &\quad = \frac{-[S_z](A_1 z + B_1)t + A(x_2 + \lambda_2 z)z + B(x_2 + \lambda_2 z)}{[A(x_2 + \lambda_2 z)z + B(x_2 + \lambda_2 z)][A(x_1 + \lambda_1 z)z + B(x_1 + \lambda_1 z)]}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \gamma(t,z)}{\partial t} &= [S_z] \times [A(x_2 + \lambda_2 z)z + B(x_2 + \lambda_2 z)]\\ &\quad \times [A(x_1 + \lambda_1 z)z + B(x_1 + \lambda_1 z)]\\ &\quad \times \{-[S_z](A_1 z + B_1)t + A(x_2 + \lambda_2 z)z + B(x_2 + \lambda_2 z)\}^{-2}. \end{aligned}$$

We compute

$$I(f)(z) = \int_0^1 \frac{f(\gamma(t,z))}{-[S_z](A_1 z + B_1)t + A(x_2 + \lambda_2 z)z + B(x_2 + \lambda_2 z)}\, dt.$$

A reformulation of $\gamma(t,z)$ gives

$$\gamma(t,z) = \frac{a_1(t)z^3 + a_2(t)z^2 + a_3(t)z + a_4(t)}{b_1(t)z^2 + b_2(t)z + b_3(t)},$$

where a_i, $i = 1, 2, 3, 4$ and b_j, $j = 1, 2, 3$ were defined in the introduction. The operator I appears in the form

$$I(f)(z) = \int_0^1 \frac{1}{b_1(t)z^2 + b_2(t)z + b_3(t)} f\left(\gamma(t, z)\right)\, dt.$$

Since by assumption $b_1(t) = a_2(t) = 0$, we easily infer $a_1(t) = 0$, and the operator is simplified to

$$I(f)(z) = \int_0^1 \frac{1}{b_2(t)z + b_3(t)} f\left(\frac{a_3(t)z + a_4(t)}{b_2(t)z + b_3(t)}\right) dt.$$

The function

$$\phi_t(z) = \frac{a_3(t)z + a_4(t)}{b_2(t)z + b_3(t)},$$

is a linear fractional transformation; thus, it transforms circles and lines to circles and lines. Since by assumption $|-b_3(t)/b_2(t)| > 1$, $t \in (0, 1)$, the function ϕ_t is analytic on the closed unit disc which implies that the image of the unit disc is a disc. Moreover, the coefficients of $\phi_t(z)$ are real; thus, $\overline{\phi_t(z)} = \phi_t(\overline{z})$ which implies that the image $\phi_t(\mathbf{D})$ is symmetrical to the real axis. Since the image of a connected set under a continuous function is a connected set, and $|\phi_t(-1)| \le 1$, $|\phi_t(1)| \le 1$ by assumption, we also get $\phi_t([-1, 1]) \subset [-1, 1]$.

The above remarks verify that ϕ_t is a well defined self map of the unit disc and, thus, the operator I is well defined in the unit disc as a weighted integral operator. An application of Lemmas 2 and 3 gives the desired result and completes the proof of Theorem 1.

4. Applications. In the introduction we remarked that well-known examples of operators of the above type that appear in previous articles is the Cesàro integral operator ([**6, 10, 11, 12**])

$$\mathcal{C}(f)(z) = \frac{1}{z} \int_0^z \frac{f(\zeta)}{1 - \zeta}\, d\zeta,$$

which is of the form (1) for $A(\zeta) = 0$, $B(\zeta) = 1 - \zeta$, $x_1 = x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 1$ and the Hilbert integral operator (as defined in [**2, 3**]),

