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A NEW RESULT ON THE SINGULAR VALUE ASYMPTOTICS OF INTEGRATION OPERATORS WITH WEIGHTS

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This paper is dedicated to Professor Rainer Kress on the occasion of his 65th birthday.

ABSTRACT. It is an interesting question for the analysis of linear ill-posed operator equations $Ax = y$ and it seems to be of some importance for regularization theory whether a non-compact linear operator with non-closed range applied to a compact linear operator mapping between Hilbert spaces can alter the degree of ill-posedness determined by the singular value decay rate $\sigma_n(A) \to 0$ as $n \to \infty$ of the compact operator A. For giving some more answer to that question we work in the space $L^2(0,1)$ and focus on non-compact multiplication operators M applied to the integration operator J such that $A = M \circ J$ determines the operator governing the equation. Compositions of this type occur as linearizations of different nonlinear inverse problems in natural sciences, engineering, and finance. Specifically, we are interested in the case of multiplication operators M generated by a multiplier function m having an essential zero in $[0,1]$. In particular, in a toy problem of inverse option pricing multipliers m with exponential-type zeros occur. By analyzing the strength of source conditions for obtaining convergence rates in regularization it was conjectured that the ill-posedness situation tends to the worse in the exponential case compared to the case of power-type zeros in m, for which we have shown in [**9**] that the degree of ill-posedness is uniformly one. Now we are going to extend this result to some family of exponential weight functions m and prove that the asymptotics $\sigma_n(A) \asymp n^{-1}$ also holds for

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that family. In this context, we emphasize that for integration operators with outer weights the use of the operator AA^* is more appropriate for the analysis of eigenvalue problems and the corresponding asymptotics of singular values than the former use of A^*A in [**9**].

1. Introduction. In this paper, for a specific situation, we are going to analyze the degree of ill-posedness of linear ill-posed operator equations

$$Ax = y \qquad (x \in X,\ y \in Y) \tag{1.1}$$

for injective, non-degenerating, compact linear operators $A : X \to Y$ mapping between infinite dimensional separable Hilbert spaces X and Y with norms $\|\cdot\|$. If preferably the smoothing properties of the operator A governing the equation (1.1) are under consideration, then the decay rate of the positive, non-increasing sequence $\{\sigma_n(A)\}_{n=1}^{\infty}$ of singular values of A tending to zero as $n \to \infty$ measures the strength of ill-posedness of (1.1) (see, e.g., Kress [**13**, p.235], Engl, Hanke, Neubauer [**3**, p.40] and Hofmann [**7**, p.31]). This strength can be expressed by a single number $\mu = \mu(A) \in (0, \infty)$ called the degree of ill-posedness of equation (1.1) if

$$\sigma_n(A) \asymp n^{-\mu}$$

is valid[1]. This a rather specific situation for A, but it plays some important role in the literature (see, e.g., Louis [**15**] and Mathé, Pereverzev [**16**]). Wide families of forward operators A in numerous inverse problems of form (1.1) have single-valued finite degrees μ of ill-posedness, for example the problem of finding the μ-th fractional derivative of a function y. With increasing μ the numerical difficulties occurring in the corresponding differentiation process systematically grow.

If, on the other hand, the linearization of a nonlinear inverse problem

$$F(x) = y \qquad (x \in D(F) \subseteq X,\ y \in Y) \tag{1.2}$$

[1]As usual we use the notation $a_n \asymp b_n$ for sequences of positive numbers a_n and b_n satisfying inequalities $c_1 \le a_n/b_n \le c_2$ for positive constants c_1 and c_2 and all $n \in \mathbb{N}$. If moreover $\lim\limits_{n\to\infty} a_n/b_n = 1$ we write $a_n \sim b_n$. If the quotients a_n/b_n are only limited from above by a constant, then we write $a_n = \mathcal{O}(b_n)$.

with continuous nonlinear forward operator $F : D(F) \subseteq X \to Y$ yields a linear operator equation (1.1) with the Fréchet derivative $A = F'(x_0)$ at an inner solution point $x_0 \in D(F)$ with a single-valued degree $\mu = \mu(F'(x_0))$ of ill-posedness, then μ can be interpreted here as local degree of ill-posedness for evaluating the local stability behavior of the nonlinear operator equation (1.2) at x_0. As an important class of nonlinear ill-posed problems (1.2) we should mention the class of equations with compact nonlinear operators F (see [**3**, Chapt. 10]) leading to compact linear operators $A = F'(x_0)$ in the linearization (see COLTON, KRESS [**2**, Theorem 4.19]).

