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CONSTANT-SIGN SOLUTIONS OF A SYSTEM OF VOLTERRA INTEGRAL EQUATIONS IN ORLICZ SPACES

RAVI P. AGARWAL, DONAL O'REGAN, AND PATRICIA J. Y. WONG

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This paper is dedicated to Professor Zuhair Nashed in recognition of his contributions to the field of integral and operator equations.

ABSTRACT. We consider the following system of Volterra intergral equations

$$u_i(t) = \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds,$$
$$a.e.\ t \in [0,T],\ 1 \le i \le n.$$

Criteria are offered for the existence of one and more *constant-sign* solutions $u = (u_1, u_2, \cdots, u_n)$ of the system in L^p and the Orlicz spaces. We say u is of *constant sign* if for each $1 \le i \le n$, $\theta_i u_i(t) \ge 0$ for $a.e.$ $t \in [0,T]$, where $\theta_i \in \{1, -1\}$ is fixed.

1. Introduction. In this paper we shall consider the system of Volterra integral equations

$$u_i(t) = \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds, \tag{1.1}$$
$$a.e.\ t \in [0,T],\ 1 \le i \le n.$$

Throughout, let $u = (u_1, u_2, \cdots, u_n)$. We are interested in establishing the existence of one and more solutions u of the system (1.1) in

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Orlicz spaces, i.e., for each $1 \le i \le n$, u_i is in the Orlicz space L_ϕ. Moreover, we are interested in *constant-sign* solutions u, by which we mean $\theta_i u_i(t) \ge 0$ for *a.e.* $t \in [0, T]$ and $1 \le i \le n$, where $\theta_i \in \{1, -1\}$ is fixed. Note that *positive* solution is a special case of constant-sign solution when $\theta_i = 1$ for $1 \le i \le n$.

We shall consider the existence of solutions in L^p $(p \ge 1)$ spaces first, since we can obtain more general results here and also the analysis will motivate the ideas later in obtaining existence results in Orlicz spaces. We remark that those results obtained for L^p spaces invariably assume a polynomial type restriction (in u) on the nonlinearity $f_i(t, u)$ (see condition (C2) later). On the other hand, seeking solutions in other Orlicz spaces will lead to restrictions that are *not* of polynomial type (see condition (H4) later), and hence will allow us to consider new classes of equations. The main tools employed in this paper are the Leray-Schauder alternative and the Krasnosel'skii's fixed point theorem.

The existence of multiple constant-sign solutions of (1.1) in $(C[0, T])^n$ has recently been tackled in [7]. However, to the knowledge of the authors, there is very little literature on the existence of solutions of Volterra integral equations in L^p spaces, the *only* paper in this area [15] applies Krasnosel'skii's fixed point theorem to obtain a positive solution of

$$y(t) = \int_0^t g(t, s) f(s, y(s)) ds, \quad a.e.\ t \in [0, T].$$

Moreover, to date very little work has been done on the existence of solutions of Volterra integral equations in *Orlicz* spaces. Hence, our present work not only generalizes and extends those of [15] to (i) *systems*, and (ii) the existence of *multiple constant-sign* solutions through the use of *two* fixed point theorems, but also investigates the existence of multiple constant-sign solutions in *Orlicz* spaces, which is *totally new* in the literature.

We note that, on the other hand, a lot of work has been done on the existence of solutions of *Fredholm* integral equations of the form

$$y(t) = \int_0^T g(t, s) f(s, y(s)) ds, \quad t \in [0, T].$$

Mostly solutions are sought in $C[0, T]$ or $L^p[0, T]$ with $p > 1$ [1, 10, 11, 13], whereas solutions in Orlicz spaces are tackled in [14, 16, 17, 19, 20]. The more recent investigation on the existence of *constant-sign* solutions of *systems* of Fredholm integral equations in continuous, L^p and Orlicz spaces can be found in Agarwal, O'Regan and Wong [2–6].

The plan of the paper is as follows. The existence of constant-sign solutions of (1.1) in L^p and Orlicz spaces will be tackled respectively in Sections 3 and 4. Examples are also presented to illustrate the usefulness of the results obtained.

2. Preliminaries The following two theorems will be needed to establish the main results later. The first theorem is known as the *Leray-Schauder alternative* and the second is usually called *Krasnosel'skii's fixed point theorem in a cone.*

Theorem 2.1. [1] *Let B be a Banach space with $E \subseteq B$ closed and convex. Assume U is a relatively open subset of E with $0 \in U$ and $S : \overline{U} \to E$ is a continuous and compact map. Then either*

(a) *S has a fixed point in $\overline{U}$, or*

(b) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda Su$.*

Theorem 2.2. [12] *Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a continuous and completely continuous operator such that, either*

(a) *$\|Su\| \le \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Su\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$, or*

(b) *$\|Su\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Su\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.*

Then, S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence in L^p Space In this section, we consider the system (1.1) where, for each $1 \le i \le n$, $g_i(t,s)$ is a real-valued function for *a.e.* $t \in [0,T]$ and *a.e.* $s \in [0,t]$, and $f_i(t,u)$ is also a real-valued function for *a.e.* $t \in [0,T]$ and all $u \in \mathbb{R}^n$. We shall establish the existence of a constant-sign solution $u \in (L^p[0,T])^n = L^p[0,T] \times L^p[0,T] \times \cdots \times L^p[0,T]$ (n times) of (1.1) where $p \ge 1$. Throughout this section, let $\theta_i \in \{1,-1\}$, $1 \le i \le n$ be fixed, also let the integers p_1 and p_2 be such that

$$1 \le p_1 \le p < \infty \qquad \text{and} \qquad \frac{1}{p_1} + \frac{1}{p_2} = 1. \tag{3.1}$$

Let the Banach space $B = (L^p[0,T])^n$ be equipped with the norm

$$\|u\|_p = \max_{1 \le i \le n} \left(\int_0^T |u_i(t)|^p dt \right)^{\frac{1}{p}} = \max_{1 \le i \le n} |u_i|_p$$

where we let $|u_i|_p = \left(\int_0^T |u_i(t)|^p dt \right)^{\frac{1}{p}}, \ 1 \le i \le n.$

To begin our discussion, let the operator $S : B \to B$ be defined by

$$Su(t) = (S_1 u(t), S_2 u(t), \cdots, S_n u(t)), \quad a.e. t \in [0,T] \tag{3.2}$$

where

$$S_i u(t) = \int_0^t g_i(t,s) f_i(s, u(s)) ds, \quad a.e. t \in [0,T], \quad 1 \le i \le n. \tag{3.3}$$

Clearly, a fixed point of the operator S is a solution of the system (1.1).

Our first two lemmas show that $S : B \to B$ is well defined, and is continuous and completely continuous.

Lemma 3.1. *Assume*

(C1) *for each* $1 \le i \le n$, $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is a Carathéodory function, i.e.,*

(i) *the map* $t \mapsto f_i(t,u)$ *is measurable for all* $u \in \mathbb{R}^n$,

(ii) *the map* $u \mapsto f_i(t,u)$ *is continuous for almost all* $t \in [0,T]$;

(C2) *for each* $1 \le i \le n$, *there exists a function* $b_i \in L^{p_2}[0,T]$ *and constants* $c_{ij} > 0, \ 1 \le j \le n$ *such that*

$$|f_i(t,u)| \le b_i(t) + \sum_{j=1}^n c_{ij} |u_j|^{\frac{p}{p_2}}, \quad a.e. \ t \in [0,T], \quad all \ u \in \mathbb{R}^n;$$

(C3) *for each* $1 \le i \le n$, *the map* $(t,s) \to g_i(t,s)$ *is measurable, and* $g_i(t,s) > 0$ *for a.e.* $t \in [0,T]$, *a.e.* $s \in [0,t]$;

(C4) *for each* $1 \le i \le n$,

$$\left[\int_0^T \left(\int_s^T |g_i(t,s)|^p dt\right)^{\frac{p_1}{p}} ds\right]^{\frac{1}{p_1}} \equiv M_i < \infty.$$

Then, the operator $S : B \to B$ is well defined.

Proof. For each $1 \le i \le n$, let

$$g_i^*(t,s) = \begin{cases} g_i(t,s), & 0 \le s \le t \le T \\ 0 \quad\quad , & 0 \le t < s \le T. \end{cases}$$

First, using Tonelli's Theorem and the fact that if a function y is measurable, then so is $|y|^p$, we see that for each $1 \le i \le n$,

$$\left(\int_0^T |g_i^*(t,s)|^{p_1} ds\right)^{\frac{p}{p_1}} = \left(\int_0^t |g_i(t,s)|^{p_1} ds\right)^{\frac{p}{p_1}}$$

is a measurable function of t on [0,T].

Applying the integral version of Minkowski's inequality [21, p.143], we obtain, noting $\frac{p}{p_1} \ge 1$ and also (C4),

$$\begin{aligned} (3.4) \quad \int_0^T \left(\int_0^t |g_i(t,s)|^{p_1} ds\right)^{\frac{p}{p_1}} dt &= \int_0^T \left(\int_0^T |g_i^*(t,s)|^{p_1} ds\right)^{\frac{p}{p_1}} dt \\ &\le \left[\int_0^T \left(\int_0^T |g_i^*(t,s)|^p dt\right)^{\frac{p_1}{p}} ds\right]^{\frac{p}{p_1}} \\ &= \left[\int_0^T \left(\int_s^T |g_i(t,s)|^p dt\right)^{\frac{p_1}{p}} ds\right]^{\frac{p}{p_1}} \\ &= M_i^p, \ 1 \le i \le n. \end{aligned}$$

Hence, $\left(\int_0^t |g_i(t,s)|^{p_1} ds\right)^{\frac{p}{p_1}}$ is integrable on $[0,T]$.

Let $u \in (L^p[0,T])^n$. In view of (C2), it is clear that

$$\left(\int_0^T |f_i(t,u(t))|^{p_2}dt\right)^{\frac{1}{p_2}} \le |b_i|_{p_2} + \sum_{j=1}^n c_{ij}\left(\int_0^T |u_j(t)|^p dt\right)^{\frac{1}{p_2}} \equiv L_i < \infty,\ 1\le i\le n. \tag{3.5}$$

Therefore,

$$f_i(t,u)\in L^{p_2}[0,T] \text{ for } u\in (L^p[0,T])^n, \quad 1\le i\le n. \tag{3.6}$$

Now, for each $1\le i\le n$, we use Hölder's inequality, (3.4) and (3.5) to get

$$\begin{aligned}\int_0^T |S_i u(t)|^p dt &\le \int_0^T \left[\int_0^t |g_i(t,s)|\cdot|f_i(s,u(s))|ds\right]^p dt\\ &\le \int_0^T\left[\left(\int_0^t |g_i(t,s)|^{p_1}ds\right)^{\frac{1}{p_1}}\left(\int_0^T |f_i(s,u(s))|^{p_2}ds\right)^{\frac{1}{p_2}}\right]^p dt\\ &\le M_i^p\, L_i^p.\end{aligned}$$

This leads to

$$|S_i u|_p \le M_i L_i < \infty \text{ for } u\in (L^p[0,T])^n, \quad 1\le i\le n. \tag{3.7}$$

Hence, $S_i u\in L^p[0,T],\ 1\le i\le n$ and so $Su\in (L^p[0,T])^n$. This shows that $S:(L^p[0,T])^n\to (L^p[0,T])^n$ is well defined. □

Lemma 3.2. Let (C1)–(C4) hold. Then, the operator $S:B\to B$ is continuous and completely continuous.