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(\zeta)}{1 - \zeta z}\, d\zeta,$$

for $A(\zeta) = -\zeta$, $B(\zeta) = 1$, $x_1 = 0$, $x_2 = 1$, $\lambda_1 = \lambda_2 = 0$. In addition, it is easy to see that the H^2 adjoint of Cesàro operator ([**10, 11**]),

$$\mathcal{A}(f)(z) = \frac{1}{z-1}\int_1^z f(\zeta)\,d\zeta,$$

is of the form (1) for $A(\zeta) = 0$, $B(\zeta) = 1$, $x_1 = 1$, $x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 1$ as well as the operator

$$\mathcal{J}(f)(z) = \frac{1}{z-1}\int_1^z \frac{f(\zeta)}{-1-\zeta}\,d\zeta,$$

which is studied in [**9**], for $A(\zeta) = 0$, $B(\zeta) = -1-\zeta$, $x_1 = 1$, $x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 1$ and the operator

$$\mathcal{H}_0(f)(z) = \frac{1}{2}\int_{-1}^1 \frac{f(\zeta)}{1-\zeta z}\,d\zeta,$$

for $A(\zeta) = -2\zeta$, $B(\zeta) = 2$, $x_1 = -1$, $x_2 = 1$, $\lambda_1 = \lambda_2 = 0$. The operator $\mathcal{H}_0$ is the H^2 equivalent to the well-known Carleman's integral operator

$$\mathcal{L}(f)(t) = \frac{1}{2}\int_0^\infty \frac{f(s)}{s+t}\,ds,$$

acting on $L^2(0,\infty)$. The equivalence of the operators $\mathcal{H}_0$ and $\mathcal{L}$, as well as the equivalence of the above operators with the operator induced by the reduced Hilbert matrix on l^2, has been described in detail in [**7**].

Standard calculations show that in all the specific cases above, the function $[A(r_z(t))z + B(r_z(t))]^{-1}$ is a bounded complex function of variable t, for any $z \in \mathbf{D}$, thus the operators are well defined when they act on Hardy spaces H^p, $p > 1$, or Bergman spaces A^p, $p > 2$.

Henceforward, we denote by X^p either Hardy space H^p, for $p > 1$, or Bergman space A^p, for $p > 2$. In most of our results a constant a appears and corresponds to either case of X^p. We keep in mind the convention that this constant is $a = 1$, when X^p denotes H^p, and $a = 2$, when X^p denotes A^p. We can also observe that the assumption $|a_3(t)+a_4(t)| \le |b_2(t)+b_3(t)|$ is weaker than the assumption $a_3(t) + a_4(t) = b_2(t) + b_3(t)$ and thus it shall not be verified when the latter holds.

Corollary 4. *The operator $\mathcal{C}$ is bounded on Hardy spaces H^p, $p > 1$, and Bergman spaces A^p, $p > 2$, and for its norm we have*

$$\|\mathcal{C}\|_{X^p \to X^p} \leq 2^{(2a/p)-1} \frac{p}{p-a}, \quad \textit{if } a < p < 2a,$$

and

$$\|\mathcal{C}\|_{X^p \to X^p} \leq \frac{p}{a}, \quad \textit{if } p \geq 2a.$$

Proof. We compute $b_1(t) = a_2(t) = 0$, $a_3(t) = t$, $a_4(t) = 0$, $b_2(t) = t - 1$ and $b_3(t) = 1$. We verify that

- $|a_4(t) - a_3(t)| = t \leq 2 - t = |b_3(t) - b_2(t)|$,
- $|b_2(t)| = 1 - t < 1 = |b_3(t)|$,
- $b_1(t) = a_2(t) = 0$.