It is an interesting question for the analysis of linear ill-posed operator equations whether a non-compact, bounded linear operator with non-closed range applied to a compact linear operator mapping between Hilbert spaces can alter the degree of ill-posedness. We asked this question in the recent paper [**9**] and gave some partial answer for the Hilbert space $X = Y = L^2(0,1)$ and for the composition $A = M \circ J$ of a multiplication operator M generated by a weight (multiplier) function m with essential zeros in $[0,1]$ and the integration operator

$$[J\,x](t) \;=\; \int_0^t x(s)ds \qquad (0 \le t \le 1). \tag{1.3}$$

Compositions of this type occur as linearizations of different nonlinear inverse problems in natural sciences, engineering and finance. For more details we refer to the paper [**8**] which was communicated by RAINER KRESS. Precisely, for A from

$$[A\,x](t) \;=\; m(t) \int_0^t x(s)ds \qquad \text{a.e. on } [0,1] \tag{1.4}$$

and weight functions m of power-type $m(t) = t^\alpha$ with $\alpha > -1$ we proved that the well-known degree $\mu(J) = 1$ of ill-posedness carries over to the composition in the form $\mu(M \circ J) = 1$. We will recall this result in detail as a proposition in Section 2.

Now we learned from KLANN, MAASS, RAMLAU that such a resistance of the degree of ill-posedness of a compact operator to additional influence factors can be advantageous, since they developed a new two-step regularization approach in [**14**], for which convergence rates results require a fixed single-valued degree of ill-posedness. So it seems to be of some interest to extend the results of [**9**] to further families of composite operators. We will do this in the following for a family of exponential weight functions $m(t) = \frac{1}{t^2} \exp\left(-\frac{c}{t}\right)$ with $c > 0$ in (1.4). The decay rate of $m(t) \to 0$ as $t \to 0$ for exponential weights is much faster than in the power-case. Nevertheless, we can formulate a theorem on the non-altering degree of ill-posedness for that exponential family in Section 4 based on an equivalence result proven in Section 3. In this context, we emphasize that for integration operators with outer weights the use of the operator AA^* is more appropriate for the analysis of eigenvalue problems and the corresponding asymptotics of singular values than the former use of A^*A in [**9**].

Example 1.1. Another specific reason for studying exponential multipliers m is due to the paper [**6**] of Hein, Hofmann, where as an inverse toy problem in finance the determination of a purely time-dependent volatility function $x(t)$ $(t \in [0,1])$ from maturity-dependent option prices $y(t)$ on the same interval can be written in the form (1.2) with $X = Y = L^2(0,1)$. In this example, the nonlinear forward operator $F = N \circ J$ with domain $D(F) = \{x \in L^2(0,1) : x(t) \geq \underline{c} > 0 \text{ a.e.}\}$ mapping in $L^2(0,1)$ is a composition of the integration operator J and a nonlinear Nemytskii operator N determined by a smooth generator function $k(t,u)$ with $(t,u) \in [0,1] \times [\underline{c}, \infty)$ of the form

$$[F\,x](t) \;=\; k(t, [J\,x](t)) \qquad (0 \leq t \leq 1). \tag{1.5}$$

The function $k(t,u)$ and its partial derivative $k_u(t,u)$ can be derived in an explicit manner from the structure of the well-known Black-Scholes formula generalized to time-varying volatlities. For an inner point $x_0 \in D(F)$ the Fréchet derivative of F then has the form

$$\begin{aligned} &[F'(x_0)\,h](t) = m(t)\,[J\,h](t) \qquad \text{with} \\ &\qquad m(t) = k_u(t, [J\,x_0](t)) \qquad (0 < t \leq 1) \end{aligned} \tag{1.6}$$

With the exception of the case of at-the-money options it could be shown in [**6**] that the weight function $m(t)$ in (1.6) has an essential zero at $t = 0$. This zero is of exponential type. Precisely, it satisfies the inequalities

$$\frac{\underline{C}}{\sqrt[4]{t}} \exp\left(-\frac{\underline{c}}{t}\right) \;\leq\; m(t) \;\leq\; \frac{\overline{C}}{\sqrt{t}} \exp\left(-\frac{\overline{c}}{\sqrt{t}}\right) \qquad (0 < t \leq 1) \tag{1.7}$$

for some positive constants $\underline{c}, \overline{c}, \underline{C}$ and $\overline{C}$.