Proof. We shall show that $S_i,\ 1\le i\le n$ is continuous and completely continuous. For each $1\le i\le n$, we observe that the operator $S_i:(L^p[0,T])^n\to L^p[0,T]$ can be written as

$$S_i = A_i F_i \tag{3.8}$$

where $F_i : (L^p[0,T])^n \to L^{p_2}[0,T]$ and $A_i : L^{p_2}[0,T] \to L^p[0,T]$ are respectively defined by

$$F_i u(t) = f_i(t, u(t)), \quad a.e.\ t \in [0,T] \tag{3.9}$$

and

$$A_i x(t) = \int_0^t g_i(t,s)x(s)ds, \quad a.e.\ t \in [0,T]. \tag{3.10}$$

Note that $F_i : (L^p[0,T])^n \to L^{p_2}[0,T]$ is well defined by (3.6). Further, for $x \in L^{p_2}[0,T]$, using (3.4) we have

$$\begin{aligned}\int_0^T |A_i x(t)|^p dt &\le \int_0^T \left[\int_0^t |g_i(t,s)| \cdot |x(s)|ds\right]^p dt \\ &\le \int_0^T \left[\left(\int_0^t |g_i(t,s)|^{p_1} ds\right)^{\frac{1}{p_1}} \left(\int_0^T |x(s)|^{p_2} ds\right)^{\frac{1}{p_2}}\right]^p dt \\ &\le M_i^p \ |x|_{p_2}^p.\end{aligned}$$

Therefore,

$$|A_i x|_p \le M_i \ |x|_{p_2} < \infty \quad \text{for} \quad x \in L^{p_2}[0,T]. \tag{3.11}$$

This shows that $A_i : L^{p_2}[0,T] \to L^p[0,T]$ is well defined. Indeed, it has been noted earlier that $S_i = A_i F_i : (L^p[0,T])^n \to L^p[0,T],\ 1 \le i \le n$, is well defined (see (3.7)).

By a result of Krasnosel'skii [13, p.22, p.27], it follows that $F_i : (L^p[0,T])^n \to L^{p_2}[0,T]$ is continuous and bounded. Thus, to prove that S_i is continuous and completely continuous, it suffices to show that A_i is continuous and completely continuous.

First, we shall prove that A_i is continuous. Let $x_m \to x$ in $L^{p_2}[0,T]$, i.e., $\lim_{m\to\infty} |x_m - x|_{p_2} = 0$. Applying Hölder's inequality and (3.4), we find

$$\begin{aligned}\int_0^T |A_i x_m(t) - A_i x(t)|^p dt &\le \int_0^T \left[\int_0^t |g_i(t,s)| \cdot |x_m(s) - x(s)| ds\right]^p dt \\ &\le \int_0^T \left[\left(\int_0^t |g_i(t,s)|^{p_1} ds\right)^{\frac{1}{p_1}} \right. \\ &\qquad \left. \left(\int_0^T |x_m(s) - x(s)|^{p_2} ds\right)^{\frac{1}{p_2}}\right]^p dt \\ &\le M_i^p \ |x_m - x|_{p_2}^p\end{aligned}$$

or

$$|A_i x_m - A_i x|_p \le M_i \ |x_m - x|_{p_2} \to 0 \quad \text{ as } \quad m \to \infty.$$

Hence, A_i is continuous.

Next, we shall show that A_i is completely continuous. Let Ω be a bounded set in $L^{p_2}[0,T]$, i.e., there exists a constant $K > 0$ such that

$$|x|_{p_2} \le K, \ \ x \in \Omega. \tag{3.12}$$

We shall use the Riesz Compactness Criteria [18, p.10] to show that $A_i\Omega$ is relatively compact. The first condition in the criteria is to have $A_i\Omega$ bounded in $L^p[0,T]$. This is satisfied in view of (3.11) and (3.12). The second condition in the criteria is to have the following satisfied

$$\int_0^T |A_i x(t+h) - A_i x(t)|^p dt \to 0 \text{ as } h \to 0, \text{ uniformly for } x \in \Omega. \tag{3.13}$$

In fact, applying Hölder's inequality, the integral version of Minkowski's inequality [21, p.143] and (3.12), we find for $x \in \Omega$,

$$
\begin{aligned}
&\int_0^T |A_i x(t+h) - A_i x(t)|^p dt \\
&\le \int_0^T \left[\int_0^t |g_i(t+h,s) - g_i(t,s)| \cdot |x(s)| ds \right.\\
&\quad \left. + \int_t^{t+h} |g_i(t+h,s)| \cdot |x(s)| ds\right]^p dt \\
&= \int_0^T \left[\int_0^T |g_i^*(t+h,s) - g_i^*(t,s)| \cdot |x(s)| ds\right]^p dt \\
&\le \int_0^T \left[\left(\int_0^T |g_i^*(t+h,s) - g_i^*(t,s)|^{p_1} ds\right)^{\frac{1}{p_1}} \left(\int_0^T |x(s)|^{p_2} ds\right)^{\frac{1}{p_2}}\right]^p dt \\
&= |x|_{p_2}^p \int_0^T \left(\int_0^T |g_i^*(t+h,s) - g_i^*(t,s)|^{p_1} ds\right)^{\frac{p}{p_1}} dt \\
&\le K^p \left[\int_0^T \left(\int_0^T |g_i^*(t+h,s) - g_i^*(t,s)|^p dt\right)^{\frac{p_1}{p}} ds\right]^{\frac{p}{p_1}}.
\end{aligned}
$$

Now noting (3.4) and the fact that translates of L^p functions ($1 \le p < \infty$) are continuous in norm, we see that (3.13) holds. Hence, the Riesz Compactness Criteria are fulfilled and we have shown that $A_i\Omega$ is relatively compact, so A_i is completely continuous. The proof is complete. □

We are now ready to employ Theorem 2.1 to get a general existence result in $(L^p[0,1])^n$.

Theorem 3.1. *Let* (C1)–(C4) *hold. Assume there is a positive constant* M_0*, independent of* λ*, with*

$$\|u\|_p \neq M_0 \tag{3.14}$$

for any solution $u \in (L^p[0,T])^n$ *of the system*

$(3.15)_\lambda$ $$u(t) = \lambda Su(t), \quad a.e.\ t \in [0,T]$$

where $0 < \lambda < 1$. *Then, the system (1.1) has a solution* $u^* \in (L^p[0,T])^n$ *with* $\|u^*\|_p \le M_0$.

Proof. By Lemma 3.2, (C1)–(C4) guarantee that $S : B \to B$ is continuous and completely continuous. In the context of Theorem 2.1, let

$$U = \{u \in B \mid \|u\|_p < M_0\}.$$

Since $\|u\|_p \ne M_0$, where u is any solution of (3.15), we cannot have conclusion (b) of Theorem 2.1, hence conclusion (a) of Theorem 2.1 must hold, i.e., the system (1.1) has a solution $u^* \in \overline{U}$ with $\|u^*\|_p \le M_0$. □

The next result applies Theorem 3.1 to get the existence of a *constant-sign* solution in $(L^p[0,1])^n$.

Theorem 3.2. Let (C1)–(C4) hold. Assume

(C5) for each $1 \le i \le n$,

$$\theta_i f_i(t,u) \ge 0, \quad a.e.\ t \in [0,T], \quad \text{all } u \in \tilde{K}$$

where

$$\tilde{K} = \left\{ u \in B \;\middle|\; \text{for each } 1 \le i \le n,\ \theta_i u_i(t) \ge 0 \text{ for } a.e.\ t \in [0,T] \right\};$$

(C6) there exists $\alpha > 0$ such that for each $1 \le i \le n$,

$$M_i \psi_i(\alpha) < \alpha$$

where $\psi_i : [0,\infty) \to [0,\infty)$ is defined by

$$\psi_i(x) = |b_i|_{p_2} + (x)^{\frac{p}{p_2}} \sum_{j=1}^n c_{ij}.$$

Then, the system (1.1) has at least one constant-sign solution $u \in (L^p[0,T])^n$ such that $\|u\|_p < \alpha$.

Proof. To begin, we consider the system

$$u_i(t) = \int_0^t g_i(t,s)\hat{f}_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds, \quad a.e.\ t \in [0,T], \ \ 1 \le i \le n \tag{3.16}$$

where $\hat{f}_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\hat{f}_i(t, u_1, u_2, \cdots, u_n) = f_i(t, \theta_1|u_1|, \theta_2|u_2|, \cdots, \theta_n|u_n|), \quad 1 \le i \le n. \tag{3.17}$$

Note that $(\theta_1|u_1|, \theta_2|u_2|, \cdots, \theta_n|u_n|) \in \tilde{K}$.

We shall prove that (3.16) has a solution. For this, we consider the system

$$u_i(t) = \lambda \int_0^t g_i(t,s)\hat{f}_i(s, u(s))ds, \quad a.e.\ t \in [0,T], \ \ 1 \le i \le n \tag{3.18$_\lambda$}$$

where $0 < \lambda < 1$. Let $u \in (L^p[0,T])^n$ be any solution of (3.18). We shall show that

$$\|u\|_p \ne \alpha, \tag{3.19}$$

then by Theorem 3.1 it follows that (3.16) has a solution.

Now, using (3.17), (C3) and (C5), we get

$$\begin{aligned}\theta_i u_i(t) &= \lambda \int_0^t g_i(t,s)\theta_i \hat{f}_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds \\ &= \lambda \int_0^t g_i(t,s)\theta_i f_i(s, \theta_1|u_1(s)|, \theta_2|u_2(s)|, \cdots, \theta_n|u_n(s)|)ds \ge 0, \\ &\qquad a.e.\ t \in [0,T],\ 1 \le i \le n\end{aligned}$$

which means that

$$|u_i(t)| = \theta_i u_i(t), \quad a.e.\ t \in [0,T], \ \ 1 \le i \le n. \tag{3.20}$$

Moreover, it is clear that for each $1 \le i \le n$,

$$\int_0^T |u_i(t)|^p dt$$
$$\le \int_0^T \left[\int_0^t |g_i(t,s)| \cdot |f_i(s, \theta_1|u_1(s)|, \theta_2|u_2(s)|, \cdots, \theta_n|u_n(s)|)|ds\right]^p dt.$$

Using the same argument as in getting (3.7), we find

$$\begin{aligned} |u_i|_p &\le M_i \left[|b_i|_{p_2} + \sum_{j=1}^n c_{ij}(|u_j|_p)^{\frac{p}{p_2}}\right] \\ &\le M_i \left[|b_i|_{p_2} + \sum_{j=1}^n c_{ij}(\|u\|_p)^{\frac{p}{p_2}}\right] = M_i\psi_i(\|u\|_p), \quad 1 \le i \le n. \end{aligned}$$

It follows that

$$\|u\|_p \le \max_{1\le j\le n} M_j\psi_j(\|u\|_p). \tag{3.21}$$

Noting (3.21) and (C6), we conclude that $\|u\|_p \ne \alpha$. Hence, (3.19) is proved.