Moreover, $a_3(t) + a_4(t) = t = b_2(t) + b_3(t)$, and Theorem 1 is applicable in both Hardy and Bergman space cases. For $p \geq 2a$, we compute,

$$Q(t) = \inf_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = \inf_{z \in \mathbf{D}} |(t-1)z + 1| = t,$$

while for $a < p < 2a$,

$$Q(t) = \sup_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = \sup_{z \in \mathbf{D}} |(t-1)z + 1| = 2 - t.$$

We also find $a_3(t)b_3(t) - a_4(t)b_2(t) = t$. A simple integral calculation gives for $p \geq 2a$,

$$\|\mathcal{C}\|_{X^p \to X^p} \leq \int_0^1 \frac{t^{2a/p-1}}{t^{a/p}} \, dt = \frac{p}{a},$$

while for $a < p < 2a$,

$$\|\mathcal{C}\|_{X^p \to X^p} \leq \int_0^1 \frac{(2-t)^{2a/p-1}}{t^{a/p}} \, dt \leq 2^{(2a/p)-1} \frac{p}{p-a}.$$

which is the desired result. □

Corollary 5. *The operator $\mathcal{A}$ is bounded on Hardy space H^p for $p > 1$, and Bergman spaces A^p for $p > 2$. Moreover,*

$$\|\mathcal{A}\|_{X^p \to X^p} \leq \frac{p}{p-a}, \quad p > a.$$

Proof. We compute $a_3(t) = t$, $a_4(t) = 1 - t$, $b_2(t) = 0$ and $b_3(t) = 1$. We verify that

- $|a_4(t) - a_3(t)| = 2t - 1 \leq 1 = |b_3(t) - b_2(t)|$,
- $|b_2(t)| = 0 < 1 = |b_3(t)|$,
- $b_1(t) = a_2(t) = 0$.

Moreover $a_3(t) + a_4(t) = 1 = b_2(t) + b_3(t)$, thus Theorem 1 is applicable in both Hardy and Bergman space cases. We further compute $a_3(t)b_3(t) - a_4(t)b_2(t) = t$, and for every $p > a$,

$$Q(t) = \sup_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = \inf_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = 1.$$

A simple integral calculation gives for $p > a$,

$$\|\mathcal{A}\|_{X^p \to X^p} \leq \int_0^1 \frac{1}{t^{a/p}}\, dt = \frac{p}{p-a},$$

which is the desired result. □

Corollary 6. *The operator $\mathcal{J}$ is bounded on Hardy space H^p for $p > 1$, and Bergman spaces A^p for $p > 2$. Moreover,*

$$\|\mathcal{J}\|_{X^p \to X^p} \leq \frac{p}{2(p-a)}, \quad \text{if } a < p < 2a,$$

and

$$\|\mathcal{J}\|_{X^p \to X^p} \leq \frac{p}{2a}, \quad \text{if } p \geq 2a.$$

Proof. We compute $a_3(t) = -t - 1$, $a_4(t) = t - 1$, $b_2(t) = t - 1$ and $b_3(t) = -t - 1$. We also verify that

- $|a_4(t) - a_3(t)| = 2t \leq 2t = |b_3(t) - b_2(t)|$,
- $|b_2(t)| = 1 - t < 1 + t = |b_3(t)|$,
- $b_1(t) = a_2(t) = 0$.

Moreover, $a_3(t) + a_4(t) = -2 = b_2(t) + b_3(t)$, and Theorem 1 is applicable in both the Hardy and Bergman space cases. We compute $a_3(t)b_3(t) - a_4(t)b_2(t) = 4t$. Moreover, for $p \geq 2a$,

$$Q(t) = \inf_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = \inf_{z \in \mathbf{D}} |(t-1)z + 1| = 2t,$$

while for $a < p < 2a$,

$$Q(t) = \sup_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = \sup_{z \in \mathbf{D}} |(t-1)z + 1| = 2.$$

A simple integral calculation gives for $p \geq 2a$,

$$\|\mathcal{J}\|_{X^p \to X^p} \leq \int_0^1 \frac{(2t)^{2a/p-1}}{(4t)^{a/p}}\, dt = \frac{p}{2a},$$

while for $a < p < 2a$,

$$\|\mathcal{J}\|_{X^p \to X^p} \leq \int_0^1 \frac{2^{2a/p-1}}{(4t)^{a/p}}\, dt = \frac{p}{2(p-a)}.$$

which is the desired result. □

Remark. The last result comprises a natural generalization of the result obtained at [**9**] for the Hardy space case.