2. A review of well-known results and conjectures for the integration operator with weights. We begin this section with a sufficient condition for the compactness of the operator $A = M \circ J$ defined in (1.4). In this context, we note that we are focused throughout the paper on injective operators M and A which occur if and only if $m(t) \neq 0$ a.e. in $[0,1]$.

Lemma 2.1. *The linear operator* $A : L^2(0,1) \to L^2(0,1)$ *defined by formula* (1.4) *is compact if* m *is a measurable function on* $[0,1]$ *satisfying the condition.*

$$\int_0^1 t\, m^2(t)\, dt < \infty\,. \tag{2.1}$$

Proof. In view of (2.1) the kernel

$$K(s,t) = \begin{cases} m(t) & \text{for} \quad 0 \le t \le s \le 1 \\ 0 & \text{for} \quad 0 \le s < t \le 1 \end{cases}$$

of the operator A (considered a linear Fredholm integral operator) has a finite double-norm

$$\int_0^1 \int_0^1 K^2(s,t)\, dt\, ds = \int_0^1 t\, m^2(t)\, dt < \infty,$$

i.e., K is a Hilbert-Schmidt kernel. This implies the compactness of A (see, e.g., [**20**, Chapter 11, §2]).

Remark 2.2. Condition (2.1) is fulfilled in the two cases

$$\text{(i)} \quad m \in L^2(0,1) \quad \text{and} \quad \text{(ii)} \quad m(t) = t^\alpha \;\; (\alpha > -1)\,,$$

which are of main importance in our study.

By using the explicit structure of the integral operator A^*A and motivated by the paper [**19**] of Vu Kim Tuan, Gorenflo we have derived in [**9**, Theorem 2.1] a result on the singular value asymptotics of A for all relevant power functions, which is recovered here in the following proposition.

Proposition 2.3. *For the singular values of a compact linear operator $A : L^2(0,1) \to L^2(0,1)$ defined by the formula* (1.4)*, where the multiplier function m is of power-type*

$$m(t) \,=\, t^{\alpha} \qquad (0 < t \le 1)$$

with some exponent $\alpha > -1$, we have

$$\sigma_n(A) \sim \frac{1}{(\alpha+1)\pi n} = \frac{1}{\pi n}\left(\int_0^1 m(t)dt\right).$$

Moreover, we had conjectured in [**9**] that the formula

$$\sigma_n(A) \sim \frac{1}{\pi n}\left(\int_0^1 m(t)dt\right) \tag{2.2}$$

implying a constant degree of ill-posedness $\mu(A) = 1$ for A from (1.4) remains valid for the whole family of weights

$$0 < m(t) \le C\,t^{\alpha} \quad \text{a.e. on } [0,1],$$

where $\alpha > -1$ and $C > 0$. This would involve the exponential case (1.7) arising in the finance application. The formula (2.2) could be fully confirmed by a series of numerical experiments of Freitag reported in [**5**], which also included exponential weight functions m.

On the other hand, source conditions

$$x_0 = A^* v \qquad (v \in Y) \tag{2.3}$$

yielding convergence rates of order

$$\|x_\beta - x_0\| = \mathcal{O}(\sqrt{\beta})$$

as $\beta \to 0$ for the method of Tikhonov regularization with

$$x_\beta = (A^*A + \beta I)^{-1} A^* y \qquad (y = Ax_0)$$

and other linear regularization methods also measure the strength of ill-posedness of an operator equation (1.1). So we can compare the strength of condition (2.3) for the case $A = J$ with the simple integration operator J defined by formula (1.3) written as

$$\text{(2.4)} \quad x_0(t) \,=\, [J^*\, v](t) \,=\, \int_t^1 v(s)\, ds \qquad (0 \le t \le 1;\; v \in L^2(0,1))$$

and the strength of condition (2.3) for the case $A = M \circ J$ with the composite integral operator from (1.4) with weights m having zeros. Provided that weight functions m occur we can write (2.3) as

$$\text{(2.5)} \quad x_0(t) \,=\, [J^* M^*\, v](s) \,=\, [J^* M\, v](t) \,=\, \int_t^1 m(s)\, v(s)\, ds \qquad (0 \le t \le 1;\; v \in L^2(0,1)).$$