It now follows from Theorem 3.1 that the system (3.16) has a solution $u^* \in (L^p[0,T])^n$ with $\|u^*\|_p \le \alpha$, and

$$u_i^*(t) = \int_0^t g_i(t,s)\hat{f}_i(s, u_1^*(s), u_2^*(s), \cdots, u_n^*(s))ds,$$
$$a.e.\ t \in [0,T],\ 1 \le i \le n.$$

Using a similar argument as above, it can be easily seen that

$$|u_i^*(t)| = \theta_i u_i^*(t), \quad a.e.\ t \in [0,T], \quad 1 \le i \le n \tag{3.22}$$

and

$$\|u^*\|_p \ne \alpha. \tag{3.23}$$

Therefore, u^* is of constant sign and $\|u^*\|_p < \alpha$. Further, using (3.17) and (3.22), we have for *a.e.* $t \in [0,T]$ and each $1 \le i \le n$,

$$\begin{aligned} u_i^*(t) &= \int_0^t g_i(t,s)\hat{f}_i(s,u_1^*(s),u_2^*(s),\cdots,u_n^*(s))ds \\ &= \int_0^t g_i(t,s)f_i(s,\theta_1|u_1^*(s)|,\theta_2|u_2^*(s)|,\cdots,\theta_n|u_n^*(s)|)ds \\ &= \int_0^t g_i(t,s)f_i(s,\theta_1^2u_1^*(s),\theta_2^2u_2^*(s),\cdots,\theta_n^2u_n^*(s))ds \\ &= \int_0^t g_i(t,s)f_i(s,u_1^*(s),u_2^*(s),\cdots,u_n^*(s))ds. \end{aligned}$$

Hence, u^* is in fact a solution of (1.1). The proof is complete. □

Remark 3.1. By examining the proof of Theorem 3.2, we realize that the function ψ_i (appeared in (C6)) must be such that

$$\left(\int_0^T |f_i(t,u(t))|^{p_2}dt\right)^{\frac{1}{p_2}} \le \psi_i(\|u\|_p).$$

Therefore, ψ_i can be defined differently from the one in (C6), so long as it satisfies the above inequality.

Remark 3.2. If $\frac{p}{p_2} < 1$, then the existence of α in (C6) is guaranteed.

We have so far used the Leray-Schauder alternative (Theorem 2.1) to obtain existence criteria in $(L^p[0,1])^n$. We shall next apply Krasnosel' skii's fixed point theorem (Theorem 2.2) to get further existence results in $(L^p[0,1])^n$.

Define a cone in B as

$$(3.24)\quad C_a = \left\{ u \in B \;\middle|\; \text{for each } 1 \le i \le n,\ \theta_i u_i(t) \ge a(t)\|u\|_p \text{ for } a.e.\ t \in [0,T] \right\}.$$

Here we assume

$$a(t) > 0,\ \ a.e.\ t \in [0,T], \qquad a \in L^p[0,T] \qquad \text{and} \qquad \|a\|_p \le 1.$$

More conditions on $a(t)$ will be presented later. It is clear that a fixed point of the operator S in C_a is a *constant-sign* solution of (1.1) in $(L^p[0,T])^n$.

Let $0 < \beta < \alpha$. Define

$$\Omega_\alpha = \{u \in B \mid \|u\|_p < \alpha\} \qquad \text{and} \qquad \Omega_\beta = \{u \in B \mid \|u\|_p < \beta\}.$$

Our next lemma shows that S maps $C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$ into C_a.

Lemma 3.3. *Let* (C1)–(C4) *hold. Assume*

(C7) *for each* $1 \le i \le n$,

$$\theta_i f_i(t, u) > 0, \quad a.e.\ t \in [0, T], \quad all\ u \in \hat{K}$$

where

$$\hat{K} = \left\{ u \in B \;\middle|\; for\ each\ 1 \le i \le n,\ \theta_i u_i(t) > 0\ for\ a.e.\ t \in [0, T] \right\};$$

moreover, $\theta_i f_i$ *is 'nondecreasing' in the sense that if* $x \le \theta_j u_j \le y$ *for some* $j \in \{1, 2, \cdots, n\}$, *then for a.e.* $t \in [0, T]$,

$$\begin{aligned}\theta_i f_i(t, u_1, \cdots, \theta_j x, \cdots, u_n) &\le \theta_i f_i(t, u_1, \cdots, u_j, \cdots, u_n)\\ &\le \theta_i f_i(t, u_1, \cdots, \theta_j y, \cdots, u_n);\end{aligned}$$

(C8) *there exists a function* $a \in L^p[0, T]$ *with* $a(t) > 0$ *for a.e.* $t \in [0, T]$ *and* $\|a\|_p \le 1$ *such that the following holds for each* $1 \le i \le n$ *and any* $R > 0$,

$$\begin{aligned}&\int_0^t g_i(t, s)\theta_i f_i(s, \theta_1 R a(s), \theta_2 R a(s), \cdots, \theta_n R a(s)) ds\\ &\qquad \ge a(t) \cdot \max_{1 \le j \le n} M_j \psi_j(R), \quad a.e.\ t \in [0, T]\end{aligned}$$

where ψ_j *is defined in (C6) (see Remark 3.1 for alternative definition of* ψ_j*).*

Then, the operator S *maps* $C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$ *into* C_a.

Proof. Let $u \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. From Lemma 3.1, we already have $Su \in B$. Moreover, in view of (C3) and (C7), it follows that

$$\theta_i(S_i u)(t) = \int_0^t g_i(t,s)\theta_i f_i(s,u(s))ds \geq 0, \qquad a.e.\ t \in [0,T],\ \ 1 \leq i \leq n. \tag{3.25}$$

We shall show that $\theta_i(S_i u)(t) \geq a(t)\|u\|_p$ for $a.e.$ $t \in [0,T]$ and each $1 \leq i \leq n$. Since $u \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$, there exists $R \in [\beta, \alpha]$ such that

$$\|u\|_p = R \quad \text{and} \quad \theta_i u_i(t) \geq a(t)R > 0, \qquad a.e.\ t \in [0,T],\ \ 1 \leq i \leq n. \tag{3.26}$$

and (C7), it is clear that for $a.e.$ $t \in [0,T]$ and $1 \leq i \leq n$,

$$\theta_i(S_i u)(t) \geq \int_0^t g_i(t,s)\theta_i f_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \cdots, \theta_n Ra(s))ds. \tag{3.27}$$

On the other hand, using Hölder's inequality we have (3.7) which provides

$$|S_i u|_p \leq M_i \left[|b_i|_{p_2} + \sum_{j=1}^n c_{ij}(|u_j|_p)^{\frac{p}{p_2}} \right] \leq M_i \psi_i(\|u\|_p) = M_i \psi_i(R), \qquad 1 \leq i \leq n$$

(note that ψ_i is defined in (C6)). It follows that

$$\|Su\|_p \leq \max_{1 \leq j \leq n} M_j \psi_j(R). \tag{3.28}$$

Now, using (3.28) in (3.27) yields

$$\begin{aligned} \theta_i(S_i u)(t) &\geq \frac{\int_0^t g_i(t,s)\theta_i f_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \cdots, \theta_n Ra(s))ds}{\max_{1 \leq j \leq n} M_j \psi_j(R)} \|Su\|_p \\ &\geq a(t)\|Su\|_p, \quad a.e.\ t \in [0,T],\ \ 1 \leq i \leq n \end{aligned}$$

where the last inequality follows from (C8). This completes the proof. □

We are now ready to apply Theorem 2.2 to get the existence of a *constant-sign* solution in $(L^p[0,1])^n$.

Theorem 3.3. *Let* (C1)–(C4), (C7) *and* (C8) *hold. Assume*

(C9) *there exists* $\alpha > 0$ *such that for each* $1 \le i \le n$,

$$M_i\psi_i(\alpha) \le \alpha$$

where ψ_i *is defined in* (C6)*;*

(C10) *there exists* $\beta\ (\neq \alpha) > 0$ *such that for each* $1 \le i \le n$,

$$\int_0^T \left[\int_0^t g_i(t,s)\theta_i f_i(s,\theta_1\beta a(s),\theta_2\beta a(s),\cdots,\theta_n\beta a(s))ds\right]^p dt \ge \beta^p.$$

Then, the system (1.1) *has at least one constant-sign solution* $u \in (L^p[0,T])^n$ *such that*

(a) $\alpha \le \|u\|_p \le \beta$ *and* $\theta_i u_i(t) \ge a(t)\alpha$, *a.e.* $t \in [0,T]$, $1 \le i \le n$ *if* $\alpha < \beta$;

(b) $\beta \le \|u\|_p \le \alpha$ *and* $\theta_i u_i(t) \ge a(t)\beta$, *a.e.* $t \in [0,T]$, $1 \le i \le n$ *if* $\beta < \alpha$.

Proof. Without any loss of generality, let $\beta < \alpha$. Since $S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_a$ is continuous and completely continuous (Lemmas 3.1–3.3), it suffices to show that (i) $\|Su\|_p \le \|u\|_p$ for $u \in C_a \cap \partial\Omega_\alpha$, and (ii) $\|Su\|_p \ge \|u\|_p$ for $u \in C_a \cap \partial\Omega_\beta$.