Corollary 7. *The operator $\mathcal{H}$ is bounded on Hardy spaces H^p for $p > 1$, and Bergman spaces A^p for $p > 2$. Moreover, for each $f \in X^p$,*

$$\|\mathcal{H}\|_{X^p \to X^p} \leq 2^{2a/p-1} \int_0^1 t^{-a/p}(1-t)^{-a/p}\, dt, \quad a < p < 2a,$$

and

$$\|\mathcal{H}\|_{X^p \to X^p} \leq \frac{\pi}{\sin(a\pi/p)}, \quad p \geq 2a.$$

Proof. We compute $a_3(t) = 0$, $a_4(t) = t$, $b_2(t) = t - 1$ and $b_3(t) = 1$. We also verify that

- $|a_4(t) - a_3(t)| = t \leq 2 - t = |b_3(t) - b_2(t)|$,
- $|b_2(t)| = 1 - t < 1 = |b_3(t)|$,
- $b_1(t) = a_2(t) = 0$.

Moreover, $a_3(t)+a_4(t) = t = b_2(t)+b_3(t)$; thus, Theorem 1 is applicable in both the Hardy and Bergman space cases. We further compute $a_3(t)b_3(t) - a_4(t)b_2(t) = t(1-t)$. We also remind the reader from the proof of Corollary 4 that for $p \geq 2a$,

$$Q(t) = \inf_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = t,$$

while for $a < p < 2a$,

$$Q(t) = \sup_{z \in \mathbf{D}} |b_2(t)z + b_3(t)| = 2 - t.$$

An integral calculation gives for $p \geq 2a$,

$$\begin{aligned} \|\mathcal{H}\|_{X^p \to X^p} &\leq \int_0^1 \frac{t^{2a/p-1}}{[t(1-t)]^{a/p}}\, dt \\ &= \int_0^1 t^{a/p-1}(1-t)^{-a/p}\, dt \\ &= \frac{\pi}{\sin(a\pi/p)}, \end{aligned}$$

where we used standard identities on beta and gamma special functions (see [**14**]), while for $a < p < 2a$,

$$\begin{aligned} \|\mathcal{H}\|_{X^p \to X^p} &\leq \int_0^1 \frac{(2-t)^{2a/p-1}}{[t(1-t)])^{a/p}}\, dt \\ &\leq 2^{2a/p-1} \int_0^1 t^{-a/p}(1-t)^{-a/p}\, dt, \end{aligned}$$

which is the desired result. □

Remark. The operator $\mathcal{H}$ is not bounded on Hardy space H^1 and Bergman space A^2. A proof of those facts appears in [**3**] and [**2**], respectively.

Corollary 8. *The operator $\mathcal{H}_0$ is bounded on Hardy space H^p for $p > 1$, and Bergman spaces A^p for $p > 2$. Moreover, for each $f \in X^p$,*

$$\|\mathcal{H}_0\|_{X^p \to X^p} \leq \int_0^{1/2} t^{a/p-1}(1-t)^{-a/p}\, dt,$$

for $p \geq 2a$, and

$$\|\mathcal{H}_0\|_{X^p \to X^p} \leq \int_0^{1/2} t^{-a/p}(1-t)^{a/p-1}\, dt,$$

for $a < p < 2a$.

Proof. We compute $a_3(t) = 2$, $a_4(t) = 4t - 2$, $b_2(t) = 4t - 2$ and $b_3(t) = 2$. We verify that

- $|a_4(t) - a_3(t)| = 4(1-t) \leq 4(1-t) = |b_3(t) - b_2(t)|$,
- $|b_2(t)| = |4t - 2| < 2 = |b_3(t)|$,
- $b_1(t) = a_2(t) = 0$.