If we assume that the multiplier function m has an essential zero only at $t = 0$, then the condition (2.4) that implies

$$\text{(2.6)} \qquad x_0 \in H^1(0,1) \quad \mathrm{w}ith \quad x_0(1) = 0$$

is weaker than the condition

$$\text{(2.7)} \qquad \frac{x_0'}{m} \in L^2(0,1) \quad \mathrm{w}ith \quad x_0(1) = 0$$

obtained from (2.5) by differentiation, since the new factor $\frac{1}{m}$ occurring in (2.7) in not in $L^\infty(0,1)$. Note that the pairs of conditions (2.4) and (2.6) on the one hand and (2.5) and (2.7) on the other hand are even equivalent.

Consequently in order to satisfy the source condition (2.5), the generalized derivative of the function x_0 has to compensate in some sense the pole of $\frac{1}{m}$ at $t = 0$. The level of compensation grows when the decay rate of $m(t) \to 0$ as $t \to 0$ gets accelerated. Hence, the strength of the requirement (2.5) imposed on x_0 grows for the families of weights m with exponential zeros compared to weights with power-type zeros. Nevertheless, the degree of ill-posedness is not altered as we will see below.

3. An equivalence lemma and its consequences. We note that the singular values $\sigma_n(A)$ of a compact operator A are the square roots of the eigenvalues of both positive definite operators A^*A and AA^*. Now we consider A from (1.4), where the corresponding adjoint operator A^* of A can be explicitly expressed by the formula

$$[A^*y](s) \;=\; \int_s^1 m(t)\,y(t)\,dt \qquad (0 \le s \le 1)\,. \tag{3.1}$$

In detail we consider for measurable m satisfying (2.1), where $m(t) \neq 0$ a.e. on $[0,1]$, the explicit structure

$$\begin{aligned}
[AA^*x](t) &= m(t) \int_0^t \left[\int_\tau^1 m(s)x(s)\,ds\right] d\tau \\
&= m(t) \int_0^t \left[\int_\tau^t m(s)x(s)ds + \int_t^1 m(s)x(s)ds\right] d\tau \\
&= m(t) \left[\int_0^t \left(\int_\tau^t m(s)x(s)ds\right) d\tau + t\int_t^1 m(s)\,x(s)\,ds\right] \\
&= \; m(t) \left[\int_0^t s\,m(s)\,x(s)\,ds + t\int_t^1 m(s)\,x(s)\,ds\right]
\end{aligned}$$

following from the expressions (1.4) for Ax, (3.1) for A^*y, and by considering the fact that interchanging the order of integration yields the identity

$$\int_0^t \left(\int_\tau^t \psi(s)\,ds\right) d\tau \;=\; \int_0^t s\,\psi(s)\,ds$$

for any integrable function $\psi(s)$ $(0 \le s \le t)$.

We search for reciprocals $\lambda > 0$ of the eigenvalues of AA^* and corresponding non-zero eigenfunctions $x \in L^2(0,1)$ satisfying the equation $\lambda AA^* x = x$. To do so we have to solve the integral equation

$$x(t) = \lambda\, m(t) \left[\int_0^t s\, m(s)\, x(s)\, ds + t \int_t^1 m(s)\, x(s)\, ds \right] . \tag{3.2}$$

Putting $u(t) = x(t)/m(t)$ from (3.2) we have the relation

$$u(t) = \lambda \left[\int_0^t s\, m^2(s)\, u(s)\, ds + t \int_t^1 m^2(s)\, u(s)\, ds \right] . \tag{3.3}$$

Differentiating (3.3) yields

$$u'(t) = \lambda \int_t^1 m^2(s)\, u(s)\, ds \,, \tag{3.4}$$

and by differentiating (3.3) a second time we obtain the second order differential equation

$$u''(t) + \lambda\, m^2(t)\, u(t) = 0 \qquad (0 < t < 1) \,. \tag{3.5}$$

Furthermore, from (3.3) and (3.4) the boundary conditions

$$u(0) = u'(1) = 0 \tag{3.6}$$

can be derived. Conversely, integrating (3.5) two times and observing (3.6) we come back to (3.2). So, we have proven the following lemma.