To verify (i), let $u \in C_a \cap \partial\Omega_\alpha$. Then,

$$\|u\|_p = \alpha \text{ and } \theta_i u_i(t) \ge a(t)\alpha > 0, \quad a.e.\ t \in [0,T], \quad 1 \le i \le n.$$

From the proof of Lemma 3.3, we obtain $(3.28)|_{R=\alpha}$ and hence noting (C9) we find

$$\|Su\|_p \le \max_{1\le j\le n} M_j\psi_j(\alpha) \le \alpha = \|u\|_p.$$

Next, to prove (ii), let $u \in C_a \cap \partial\Omega_\beta$. So

$$\|u\|_p = \beta \ \text{ and } \ \theta_i u_i(t) \geq a(t)\beta > 0, \quad a.e. \ t \in [0,T], \ \ 1 \leq i \leq n.$$

Now $\|Su\|_p = |S_i u|_p$ for some $i \in \{1, 2, \cdots, n\}$. Thus, using (3.25), $(3.27)|_{R=\beta}$ and (C10) gives

$$\begin{aligned}
\|Su\|_p = |S_i u|_p \ &= \ \left\{ \int_0^T [\theta_i (S_i u)(t)]^p dt \right\}^{\frac{1}{p}} \\
\geq \left\{ \int_0^T \left[\int_0^t g_i(t,s)\theta_i f_i(s, \theta_1 \beta a(s), \theta_2 \beta a(s), \cdots, \theta_n \beta a(s)) ds \right]^p dt \right\}^{\frac{1}{p}} \\
\geq \beta \ = \ \|u\|_p.
\end{aligned}$$

Having obtained (i) and (ii), it follows from Theorem 2.2 that S has a fixed point $u \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. Therefore, conclusion (b) follows immediately. ◻

Remark 3.3. In (C9) if we have *strict* inequality instead, i.e., condition (C6), then from the proof of Theorem 3.3 we see that a fixed point u of S must satisfy $\|u\|_p \neq \alpha$. Similarly, if the inequality in (C10) is *strict*, i.e.,

$$\int_0^T \left[\int_0^t g_i(t,s)\theta_i f_i(s, \theta_1 \beta a(s), \theta_2 \beta a(s), \cdots, \theta_n \beta a(s)) ds \right]^p dt > \beta^p,$$

then a fixed point u of S must fulfill $\|u\|_p \neq \beta$. Hence, with *strict* inequalities in (C9) and (C10), the conclusion of Theorem 3.3 becomes: the system (1.1) has at least one constant-sign solution $u \in (L^p[0,T])^n$ such that

(a) $\alpha < \|u\|_p < \beta$ and $\theta_i u_i(t) > a(t)\alpha$, $a.e. t \in [0,T]$, $1 \leq i \leq n$ if $\alpha < \beta$;

(b) $\beta < \|u\|_p < \alpha$ and $\theta_i u_i(t) > a(t)\beta$, $a.e.\ t \in [0,T]$, $1 \leq i \leq n$ if $\beta < \alpha$.

Our next result gives the existence of *multiple* constant-sign solutions of (1.1) in $(L^p[0,T])^n$.

Theorem 3.4. *Assume* (C1)–(C5), (C7) *and* (C8) *hold. Let* (C6) *be satisfied for* $\alpha = \alpha_0$, (C9) *be satisfied for* $\alpha = \alpha_\ell,\ \ell = 1, 2, \cdots, k$, *and* (C10) *be satisfied for* $\beta = \beta_\ell,\ \ell = 1, 2, \cdots, m$.

(a) *If* $m = k + 1$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1}$, *then* (1.1) *has (at least)* $2k$ *constant-sign solutions* $u^1, \cdots, u^{2k} \in (L^p[0,T])^n$ *such that*

$$\beta_1 \le \|u^1\|_p \le \alpha_1 \le \|u^2\|_p \le \beta_2 \le \cdots \le \alpha_k \le \|u^{2k}\|_p \le \beta_{k+1}.$$

(b) *If* $m = k$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k$, *then* (1.1) *has (at least)* $2k - 1$ *constant-sign solutions* $u^1, \cdots, u^{2k-1} \in (L^p[0,T])^n$ *such that*

$$\beta_1 \le \|u^1\|_p \le \alpha_1 \le \|u^2\|_p \le \beta_2 \le \cdots \le \beta_k \le \|u^{2k-1}\|_p \le \alpha_k.$$

(c) *If* $m = k+1$ *and* $0 < \alpha_0 < \beta_1 < \alpha_1 < \cdots < \alpha_{m-1} < \beta_m$, *then* (1.1) *has (at least)* $2m$ *constant-sign solutions* $u^0, \cdots, u^{2m-1} \in (L^p[0,T])^n$ *such that*

$$0 \le \|u^0\|_p < \alpha_0 < \|u^1\|_p \le \beta_1 \le \|u^2\|_p \le \alpha_1 \le \cdots \le \alpha_{m-1} \le \|u^{2m-1}\|_p \le \beta_m.$$

(d) *If* $m = k$ *and* $0 < \alpha_0 < \beta_1 < \alpha_1 < \cdots < \beta_m < \alpha_m$, *then* (1.1) *has (at least)* $2m+1$ *constant-sign solutions* $u^0, \cdots, u^{2m} \in (L^p[0,T])^n$ *such that*

$$0 \le \|u^0\|_p < \alpha_0 < \|u^1\|_p \le \beta_1 \le \|u^2\|_p \le \alpha_1 \le \cdots \le \beta_m \le \|u^{2m}\|_p \le \alpha_m.$$

Proof. In (a) and (b), we just apply Theorem 3.3 repeatedly. In (c) and (d), Theorem 3.2 is used to obtain the existence of $u^0 \in (L^p[0,T])^n$ with $0 \le \|u^0\|_p < \alpha_0$, the results then follow by repeated use of Theorem 3.3. □

Remark 3.4. Suppose in Theorem 3.4 we have some *strict* inequalities in (C9) and (C10), say, involving α_i and β_j for some $i \in \{1, 2, \cdots, k\}$

and some $j \in \{1, 2, \cdots, m\}$. Then, noting Remark 3.3, those inequalities in the conclusion involving α_i and β_j will also be *strict*.

We shall now illustrate two applications of Theorem 3.3.

Example 3.1. In nonlinear diffusion and percolation problems (see [8, 9] and the references cited therein), the system (1.1) arises where g_i is a convolution kernel, i.e.,

$$u_i(t)=\int_0^t g_i(t-s) f_i(s, u_1(s), u_2(s), \cdots, u_n(s))ds, \;\; a.e.\; t \in [0, T],\; 1 \le i \le n.$$

In particular, Bushell and Okrasiński [8] investigated a special case of the above system given by

$$y(t) = \int_0^t (t-s)^{\gamma-1} f(y(s))ds, \;\; a.e.\; t \in [0, T]$$

where $\gamma > 1$. To generalize their problem and also to illustrate the usefulness of the results obtained for (1.1), we shall consider the system

$$(3.29) \qquad u_i(t) = \int_0^t (t-s)^{\gamma_i-1} f(s, u_1(s), u_2(s), \cdots, u_n(s))ds, \\ a.e.\; t \in [0, T],\; 1 \le i \le n$$

where

$$(3.30) \quad n = 2, \quad T = 1, \quad \gamma_1 = 2, \quad \gamma_2 = 3 \quad \text{and} \quad f(t, u) = u_1^{\frac{1}{4}} + u_2^{\frac{1}{4}}.$$

Suppose we are interested in seeking positive solutions of (3.29), (3.30) in $(L^2[0, 1])^2$. So we fix $\theta_i = 1,\; 1 \le i \le n$ and $p = 2$. Choose $p_1 = p_2 = 2$.

Clearly, conditions (C1), (C3) and (C7) are satisfied. Condition (C2) also holds with $b_i = 0$ and $c_{ij} = 1$. In (C4), we compute that

$$M_1 = \left[\int_0^1 \left(\int_s^1 (t-s)^2 dt\right) ds\right]^{\frac{1}{2}} = \sqrt{\frac{1}{12}}$$

$$\text{and} \qquad M_2 = \left[\int_0^1 \left(\int_s^1 (t-s)^4 dt\right) ds\right]^{\frac{1}{2}} = \sqrt{\frac{1}{30}}.$$

Further, using Hölder's inequality, we obtain

$$\begin{aligned}\left\{\int_0^1 [f(t,u(t))]^{p_2} dt\right\}^{\frac{1}{p_2}} &= \left\{\int_0^1 \left[(u_1(t))^{\frac14} + (u_2(t))^{\frac14}\right]^2 dt\right\}^{\frac12}\\ &\le \left\{\int_0^1 2\left[(u_1(t))^{\frac12} + (u_2(t))^{\frac12}\right] ds\right\}^{\frac12}\\ &\le 2^{\frac12}\left\{\left[\int_0^1 (u_1(t))^2\ dt\right]^{\frac14} + \left[\int_0^1 (u_2(t))^2\ dt\right]^{\frac14}\right\}^{\frac12}\\ &= 2^{\frac12}\left\{|u_1|_2^{\frac12} + |u_2|_2^{\frac12}\right\}^{\frac12}\\ &\le 2^{\frac12}\left\{2\|u\|_2^{\frac12}\right\}^{\frac12}\\ &= 2\|u\|_2^{\frac14}\ \equiv\ \psi(\|u\|_2)\end{aligned}$$

where the function ψ is defined by

$$\psi(x) = 2x^{\frac14}. \tag{3.31}$$

In view of Remark 3.1, we shall use the ψ defined in (3.31) in conditions (C8) and (C9).

Now, the inequality in (C8) reduces to

$$\int_0^t g_i(t,s) 2(Ra(s))^{\frac14}\ ds \ge a(t)\cdot \max_{j\in\{1,2\}} M_j\psi(R) = a(t)\cdot M_1(2R^{\frac14}),$$
$$a.e.\ \ t\in[0,1],\ \ i=1,2$$

or

$$\int_0^t g_i(t,s)(a(s))^{\frac14}\ ds \ge a(t)\cdot M_1, \quad a.e.\ \ t\in[0,1],\ \ i=1,2. \tag{3.32}$$

Let

$$a(t) = \left(\frac{1}{12}\right)^{\frac{2}{3}} t^4. \tag{3.33}$$

It is clear that $a(t) > 0$ for *a.e.* $t \in [0,1]$ and $\|a\|_2 = 0.063595 \leq 1$. We shall check that $a(t)$ fulfills (3.32). Indeed, by direct integration $(3.32)|_{i=1}$ $(g_1(t,s) = t - s)$ and $(3.32)|_{i=2}$ $(g_2(t,s) = (t-s)^2)$ respectively reduce to

$$2 \geq t, \quad a.e.\ t \in [0,1]$$

and

$$t^4 \geq t^4, \quad a.e.\ t \in [0,1]$$

which are trivially true. Hence, we have shown that (C8) is satisfied with $a(t)$ defined in (3.33).

Next, the condition (C9) reduces to

$$M_i \psi(\alpha) \leq \alpha, \quad i = 1, 2$$

or equivalently $2M_1 \leq \alpha^{\frac{3}{4}}$, which yields $\alpha \geq 0.48075$. Finally, (C10) leads to

$$\int_0^1 \left[\int_0^t g_i(t,s) 2(\beta a(s))^{\frac{1}{4}}\ ds\right]^2 dt \geq \beta^2,\ i = 1, 2$$

which gives $\beta \leq 0.010461$. Hence, (C9) and (C10) are satisfied if

$$\alpha \geq 0.48075 \qquad \text{and} \qquad \beta \leq 0.010461. \tag{3.34}$$

Since (C1)–(C4) and (C7)–(C10) are satisfied, Theorem 3.3 is applicable and we conclude that (3.29), (3.30) has at least one positive solution $u \in (L^2[0,1])^2$ with

$$\beta \leq \|u\|_2 \leq \alpha \qquad \text{and} \qquad u_i(t) \geq \left[\left(\frac{1}{12}\right)^{\frac{2}{3}} t^4\right]\beta, \tag{3.35}$$
$$a.e.\ t \in [0,1],\ i = 1, 2.$$

Noting the ranges in (3.34), it follows from (3.35) that

$$0.010461 \le \|u\|_2 \le 0.48075 \quad \text{and}$$
$$u_i(t) \ge \left[\left(\frac{1}{12}\right)^{\frac{2}{3}} t^4\right](0.010461), \quad a.e.\ t \in [0,1],\ i = 1,2.$$

□

Example 3.2. Consider the system

$$\text{(3.36)} \quad \begin{cases} u_1(t) = \int_0^t st^{-\frac{3}{2}} \left[(u_1(s))^{\frac{1}{4}} + (u_2(s))^{\frac{1}{4}}\right] ds, & a.e.\ t \in [0,1] \\ u_2(t) = \int_0^t s^2t^{-\frac{3}{2}} \left[(u_1(s))^{\frac{1}{4}} + (u_2(s))^{\frac{1}{4}}\right] ds, & a.e.\ t \in [0,1]. \end{cases}$$

Here,

$$n = 2, \quad T = 1, \quad g_1(t,s) = st^{-\frac{3}{2}},$$
$$g_2(t,s) = s^2t^{-\frac{3}{2}} \quad \text{and} \quad f(t,u) = u_1^{\frac{1}{4}} + u_2^{\frac{1}{4}}.$$

Suppose we are interested in seeking positive solutions of (3.36) in $(L^2[0,1])^2$. Thus, we fix $\theta_i = 1,\ 1 \le i \le n$ and $p = p_1 = p_2 = 2$.