Moreover, $a_3(t)+a_4(t) = 4t = b_2(t)+b_3(t)$, and Theorem 1 is applicable in both the Hardy and Bergman space cases.

We further compute $a_3(t)b_3(t)-a_4(t)b_2(t) = 16t(1-t)$, and for $p \geq 2a$,

$$Q(t) = \inf_{z\in\mathbf{D}} |(4t-2)z+2| = \begin{cases} 4(1-t) & 0 < t < 1/2, \\ 4t & 1/2 \leq t < 1, \end{cases}$$

while for $a < p < 2a$,

$$Q(t) = \sup_{z\in\mathbf{D}} |(4t-2)z+2| = \begin{cases} 4t & 0 < t < 1/2, \\ 4(1-t) & 1/2 \leq t < 1. \end{cases}$$

From Theorem 1 we obtain that for any $t \in (0,1)$, the operator $\mathcal{H}_0$ is bounded on Hardy spaces H^p, $1 < p < +\infty$, and Bergman spaces A^p, $p > 2$ and for its norm we have

$$\|\mathcal{H}_0\|_{X^p \to X^p} \leq \int_0^1 \frac{Q(t)^{1-2a/p}}{[16t(1-t)]^{a/p}}\, dt. \tag{2}$$

From the calculations above and estimate (2), we get for $p \geq 2a$,

$$\begin{aligned}\|\mathcal{H}_0\|_{X^p\to X^p} &\leq \frac{1}{2}\int_0^{1/2} t^{a/p-1}(1-t)^{-a/p}\,dt \\ &\quad+\frac{1}{2}\int_{1/2}^{1} t^{-a/p}(1-t)^{a/p-1}\,dt \\ &= \int_0^{1/2} t^{a/p-1}(1-t)^{-a/p}\,dt,\end{aligned}$$

while for $a < p < 2a$ we end at

$$\begin{aligned}\|\mathcal{H}_0\|_{X^p\to X^p} &\leq \frac{1}{2}\int_0^{1/2} t^{-a/p}(1-t)^{a/p-1}\,dt \\ &\quad+\frac{1}{2}\int_{1/2}^{1} t^{a/p-1}(1-t)^{-a/p}\,dt \\ &= \int_0^{1/2} t^{-a/p}(1-t)^{a/p-1}\,dt. \qquad \square\end{aligned}$$

Final remarks. In the recent article [**4**], the authors prove that the norm estimate of the Hilbert operator of Corollary 7 is sharp for $p \geq 2a$, in both the Hardy and Bergman space cases. Moreover, they show that the same estimate still holds in the Hardy space case for $1 < p < 2$, which implies that Theorem 1 does not provide a sharp estimate in this case.

On the other hand, Theorem 1 is proven to be sharp for other cases we exhibit in the last section. Actually, for $p \geq 2a$, using a semigroup theory argument, it is possible to show that the norm estimates are sharp for the Cesàro integral operator on Hardy and Bergman spaces (see [**10, 12**] respectively). Analogous arguments show that for $p > a$, the estimates are sharp for the operator $\mathcal{A}$, on Hardy and Bergman spaces as well (see [**10, 11**] respectively).

Finally, since the spectrum of Carleman's Integral operator on $L^2(0,1)$ is the interval $[0,\pi/2]$ ([**1**, page 169]) and

$$\|\mathcal{H}_0\|_{H^2\to H^2} \leq \int_0^{1/2} t^{-1/2}(1-t)^{-1/2}\,dt = \frac{\pi}{2},$$

we derive that the norm estimate of Corollary 8 for the Hardy space case and $p = 2$, is best possible, in particular

$$\|\mathcal{H}_0\|_{H^2 \to H^2} = \frac{\pi}{2}.$$

The final question that naturally arises for future research is the detection of the appropriate assumptions for Theorem 1 so that it can be a sharp norm estimate of the integral operator (1).

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