Lemma 3.1. *The integral equation* (3.2) *and the eigenvalue problem* (3.5 - 3.6) *are equivalent with respect to the substitution $x = mu$.*

Remark 3.2. In accordance with the boundary conditions (3.6) we are looking for solutions $u \in C[0,1]$ of problem (3.5 - 3.6). In case (i) $m \in L^2(0,1)$ then it follows $x = mu \in L^2(0,1)$. In case (ii) $m(t) = t^\alpha$ $(\alpha > -1)$ the functions $u(t)$ behave like t as $t \to 0$ (see

Example 3.3 below) so that the functions $x(t)$ behave like $t^{1+\alpha}$ as $t \to 0$ and we obtain $x \in C[0,1]$. In general, by assumption (2.1), we have $x \in L^2(0,1)$ if $u \in C[0,1]$ with $u(t) = \mathcal{O}(t^{1/2})$ as $t \to 0$. We also mention that the condition $m(t) \neq 0$ a.e. in $[0,1]$ can be omitted if we are only interested in the construction of $x = mu$ via the solutions u of (3.5 - 3.6).

Example 3.3. First we apply Lemma 3.1 to power functions

$$m(t) = t^\alpha \qquad (0 < t \le 1) \qquad \text{with exponents} \quad \alpha > -1 \tag{3.7}$$

as multiplier functions in (1.4). In that case we can rewrite the differential equation (3.5) by multiplying t^2 on both sides in the form

$$t^2\, u''(t) + \lambda\, t^{2(\alpha+1)}\, u(t) = 0\,. \tag{3.8}$$

This is useful, because the equation (3.8) has an explicit general solution (cf. Erdélyi [**4**, p.13, formula (62)]). Setting $\sigma := 1/\sqrt{\lambda}$ this solution can be verified as

$$\begin{aligned} u(t) &= t^{1/2} Z_\varrho \left(\frac{1}{\sigma(\alpha+1)}\, t^{\alpha+1} \right) \\ &= t^{1/2} \left[C_1 J_\varrho \left(\frac{1}{\sigma(\alpha+1)}\, t^{\alpha+1} \right) + C_2 J_{-\varrho} \left(\frac{1}{\sigma(\alpha+1)}\, t^{\alpha+1} \right) \right], \end{aligned}$$

where Z_ϱ denotes the general cylinder function and $J_\varrho, J_{-\varrho}$ are the Bessel functions of first kind and order $\varrho = \frac{1}{2(\alpha+1)} > 0$. For simplicity, we have taken $\varrho \neq 1, 2 \ldots$. The boundary condition $u(0) = 0$ leads to $C_2 = 0$ and the other boundary condition $u'(1) = 0$ yields the eigenvalue equation

$$\varrho J_\varrho(z) + z J'_\varrho(z) = 0 \qquad \text{with} \qquad z = \frac{1}{\sigma(\alpha+1)}\,,$$

which by the relation $\varrho J_\varrho + z J'_\varrho = z J_{\varrho-1}$ (cf. [**4**, p.11, formula (54)]) is equivalent to the equation

$$J_{-\nu} \left(\frac{1}{\sigma(\alpha+1)} \right) = 0 \qquad \text{with} \qquad \nu = \frac{2\alpha+1}{2\alpha+2}\,. \tag{3.9}$$

Equation (3.9) was also obtained in [**9**] by working with the operator A^*A and implies the asymptotics (2.2) for the singular values of A in the case of weights m from (3.7) (cf. [**9**, Theorem 2.1]).[2]

Example 3.4. Our main interest in this paper is focused on the case of exponential functions m, which was missing up to now. So let us consider as a specific family of this type the multiplier functions

$$m(t) = \frac{1}{t^2} \exp\left(-\frac{c}{t}\right) \quad (0 < t \leq 1) \qquad \text{with constants} \quad c > 0 \tag{3.10}$$

and taking into account Lemma 3.1 the associated differential equation

$$t^4\, u''(t) + \lambda \exp\left(-\frac{2c}{t}\right) u(t) = 0\,. \tag{3.11}$$

By substituting $y := \frac{2c}{t}$ in (3.11), for the function $v(y) = u(t)$ we then have the differential equation

$$v''(y) + \frac{2}{y}\, v'(y) + \eta \exp(-y)\, v(y) = 0 \qquad \text{with} \qquad \eta = \frac{\lambda}{4\,c^2}\,, \tag{3.12}$$

which has the general solution (cf. Kamke [**12**, p.442, formula (23)])