Clearly, conditions (C1), (C3) and (C7) are satisfied. Condition (C2) also holds with $b_i = 0$ and $c_{ij} = 1$. Condition (C4) is satisfied with $M_1 = \sqrt{\frac{1}{3}}$ and $M_2 = \sqrt{\frac{1}{15}}$.

From Example 3.1, we shall use the ψ defined in (3.31) in conditions (C8) and (C9).

As before, we see that the inequality in (C8) reduces to (3.32). Let

$$\text{(3.37)} \qquad a(t) = \left(\frac{3}{16}\right)^{\frac{2}{3}} t^4.$$

It is clear that $a(t) > 0$ for $a.e.\ t \in [0,1]$ and $\|a\|_2 = 0.10920 \le 1$. We shall check that $a(t)$ fulfills (3.32). Indeed, by direct integration $(3.32)|_{i=1}$ and $(3.32)|_{i=2}$ respectively reduce to

$$\frac{4}{3} \geq t^{\frac{5}{2}}, \quad a.e.\ t \in [0,1]$$

and

$$1 \geq t^{\frac{3}{2}}, \quad a.e.\ t \in [0,1]$$

which are trivially true. Hence, we have shown that (C8) is satisfied with $a(t)$ defined in (3.37).

Finally, by direct computation we see that (C9) and (C10) are satisfied if

$$\alpha \geq 1.2114 \qquad \text{and} \qquad \beta \leq 0.082852. \tag{3.38}$$

Now that (C1)–(C4) and (C7)–(C10) are satisfied, we can apply Theorem 3.3 to conclude that (3.36) has at least one positive solution $u \in (L^2[0,1])^2$ with

$$\beta \leq \|u\|_2 \leq \alpha \quad \text{and} \quad u_i(t) \geq \left[\left(\frac{3}{16}\right)^{\frac{2}{3}} t^4\right]\beta, \quad a.e.\ t \in [0,1],\ i = 1,2. \tag{3.39}$$

Noting the ranges in (3.38), it follows from (3.39) that

$$0.082852 \leq \|u\|_2 \leq 1.2114 \quad \text{and} \quad u_i(t) \geq \left[\left(\frac{3}{16}\right)^{\frac{2}{3}} t^4\right](0.082852), \quad a.e.\ t \in [0,1],\ i = 1,2.$$

□

4. Existence in Orlicz Space We shall consider the system (1.1) where, for each $1 \leq i \leq n$, $g_i : [0,T] \times [0,t] \to \mathbb{R}^{N\times N}$ is a matrix valued kernel function and $f_i : [0,T] \times (\mathbb{R}^N)^n \to \mathbb{R}^N$ is a single-valued nonlinear function. We shall establish the existence of a constant-sign solution $u \in (L_\phi)^n$ of (1.1) where L_ϕ is an Orlicz space. Throughout this section, let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ be fixed.

Let $x = (x_1, x_2, \cdots, x_N)^T$ and $y = (y_1, y_2, \cdots, y_N)^T$ be in $\mathbb{R}^N$. Throughout, by $x \geq y$ we shall mean $x_i \geq y_i$ for each $1 \leq i \leq N$. Similarly, if $x, y \in \mathbb{R}^{N\times N}$ (real $N \times N$ matrices), then $x \geq y$ also means inequality in the componentwise sense.

Let B be a Banach space. Let the operator $S : B \to (\mathbb{R}^N)^n$ be defined by

$$Su(t) = (S_1u(t), S_2u(t), \cdots, S_nu(t)), \quad a.e.\ t \in [0, T] \tag{4.1}$$

where

$$S_iu(t) = \int_0^t g_i(t, s)f_i(s, u(s))ds, \quad a.e.\ t \in [0, T], \quad 1 \leq i \leq n. \tag{4.2}$$

Clearly, a fixed point of the operator S is a solution of the system (1.1). We observe that the operator S_i can be written as

$$S_i = A_iF_i \tag{4.3}$$

where $F_i : B \to \mathbb{R}^N$ is defined by

$$F_iu(t) = f_i(t, u(t)), \quad a.e.\ t \in [0, T] \tag{4.4}$$

and $A_i : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$A_ix(t) = \int_0^t g_i(t, s)x(s)ds, \quad a.e.\ t \in [0, T]. \tag{4.5}$$

Our first result is a general existence principle in B. The main tool used is the Leray-Schauder alternative (Theorem 2.1).

Theorem 4.1. Let $X = (X, |\cdot|_X)$ be a Banach space and let $X^n = X \times X \times \cdots \times X$ (n times) be equipped with the norm $\|\cdot\|$ where

$$\|u\| = \max_{1\leq i\leq n} |u_i|_X, \quad u \in X^n.$$

Let Y be a Banach space. For each $1 \leq i \leq n$, suppose

$$F_i : X^n \to Y \qquad \text{and} \qquad A_i : Y \to X \tag{4.6}$$

and

(4.7) $\quad A_iF_i : X^n \to X$ is continuous and completely continuous.

Moreover, assume there is a positive constant M_0, independent of λ, with

$$\|u\| \neq M_0 \tag{4.8)}$$

for any solution $u \in X^n$ of the system

$$u_i = \lambda A_i F_i u, \quad a.e. \tag{4.9$_\lambda$}$$

where $1 \leq i \leq n$ and $0 < \lambda < 1$. Then, the system (1.1) has a solution $u^* \in X^n$ with $\|u^*\| \leq M_0$.

Proof. Clearly, a solution of $(4.9)_\lambda$ is a fixed point of the equation $u = \lambda Su$ where S is defined in (4.1), (4.2). Now (4.7) guarantees that S is continuous and completely continuous. In the context of Theorem 2.1, let

$$U = \{u \in X^n \mid \|u\| < M_0\}.$$

Since $\|u\| \neq M_0$, where u is any solution of $(4.9)_\lambda$, we cannot have conclusion (b) of Theorem 2.1, hence conclusion (a) of Theorem 2.1 must hold, i.e., the system (1.1) has a solution $u^* \in \overline{U}$ with $\|u^*\| \leq M_0$. □

We shall now tackle the existence of a solution u of (1.1) with $u_i \in X,\ 1 \leq i \leq n$ where X is an *Orlicz* space. To introduce Orlicz spaces, we require the following definition.

Definition 4.1. A function P is called an *N-function* if it admits a representation

$$P(u) = \int_0^{|u|} p(t)dt$$

where the function p is right continuous for $t \geq 0$, positive for $t > 0$ and nondecreasing and satisfies the conditions

$$p(0) = 0, \qquad \lim_{t\to\infty} p(t) = \infty.$$

The functions $P_1(u) = \frac{|u|^\alpha}{\alpha}$, $\alpha > 1$ and $P_2(u) = e^{|u|} - |u| - 1$ are examples of N-functions.

Let p be as above. Let $q(s) = \sup_{p(t)\leq s} t$. The functions

$$P(u) = \int_0^{|u|} p(t)dt, \qquad Q(v) = \int_0^{|v|} q(s)ds$$

are called *complementary N-functions.* Note $N_1(v) = \frac{|v|^\beta}{\beta}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, is the N-function complementary to the N-function P_1 whereas $N_2(v) = (1 + |v|)\ln(1 + |v|) - |v|$ is the N-function complementary to the N-function P_2.

Now, let P and Q be complementary N-functions. The *Orlicz class*, denoted by $\mathcal{O}_P$, consists of measurable functions $y : [0, d] \to \mathbb{R}^N$ for which

$$\rho(y; P) = \int_0^d P(y(x))dx < \infty.$$

We shall denote by $L_P([0, d], \mathbb{R}^N)$ the *Orlicz space* of all measurable functions $y : [0, d] \to \mathbb{R}^N$ for which

$$|y|_{P,d} = \sup_{\substack{\rho(y;Q)\leq 1 \\ v\in\mathcal{O}_Q}} \left| \int_0^d y(x) \cdot v(x)dx \right| < \infty.$$

It is known that $\big(L_P([0, d], \mathbb{R}^N),\ |\cdot|_{P,d}\big)$ is a Banach space [14]. Let $E_P([0, d], \mathbb{R}^N)$ be the closure in $L_P([0, d], \mathbb{R}^N)$ of the set of all bounded functions. Note that $E_P \subseteq L_P \subseteq \mathcal{O}_P$. We have $E_P = L_P = \mathcal{O}_P$ if P satisfies the $(\triangle_2)$ condition, which is

$(\triangle_2)$ there exist $\omega,\ y_0 \geq 0$ such that for $y \geq y_0$, we have $P(2y) \leq \omega P(y)$.

Our first existence result in *Orlicz* space employs Theorem 4.1.

Theorem 4.2. Let P and Q be complementary N-functions. Assume

(H1) ϕ is an N-function, and Q satisfies the $(\triangle_2)$ condition;

(H2) for each $1 \le i \le n$, $g_i(t,\cdot) \in E_P([0,T],\mathbb{R}^{N\times N})$ for $a.e.$ $t \in [0,T]$, and the function $t \mapsto |g_i(t,\cdot)|_{P,T}$ belongs to $E_\phi([0,T],\mathbb{R})$;

(H3) for each $1 \le i \le n$, $f_i : [0,T] \times (\mathbb{R}^N)^n \to \mathbb{R}^N$ is a Carathéodory function, i.e.,

(i) the map $t \mapsto f_i(t,u)$ is measurable for all $u \in (\mathbb{R}^N)^n$,

(ii) the map $u \mapsto f_i(t,u)$ is continuous for almost all $t \in [0,T]$;

(H4) for each $r > 0$ and $1 \le i \le n$, there exists $\eta_{r,i} \in L_Q([0,T],\mathbb{R})$ and $K_{r,i} \ge 0$ such that

$$|f_i(t,u)| \le \eta_{r,i}(t) + K_{r,i}\, Q^{-1}\left(\phi\left(\frac{|u_i|}{r}\right)\right),\ a.e.\ t \in [0,T],\ \text{all } u \in (\mathbb{R}^N)^n.$$

Moreover, assume there is a positive constant M_0, independent of λ, with

$$\|u\|_{\phi,T} = \max_{1\le i\le n} |u_i|_{\phi,T} \ne M_0 \tag{4.10}$$

for any solution u of $(4.9)_\lambda$. Then, the system (1.1) has a solution $u^* \in \big(E_\phi([0,T],\mathbb{R}^N)\big)^n$ with $\|u^*\|_{\phi,T} \le M_0$.

Proof. It follows immediately from Lemma 16.3 and Theorem 16.3 of [14] (take $M_1 = Q,\ M_2 = \phi$ and $N_1 = P$) that $A_i : E_Q([0,T],\mathbb{R}^N) \to E_\phi([0,T],\mathbb{R}^N)$ is continuous and completely continuous. Let

$$U = \left\{u \in \big(E_\phi([0,T],\mathbb{R}^N)\big)^n \;\Big|\; \|u\|_{\phi,T} < M_0\right\}. \tag{4.11}$$

Applying Theorem 17.6 in [14] (with $M_2 = Q$ and $M_1 = \phi$), we deduce that $F_i : \overline{U} \to E_Q([0,T],\mathbb{R}^N)$ is continuous and F_i maps bounded sets into bounded sets. Thus $A_iF_i : \overline{U} \to E_\phi([0,T],\mathbb{R}^N)$ is continuous (A_i is continuous and F_i is also continuous) and completely continuous (A_i is completely continuous and F_i maps bounded sets into bounded sets). With $X = E_\phi([0,T],\mathbb{R}^N)$ and $Y = E_Q([0,T],\mathbb{R}^N)$, the result now follows from Theorem 4.1. □

Remark 4.1. By placing other conditions on g_i and f_i (see [14, Sections 15, 16, 17]) we may deduce other existence results in an Orlicz space.

Our next two results apply Theorem 4.2.

Theorem 4.3. Let P and Q be complementary N-functions. Let (H1)–(H4) hold. Assume

(H5) for each $1 \le i \le n$, $\hat{g}_i(t) \equiv |g_i(t,\cdot)|_{P,t}$ is bounded for $t \in [0,T]$;

(H6) for each $r > 0$ and $1 \le i \le n$,

$$\frac{1}{2}\int_0^T [1+\phi(b_{r,i}(t))]\,dt < \int_0^\infty \left[1+\phi\left(\frac{2\hat{g}_i K_{r,i}}{r}\ x\right)\right]^{-1} dx$$

where

$$\hat{g}_i = \sup_{t\in[0,T]} \hat{g}_i(t) \qquad \text{and} \qquad b_{r,i}(t) = \frac{2\hat{g}_i(t)}{r}\left(|\eta_{r,i}|_{Q,t} + K_{r,i}\right);$$

(H7) for each $1 \le i \le n$,

$$\sup_{r\in(0,\infty)} \frac{r}{|q_i|_{\phi,T}\left\{|\eta_{r,i}|_{Q,T}+K_{r,i}+K_{r,i}\ I_{r,i}^{-1}\left(\frac{1}{2}\int_0^T[1+\phi(b_{r,i}(t))]\,dt\right)\right\}} > 1$$

where

$$q_i(t) = |g_i(t,\cdot)|_{P,T} \qquad \text{and} \qquad I_{r,i}(z) = \int_0^z \left[1+\phi\left(\frac{2\hat{g}_i K_{r,i}}{r}\ x\right)\right]^{-1} dx.$$

Then, the system (1.1) has a solution $u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$.

Proof. In view of (H7), for any $1 \le i \le n$ there exists a positive constant M_0 such that

$$\frac{M_0}{|q_i|_{\phi,T}\left\{|\eta_{M_0,i}|_{Q,T}+K_{M_0,i}+K_{M_0,i}\ I_{M_0,i}^{-1}\left(\frac{1}{2}\int_0^T[1+\phi(b_{M_0,i}(t))]\,dt\right)\right\}} > 1. \tag{4.12}$$

For each $1 \le i \le n$, define $A_i : E_Q([0,T],\mathbb{R}^N) \to E_\phi([0,T],\mathbb{R}^N)$ by (4.5), and $F_i : \left(E_\phi([0,T],\mathbb{R}^N)\right)^n \to E_Q([0,T],\mathbb{R}^N)$ by (4.4). Let u be a solution of $(4.9)_\lambda$ for some $\lambda \in (0,1)$ with $\|u\|_{\phi,T} = M_0$. Then, there exists some $j \in \{1, 2, \cdots, n\}$ such that $|u_j|_{\phi,T} = M_0$.

By Lemma 16.3(a) of [14] (with $M_2 = \phi,\ N_1 = P,\ M_1 = Q$), we have

$$|A_i v|_{\phi,T} \le |q_i|_{\phi,T} \cdot |v|_{Q,T}, \quad 1 \le i \le n. \tag{4.13}$$

Moreover, using [14, Theorem 10.5 with $k = 1$] gives

$$\left| Q^{-1}\left(\phi\left(\frac{|u_i|}{M_0}\right)\right)\right|_{Q,t} \le 1 + \int_0^t \phi\left(\frac{|u_i(s)|}{M_0}\right) ds, \ \ 1 \le i \le n. \tag{4.14}$$

Now applying $(\mathrm{H4})|_{r=M_0}$, (4.13) and (4.14), we find

$$\begin{aligned} |u_j|_{\phi,T} &\le \left| \int_0^t g_j(t,s) f_j(s,u(s)) ds \right|_{\phi,T} \\ &\le \left| \int_0^t g_j(t,s) \left\{ \eta_{M_0,j}(s) + K_{M_0,j} Q^{-1}\left(\phi\left(\frac{|u_j|}{M_0}\right)\right)\right\} ds \right|_{\phi,T} \\ &\le |q_j|_{\phi,T} \left\{ |\eta_{M_0,j}|_{Q,T} + K_{M_0,j} \left| Q^{-1}\left(\phi\left(\frac{|u_j|}{M_0}\right)\right)\right|_{Q,T} \right\} \\ &\le |q_j|_{\phi,T} \left\{ |\eta_{M_0,j}|_{Q,T} + K_{M_0,j} \left[1 + \int_0^T \phi\left(\frac{|u_j(s)|}{M_0}\right) ds \right] \right\}. \end{aligned} \tag{4.15}$$

On the other hand, using (H4), Hölder's inequality and (4.14), it follows from $(4.9)_\lambda$ that for *a.e.* $t \in [0,T]$,

$$\begin{aligned} |u_j(t)| &\le \left| \int_0^t g_j(t,s) f_j(s,u(s)) ds \right| \\ &\le \hat{g}_j(t) \left\{ |\eta_{M_0,j}|_{Q,t} + K_{M_0,j} \left| Q^{-1}\left(\phi\left(\frac{|u_j|}{M_0}\right)\right)\right|_{Q,t} \right\} \\ &\le \hat{g}_j(t) \left(|\eta_{M_0,j}|_{Q,t} + K_{M_0,j} \right) + \hat{g}_j K_{M_0,j} \int_0^t \phi\left(\frac{|u_j(s)|}{M_0}\right) ds. \end{aligned} \tag{4.16}$$

Noting that $\phi(x+y) \le \frac{1}{2}[\phi(2x)+\phi(2y)]$ for $x, y \ge 0$, from (4.16) we obtain for *a.e.* $t \in [0,T]$,

$$\phi\left(\frac{|u_j(t)|}{M_0}\right) \le \frac{1}{2}\,\phi\left(b_{M_0,j}(t)\right) + \frac{1}{2}\,\phi\left(\frac{2\hat{g}_j K_{M_0,j}}{M_0}\int_0^t \phi\left(\frac{|u_j(s)|}{M_0}\right) ds\right). \tag{4.17}$$

Let

$$w(t) = \int_0^t \phi\left(\frac{|u_j(s)|}{M_0}\right) ds.$$

Then, (4.17) gives

$$\begin{aligned} 2w'(t) &\le \phi\left(b_{M_0,j}(t)\right) + \phi\left(\frac{2\hat{g}_j K_{M_0,j}}{M_0}\; w(t)\right) \\ &\le \left[1+\phi\left(b_{M_0,j}(t)\right)\right] \cdot \left[1+\phi\left(\frac{2\hat{g}_j K_{M_0,j}}{M_0}\; w(t)\right)\right] \end{aligned}$$

or

$$w'(t)\left[1+\phi\left(\frac{2\hat{g}_j K_{M_0,j}}{M_0}\; w(t)\right)\right]^{-1} \le \frac{1}{2}\left[1+\phi\left(b_{M_0,j}(t)\right)\right]$$

which, upon integrating from 0 to T, provides

$$\begin{aligned} \int_0^{w(T)} \left[1+\phi\left(\frac{2\hat{g}_j K_{M_0,j}}{M_0}\; x\right)\right]^{-1} dx \;&=\; I_{M_0,j}(w(T)) \\ &\le \frac{1}{2}\int_0^T \left[1+\phi\left(b_{M_0,j}(t)\right)\right] dt < I_{M_0,j}(\infty) \text{ (by (H6))}. \end{aligned}$$

Since $I_{M_0,j}$ is strictly increasing, it follows immediately from above that

$$w(T) = \int_0^T \phi\left(\frac{|u_j(s)|}{M_0}\right) ds \le I^{-1}_{M_0,j}\left(\frac{1}{2}\int_0^T \left[1+\phi\left(b_{M_0,j}(t)\right)\right] dt\right). \tag{4.18}$$

Now, we substitute (4.18) into (4.15) to get

$$\begin{aligned} M_0 = |u_j|_{\phi,T} &\le |q_j|_{\phi,T}\left\{|\eta_{M_0,j}|_{Q,T} + K_{M_0,j}[1+w(T)]\right\} \\ &\le |q_j|_{\phi,T}\Bigg\{|\eta_{M_0,j}|_{Q,T} \\ &\qquad + K_{M_0,j} + K_{M_0,j}\; I^{-1}_{M_0,j}\left(\frac{1}{2}\int_0^T \left[1+\phi\left(b_{M_0,j}(t)\right)\right] dt\right)\Bigg\} \\ &< M_0 \text{ (by (4.12))} \end{aligned}$$

which is a contradiction. Hence, any solution u of $(4.9)_\lambda$ must satisfy $\|u\|_{\phi,T} \neq M_0$. The condition (4.10) is satisfied and the conclusion is now immediate from Theorem 4.2. ◻

Theorem 4.4. *Let P and Q be complementary N-functions. Let* (H1)–(H4) *hold. Assume* (H8) *for each $1 \le i \le n$,*

$$\sup_{r\in(0,\infty)} \frac{r}{|q_i|_{\phi,T}\left(|\eta_{r,i}|_{Q,T} + 2K_{r,i}\right)} > 1$$

where $q_i(t) = |g_i(t,\cdot)|_{P,T}$. Then, the system (1.1) *has a solution $u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$.*

Proof. In view of (H8), for any $1 \le i \le n$ there exists a positive constant M_0 such that

$$\frac{M_0}{|q_i|_{\phi,T}\left(|\eta_{M_0,i}|_{Q,T} + 2K_{M_0,i}\right)} > 1. \tag{4.19}$$

As in the proof of Theorem 4.3, let u be a solution of $(4.9)_\lambda$ for some $\lambda \in (0,1)$ with $\|u\|_{\phi,T} = M_0$. Also, let $j \in \{1,2,\cdots,n\}$ be such that $\|u\|_{\phi,T} = |u_j|_{\phi,T} = M_0$.