$$\begin{aligned} v(y) &= \frac{1}{y}\, Z_0(2\sqrt{\eta}\, e^{-y/2}) \\ &= \frac{C_1}{y}\, J_0(2\sqrt{\eta}\, e^{-y/2}) + \frac{C_2}{y}\, Y_0(2\sqrt{\eta}\, e^{-y/2}) \qquad (2c < y < \infty)\,, \end{aligned} \tag{3.13}$$

where Z_0, J_0, Y_0 denote the general, first kind and second kind Bessel function of zero order, respectively. The boundary condition $u(0) = 0$ means $v(\infty) = \lim\limits_{y\to\infty} v(y) = 0$. As $y \to \infty$ it holds $e^{-y/2} \to 0$, and therefore

$$v(y) \sim \frac{C_1}{y} + \frac{C_2}{y}\, \frac{2}{\pi} \ln\left[\sqrt{\eta}\, e^{-y/2}\right] \sim -\frac{C_2}{\pi} \quad \text{as} \quad y \to \infty\,,$$

since $J_0(z) \sim 1$ and $Y_0(z) \sim \frac{2}{\pi} \ln(\frac{z}{2})$ as $z \to 0$ (cf. [**4**, p.8, formula (33)]). This implies $C_2 = 0$. Further, taking $C_1 = 1$ we have

$$v(y) = \frac{1}{y}\, J_0\left(2\sqrt{\eta}\, e^{-y/2}\right)$$

[2]We take the opportunity to correct a typo in the verfication of the asymptotic relation (28) in [**9**, p.431]. In the second term of the asymptotic formula for $J'_{-\nu}(t)$ as $t \to 0$ above formula (28) of [**4**] the factor $(1 - \frac{2}{\nu})$ is missing.

and

$$v'(y) = \frac{\sqrt{\eta}}{y}\, e^{-y/2}\, J_1\left(2\sqrt{\eta}\, e^{-y/2}\right) - \frac{1}{y^2}\, J_0\left(2\sqrt{\eta}\, e^{-y/2}\right)$$

since $J_0'(z) = -J_1(z)$. The boundary condition $u'(1) = 0$ is equivalent to the condition $v'(2c) = 0$, i.e.,

$$c\, z\, J_1(z) - J_0(z) = 0 \qquad \text{with} \qquad z = 2\sqrt{\eta}\, e^{-c} = \frac{e^{-c}}{c}\, \frac{1}{\sigma}\,. \tag{3.14}$$

For $\sigma \to 0$ we have $z \to \infty$ and (cf. [**4**, p.85, formula (3)])

$$J_0(z) = \left(\frac{1}{2}\, \pi z\right)^{-1/2} \cos\left(z - \frac{\pi}{4}\right) + \mathcal{O}\left(z^{-3/2}\right) \qquad \text{as} \quad z \to \infty$$

and

$$J_1(z) = \left(\frac{1}{2}\, \pi z\right)^{-1/2} \left[\cos\left(z - \frac{3}{4}\, \pi\right) - \frac{3}{8z}\, \sin\left(z - \frac{3}{4}\, \pi\right)\right] + \mathcal{O}\left(z^{-5/2}\right) \qquad \text{as} \quad z \to \infty.$$

Hence, as $n \to \infty$ the eigenvalue equation (3.14) is asymptotically equal to the equation $J_1(z_n) = 0$ which yields the asymptotic relation (cf. Jahnke-Ende [**10**, p.146])

$$z_n = \frac{e^{-c}}{c}\, \frac{1}{\sigma_n} \sim \pi n\,,$$

and consequently the result

$$\sigma_n(A) \sim \frac{S}{\pi n} \qquad \text{with} \quad S = \int_0^1 m(t)dt = \frac{1}{c}\, e^{-c} \tag{3.15}$$

for the exponential family of weights m from (3.10), which again is in correspondence with the conjectured formula (2.2).

Based on Lemma 3.1 the conjecture (2.2) for general m follows from results by Kac and Krein [**11**] (cf. also [**17**]) on weighted Sturm-Liouville problems for the string applied to problem (3.5 - 3.6). In the examples above, we have shown this explicitly for families of power-type and exponential-type functions, respectively.