Using a similar argument as before, we obtain (4.15). Now, we employ Lemma 9.2 in [14] to get

$$\int_0^T \phi\left(\frac{|u_j(s)|}{M_0}\right) ds \le \frac{|u_j|_{\phi,T}}{M_0} = \frac{M_0}{M_0} = 1. \tag{4.20}$$

Substituting (4.20) into (4.15) immediately leads to

$$M_0 = |u_j|_{\phi,T} \le |q_j|_{\phi,T}\left(|\eta_{M_0,j}|_{Q,T} + 2K_{M_0,j}\right) < M_0 \text{ (by (4.19))},$$

a contradiction.

Hence, any solution u of $(4.9)_\lambda$ must satisfy $\|u\|_{\phi,T} \neq M_0$, i.e., we have the condition (4.10). The conclusion is now immediate from Theorem 4.2. ◻

Remark 4.2. Let p (>1) and q be integers such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the case $n = 1$. In Theorem 3.2, the existence of a solution in $L^p[0,T]$ is established using the conditions (see (C2) and (C6))

(4.21) $$|f(t,u)| \le \eta_r(t) + M_r\ |u|^{\frac{p}{q}}, \quad a.e.\ t \in [0,T],\ \text{all } u \in \mathbb{R}$$

and

(4.22) $$\sup_{r\in(0,\infty)} \left(\frac{r}{a_0 + a_1\ r^{\frac{p}{q}}} \right) > 1$$

where a_0 and a_1 are some fixed constants. We remark that our Theorems 4.3 and 4.4, which give the existence of a solution in an *Orlicz* space, are 'analogous' to Theorem 3.2 in the sense that if we let $\phi(x) = \frac{|x|^p}{p}$, then $Q(x) = \frac{|x|^q}{q}$, and so $Q^{-1}(x) = (qx)^{\frac{1}{q}}$ for $x \ge 0$. Then, (H4) with

$$K_r = M_r \cdot r^{\frac{p}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}}$$

reduces to (4.21), since with this K_r we have

$$K_r\ Q^{-1}\left(\phi\left(\frac{|u|}{r}\right)\right) = M_r\ |u|^{\frac{p}{q}}.$$

Moreover, condition (H8) is 'parallel' to (4.22).

Remark 4.3. It is also possible to prove Theorem 4.4 using Schauder's fixed point theorem.

Remark 4.4. In Theorems 4.3 and 4.4, one could replace (H4) with the following condition:

(H4)$'$ there exists $\epsilon > 0$ and for each $1 \le i \le n$, there exists $\eta_{M_0+\epsilon,i} \in L_Q([0,T],\mathbb{R})$ and $K_{M_0+\epsilon,i} \ge 0$ such that

$$|f_i(t,u)| \le \eta_{M_0+\epsilon,i}(t) + K_{M_0+\epsilon,i}\ Q^{-1}\left(\phi\left(\frac{|u_i|}{M_0+\epsilon}\right)\right), \quad a.e.\ t \in [0,T],\ \text{all } u \in (\mathbb{R}^N)^n$$

where M_0 is as in (4.12) (for Theorem 4.3) or as in (4.19) (for Theorem 4.4).

Remark 4.5. The conditions (H7) and (H8) are respectively equivalent to

(H7)$'$ there exists $\alpha > 0$ such that for each $1 \le i \le n$,

$$|q_i|_{\phi,T} \left\{ |\eta_{\alpha,i}|_{Q,T} + K_{\alpha,i} + K_{\alpha,i}\ I_{\alpha,i}^{-1} \left(\frac{1}{2} \int_0^T [1 + \phi(b_{\alpha,i}(t))]\, dt \right) \right\} < \alpha$$

where q_i and $I_{\alpha,i}$ are defined in (H7);

(H8)$'$ there exists $\alpha > 0$ such that for each $1 \le i \le n$,

$$|q_i|_{\phi,T} \left(|\eta_{\alpha,i}|_{Q,T} + 2K_{\alpha,i} \right) < \alpha.$$

Moreover, with (H7)$'$ and (H8)$'$ the conclusion of Theorems 4.3 and 4.4 becomes: the system (1.1) has a solution $u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ such that $\|u\|_{\phi,T} < \alpha$.

Till now we have employed the Leray-Schauder alternative (Theorem 2.1) to obtain existence criteria in $\left(E_\phi([0,T],\mathbb{R}^N)\right)^n$. We shall next apply Krasnosel'skii's fixed point theorem (Theorem 2.2) to get further existence results for *constant-sign* solutions in $\left(E_\phi([0,T],\mathbb{R}^N)\right)^n$.

Consider the Banach space $\left(\left(E_\phi\left([0,T],\right)^N\right)^n, \|\cdot\|_{\phi,T}\right)$. Define a cone in $\left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ as

$$C_\gamma = \left\{ u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n \;\middle|\; \text{for each } 1 \le i \le n, \\ \theta_i u_i(t) \ge \gamma(t)\|u\|_{\phi,T} \text{ for } a.e.\ t \in [0,T] \right\}. \tag{4.23}$$

Here we assume

$$\gamma(t) > 0,\ a.e.\ t \in [0,T], \qquad \gamma \in E_\phi([0,T],\mathbb{R}^N) \qquad \text{and} \qquad |\gamma|_{\phi,T} \le 1.$$

More conditions on $\gamma(t)$ will be presented later. It is clear that a fixed point of the operator S in C_γ is a *constant-sign* solution of (1.1) in $\left(E_\phi([0,T],\mathbb{R}^N)\right)^n$.

Let $0 < \beta < \alpha$. Define

$$\Omega_\alpha = \left\{ u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n \;\middle|\; \|u\|_{\phi,T} < \alpha \right\}$$

and

$$\Omega_\beta = \left\{ u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n \;\middle|\; \|u\|_{\phi,T} < \beta \right\}.$$

Our next lemma shows that S maps $C_\gamma \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$ into C_γ.

Lemma 4.1. *Let P and Q be complementary N-functions. Let (H1)–(H4) hold. Assume*

(H9) *for each $1 \le i \le n$, $g_i(t,s) : [0,T] \times [0,t] \to [0,\infty)^{N\times N}$ is such that the map $(t,s) \to g_i(t,s)$ is measurable, and $g_i(t,s) > 0$, a.e. $t \in [0,T]$, a.e. $s \in [0,t]$;*

(H10) *for each $1 \le i \le n$,*

$$\theta_i f_i(t,u) > 0, \quad \text{a.e. } t \in [0,T], \text{ all } u \in \hat{K}$$

where

$$\hat{K} = \left\{ u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n \;\middle|\; \text{for each } \ 1 \le i \le n, \right.$$
$$\left. \theta_i u_i(t) > 0 \text{ for } \quad \text{a.e. } t \in [0,T] \right\};$$

moreover, $\theta_i f_i$ is 'nondecreasing' in the sense that if $x \le \theta_j u_j \le y$ for some $j \in \{1,2,\cdots,n\}$, then for a.e. $t \in [0,T]$,

$$\begin{aligned}\theta_i f_i(t,u_1,\cdots,\theta_j x,\cdots,u_n) &\le \theta_i f_i(t,u_1,\cdots,u_j,\cdots,u_n)\\ &\le \theta_i f_i(t,u_1,\cdots,\theta_j y,\cdots,u_n);\end{aligned}$$

(H11) *there exists a function $\gamma \in E_\phi([0,T],\mathbb{R}^N)$ with $\gamma(t) > 0$ for a.e. $t \in [0,T]$ and $|\gamma|_{\phi,T} \le 1$ such that the following holds for each $1 \le i \le n$ and any $R > 0$,*

$$\begin{aligned}&\int_0^t g_i(t,s)\theta_i f_i(s,\theta_1 R\gamma(s),\theta_2 R\gamma(s),\cdots,\theta_n R\gamma(s))ds\\ &\qquad \ge \gamma(t)\cdot \max_{1\le j\le n} |q_j|_{\phi,T}\left(|\eta_{R,j}|_{\phi,T} + 2K_{R,j}\right), \quad \text{a.e. } t \in [0,T].\end{aligned}$$

Then, the operator S maps $C_\gamma \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$ into C_γ.

Proof. Let $u \in C_\gamma \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. We already have $Su \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ (Theorem 4.2). Moreover, in view of (H9) and (H10), it follows that

$$\theta_i(S_i u)(t) = \int_0^t g_i(t,s)\theta_i f_i(s,u(s))ds \geq 0, \qquad a.e.\ t \in [0,T],\ 1 \leq i \leq n. \tag{4.24}$$

Since $u \in C_\gamma \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$, there exists $R \in [\beta, \alpha]$
such that

$$\|u\|_{\phi,T} = R \qquad \text{and} \qquad \theta_i u_i(t) \geq \gamma(t)R > 0, \qquad a.e.\ \ t \in [0,T],\ 1 \leq i \leq n. \tag{4.25}$$

It follows from (H9) and (H10) that for $a.e.$ $t \in [0,T]$ and $1 \leq i \leq n$,

$$\theta_i(S_i u)(t) \geq \int_0^t g_i(t,s)\theta_i f_i(s, \theta_1 R\gamma(s), \theta_2 R\gamma(s), \cdots, \theta_n R\gamma(s))ds. \tag{4.26}$$

On the other hand, using (H4)$|_{r=R}$, (4.13) and (4.14)$|_{M_0=R}$, we find for $1 \leq i \leq n$,

$$\begin{aligned} |S_i u|_{\phi,T} &= \left| \int_0^t g_i(t,s) f_i(s,u(s))ds \right|_{\phi,T} \\ &\leq \left| \int_0^t g_i(t,s) \left\{ \eta_{R,i}(s) + K_{R,i}\ Q^{-1}\left(\phi\left(\frac{|u_i|}{R}\right)\right)\right\} ds \right|_{\phi,T} \\ &\leq |q_i|_{\phi,T} \left\{ |\eta_{R,i}|_{Q,T} + K_{R,i} \left| Q^{-1}\left(\phi\left(\frac{|u_i|}{R}\right)\right)\right|_{Q,T} \right\} \\ &\leq |q_i|_{\phi,T} \left\{ |\eta_{R,i}|_{Q,T} + K_{R,i} \left[1 + \int_0^T \phi\left(\frac{|u_i(s)|}{R}\right) ds \right] \right\}. \end{aligned} \tag{4.27}$$

Now, we employ Lemma 9.2 in [14] to get

$$(4.28) \qquad \int_0^T \phi\left(\frac{|u_i(s)|}{R}\right) ds \le \frac{|u_i|_{\phi,T}}{R} \le 1.$$

Hence, from (4.27) we get

$$|S_i u|_{\phi,T} \le |q_i|_{\phi,T} \left(|\eta_{R,i}|_{Q,T} + 2K_{R,i}\right), \ 1 \le i \le n$$

which implies

$$(4.29) \qquad \|Su\|_{\phi,T} \le \max_{1\le j\le n} |q_j|_{\phi,T} \left(|\eta_{R,j}|_{Q,T} + 2K_{R,j}\right).$$

Now, using (4.29) in (4.26) yields

$$\begin{aligned}\theta_i(S_i u)(t) &\ge \frac{\int_0^t g_i(t,s)\theta_i f_i(s,\theta_1 R\gamma(s),\theta_2 R\gamma(s),\cdots,\theta_n R\gamma(s))ds}{\max_{1\le j\le n} |q_j|_{\phi,T}\left(|\eta_{R,j}|_{Q,T}+2K_{R,j}\right)}\|Su\|_{\phi,T}\\ &\ge \gamma(t)\|Su\|_{\phi,T}, \quad a.e.\ t\in[0,T],\ 1\le i\le n\end{aligned}$$

where the last inequality follows from (H11). This completes the proof. □

We are now ready to apply Theorem 2.2 to get the existence of a *constant-sign* solution in $\left(E_\phi([0,T],\mathbb{R}^N)\right)^n$.