4. The main theorem. Now we are ready to formulate the main theorem of this paper that extends, based on both examples of Section 3, the Corollary 2.2 of [**9**] concerning wider classes of weight functions m in (1.4) implying $\sigma_n(A) \asymp n^{-1}$ and hence a non-changing degree of ill-posedness of corresponding equations (1.1).

Theorem 4.1. *For the singular values of a compact linear operator* $A : L^2(0,1) \to L^2(0,1)$ *defined by the formulae* (1.4), *where the multiplier function* m *satisfies for some exponent* $\alpha > -1$ *and for some positive constants* c, $\underline{C}$, *and* $\overline{C}$ *the inequalities*

$$\frac{\underline{C}}{t^2}\,\exp\left(-\frac{\underline{c}}{t}\right) \;\le\; m(t) \;\le\; \overline{C}\,t^{\alpha} \qquad \text{a.e. on } [0,1]\,, \tag{4.1}$$

we have

$$\sigma_n(A) \,\asymp\, \frac{1}{n}\,. \tag{4.2}$$

Proof. For Ax from (1.4),

$$[A_{down}\,x](t) \;=\; \frac{\underline{C}}{t^2}\,\exp\left(-\frac{\underline{c}}{t}\right)\,\int_0^t x(s)ds \qquad \text{a.e. on } [0,1]$$

and

$$[A_{up}\,x](t) \;=\; \overline{C}\,t^{\alpha}\,\int_0^t x(s)ds \qquad \text{a.e. on } [0,1]$$

from (4.1) we directly obtain

$$\|A_{down}\,x\| \le \|Ax\| \le \|A_{up}\,x\| \qquad \text{for all} \quad x \in L^2(0,1). \tag{4.3}$$

Now the Poincaré-Fischer extremum principle (see, e.g., [**1**, Lemma 4.18]) yields the representation

$$\sigma_n(A) \;=\; \max_{X_n \subset L^2(0,1)}\;\min_{x\in X_n,\, x\neq 0}\;\frac{\|A\,x\|}{\|x\|}$$

for the n-th singular value of the compact operator A, where X_n denotes an arbitrary n-dimensional subspace of the Hilbert space $L^2(0,1)$. Both the existence of a minimum of $\|Ax\|/\|x\|$ over all non-zero elements from X_n and the existence of a maximum of $\min\limits_{x\in X_n,\, x\neq 0} \frac{\|A\,x\|}{\|x\|}$ over all finite dimensional subspaces X_n are shown in the context of the proof of this principle. As a consequence we have for compact operators A and B mapping in $L^2(0,1)$ which satisfy the inequality $\|Ax\| \le \|Bx\|$ for all $x \in L^2(0,1)$ that

$$\min_{x\in X_n,\, x\neq 0}\,\frac{\|A\,x\|}{\|x\|} \le \min_{x\in X_n,\, x\neq 0}\,\frac{\|B\,x\|}{\|x\|} \qquad \text{and} \qquad \sigma_n(A) \le \sigma_n(B).$$

This fact was already mentioned in [**7**, Lemma 2.46]. Then the results $\sigma_n(A_{up}) \asymp \frac{1}{n}$ from Example 3.3 and $\sigma_n(A_{down}) \asymp \frac{1}{n}$ from Example 3.4 together with (4.3) prove the assertion of the theorem. □

Finally we note that Theorem 4.1 also implies $\mu(A) = 1$ for the situation of Example 1.1. Precisely, with m from (1.7) the hypothesis (4.1) can be verified for appropriate constants. On the one hand, the upper bound $\frac{\overline{C}}{\sqrt{t}}\,\exp\left(-\frac{\overline{c}}{\sqrt{t}}\right)$ $(0 <$

$t \leq 1)$ in (1.7) can be extended to a continuous function on $[0,1]$ by setting its function value zero for $t = 0$. Hence $m(t) \leq \hat{C}$ $(0 < t \leq 1)$ for some constant $0 < \hat{C} < \infty$. On the other hand, given positive constants $\underline{C}$ and $\underline{c}$ there exist other positive constants C and c such that we can estimate the lower bound of (1.7) as

$$\frac{C}{t^2} \exp\left(-\frac{c}{t}\right) \leq \frac{\underline{C}}{\sqrt[4]{t}} \exp\left(-\frac{\underline{c}}{t}\right) \leq m(t) \qquad (0 < t \leq 1)$$

with some $c > \underline{c}$, since the exponential decay is always faster than a power-type decay of arbitrary order.

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