Theorem 4.5. *Let P and Q be complementary N-functions. Let (H1)–(H4) and (H9)–(H11) hold. Assume*

(H12) *there exists $\delta > 0$ such that $\phi\left(\frac{x}{y}\right) \ge \delta\,\frac{\phi(x)}{\phi(y)}$ for $x, y \ge 0$;*

(H13) *there exists $\alpha > 0$ such that for each $1 \le i \le n$,*

$$|q_i|_{\phi,T}\left(|\eta_{\alpha,i}|_{Q,T} + 2K_{\alpha,i}\right) \le \alpha;$$

(H14) *there exists $\beta > 0$ such that for each $1 \le i \le n$,*

$$\phi^{-1}\left(\delta\int_0^T \phi\left(\int_0^s g_i(s,\tau)\theta_i f_i(\tau,\theta_1\beta\gamma(\tau),\theta_2\beta\gamma(\tau),\cdots,\theta_n\beta\gamma(\tau))d\tau\right)ds\right) \ge \beta$$

(of course we also assume the above integral exists).

Then, the system (1.1) *has at least one constant-sign solution* $u \in \big(E_\phi([0,T],\mathbb{R}^N)\big)^n$ *such that*

(a) $\alpha \le \|u\|_{\phi,T} \le \beta$ *and* $\theta_i u_i(t) \ge \gamma(t)\alpha,$ *a.e.* $t \in [0,T],\ 1 \le i \le n$ *if* $\alpha < \beta$;

(a) $\beta \le \|u\|_{\phi,T} \le \alpha$ *and* $\theta_i u_i(t) \ge \gamma(t)\beta,$ *a.e.* $t \in [0,T],\ 1 \le i \le n$ *if* $\beta < \alpha$.

Proof. Without any loss of generality, let $\beta < \alpha$. Since $S : C_\gamma \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_\gamma$ is continuous and completely continuous (Theorem 4.2 and Lemmas 4.1), it suffices to show that (i) $\|Su\|_{\phi,T} \le \|u\|_{\phi,T}$ for $u \in C_\gamma \cap \partial\Omega_\alpha$, and (ii) $\|Su\|_{\phi,T} \ge \|u\|_{\phi,T}$ for $u \in C_\gamma \cap \partial\Omega_\beta$.

To prove (i), let $u \in C_\gamma \cap \partial\Omega_\alpha$. Then,

$$\|u\|_{\phi,T} = \alpha \quad \text{and} \quad \theta_i u_i(t) \ge \gamma(t)\alpha > 0, \quad a.e.\ t \in [0,T],\ 1 \le i \le n.$$

As in the proof of Lemma 4.1, we obtain $(4.29)|_{R=\alpha}$, which, together with (H13) yields

$$\|Su\|_{\phi,T} \le \max_{1\le j\le n} |q_j|_{\phi,T}\,(|\eta_{\alpha,j}|_{Q,T} + 2K_{\alpha,j}) \le \alpha = \|u\|_{\phi,T}.$$

Next, we shall verify (ii). Let $u \in C_\gamma \cap \partial\Omega_\beta$. Then,

$$\|u\|_{\phi,T} = \beta \qquad \text{and} \qquad \theta_i u_i(t) \ge \gamma(t)\beta > 0,\ a.e.\ t \in [0,T],\ 1 \le i \le n.$$

Fix $1 \le i \le n$. As in the proof of Lemma 4.1, we get $(4.26)|_{R=\beta}$. Since $\phi(x)$ is an increasing function of x for $x \ge 0$, it follows that

$$\begin{aligned}(4.30)\quad &\phi(|S_i u(t)|)\\ &\ge \phi\left(\int_0^t g_i(t,s)\theta_i f_i(s,\theta_1\beta\gamma(s),\theta_2\beta\gamma(s),\cdots,\theta_n\beta\gamma(s))ds\right),\ a.e.\ t \in [0,T].\end{aligned}$$

Now, applying Lemma 9.2 in [14] gives

$$\int_0^T \phi\left(\frac{|S_i u(s)|}{\|Su\|_{\phi,T}}\right) ds \le \frac{|S_i u|_{\phi,T}}{\|Su\|_{\phi,T}} \le 1. \tag{4.31}$$

Also, using (H12) we find

$$\int_0^T \phi\left(\frac{|S_i u(s)|}{\|Su\|_{\phi,T}}\right) ds \ge \delta \int_0^T \frac{\phi(|S_i u(s)|)}{\phi(\|Su\|_{\phi,T})}\, ds. \tag{4.32}$$

Coupling (4.31) and (4.32) leads to

$$1 \ge \delta \int_0^T \frac{\phi(|S_i u(s)|)}{\phi(\|Su\|_{\phi,T})}\, ds$$

which, together with (4.30), provides

$$\phi(\|Su\|_{\phi,T}) \ge \delta \int_0^T \phi(|S_i u(s)|) ds$$
$$\ge \delta \int_0^T \phi\left(\int_0^s g_i(s,\tau)\theta_i f_i(\tau, \theta_1\beta\gamma(\tau), \theta_2\beta\gamma(\tau), \cdots, \theta_n\beta\gamma(\tau))d\tau\right) ds.$$

In view of (H14), it follows that

$$\|Su\|_{\phi,T}$$
$$\ge \phi^{-1}\left(\delta \int_0^T \phi\left(\int_0^s g_i(s,\tau)\theta_i f_i(\tau, \theta_1\beta\gamma(\tau), \theta_2\beta\gamma(\tau), \cdots, \theta_n\beta\gamma(\tau))d\tau\right) ds\right)$$
$$\ge \beta \ = \ \|u\|_{\phi,T}.$$

Having established (i) and (ii), we conclude from Theorem 2.2 that S has a fixed point $u \in C_\gamma \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. Therefore, conclusion (b) follows immediately. □

Remark 4.6. In (H13) if we have *strict* inequality instead, i.e., condition $(\text{H8})'$, then from the proof of Theorem 4.5 we see that a fixed point u of S must satisfy $\|u\|_{\phi,T} \ne \alpha$. Likewise, if the inequality in (H14) is *strict*, i.e.,

$$\phi^{-1}\left(\delta\int_0^T\phi\left(\int_0^s g_i(s,\tau)\theta_i f_i(\tau,\theta_1\beta\gamma(\tau),\theta_2\beta\gamma(\tau),\cdots,\theta_n\beta\gamma(\tau))d\tau\right)ds\right)>\beta,$$

then a fixed point u of S must fulfill $\|u\|_{\phi,T} \neq \beta$. Hence, with *strict* inequalities in (H13) and (H14), the conclusion of Theorem 4.5 becomes: the system (1.1) has at least one constant-sign solution $u \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ such that

(a) $\alpha < \|u\|_{\phi,T} < \beta$ and $\theta_i u_i(t) > \gamma(t)\alpha, \quad a.e.\ t \in [0,T],\ 1 \le i \le n$ if $\alpha < \beta$;

(b) $\beta < \|u\|_{\phi,T} < \alpha$ and $\theta_i u_i(t) > \gamma(t)\beta, \quad a.e.\ t \in [0,T],\ 1 \le i \le n$ if $\beta < \alpha$.

Our next two result gives the existence of *multiple* constant-sign solutions of (1.1) in $\left(E_\phi([0,T],\mathbb{R}^N)\right)^n$.

Theorem 4.6. *Let P and Q be complementary N-functions. Assume* (H1)–(H6) *and* (H9)–(H12) *hold. Let* (H7)′ *be satisfied for $\alpha = \alpha_0$,* (H13) *be satisfied for $\alpha = \alpha_\ell,\ \ell = 1,2,\cdots,k$, and* (H14) *be satisfied for $\beta = \beta_\ell,\ \ell = 1,2,\cdots,m$.*

(a) *If $m = k+1$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1}$, then (1.1) has (at least) $2k$ constant-sign solutions $u^1,\cdots,u^{2k} \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ such that*

$$\beta_1 \le \|u^1\|_{\phi,T} \le \alpha_1 \le \|u^2\|_{\phi,T} \le \beta_2 \le \cdots \le \alpha_k \le \|u^{2k}\|_{\phi,T} \le \beta_{k+1}.$$

(b) *If $m = k$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k$, then (1.1) has (at least) $2k-1$ constant-sign solutions $u^1,\cdots,u^{2k-1} \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ such that*

$$\beta_1 \le \|u^1\|_{\phi,T} \le \alpha_1 \le \|u^2\|_{\phi,T} \le \beta_2 \le \cdots \le \beta_k \le \|u^{2k-1}\|_{\phi,T} \le \alpha_k.$$

(c) *If $m = k+1$ and $0 < \alpha_0 < \beta_1 < \alpha_1 < \cdots < \alpha_{m-1} < \beta_m$, then (1.1) has (at least) $2m$ constant-sign solutions $u^0,\cdots,u^{2m-1} \in \left(E_\phi([0,T],\mathbb{R}^N)\right)^n$ such that*

$$0 \le \|u^0\|_{\phi,T} < \alpha_0 < \|u^1\|_{\phi,T} \le \beta_1 \le \|u^2\|_{\phi,T} \\ \le \alpha_1 \le \cdots \le \alpha_{m-1} \le \|u^{2m-1}\|_{\phi,T} \le \beta_m.$$

(d) *If* $m = k$ *and* $0 < \alpha_0 < \beta_1 < \alpha_1 < \cdots < \beta_m < \alpha_m$, *then (1.1) has (at least)* $2m+1$ *constant-sign solutions* $u^0, \cdots, u^{2m} \in \left(E_\phi([0,T], \mathbb{R}^N)\right)^n$ *such that*

$$0 \le \|u^0\|_{\phi,T} < \alpha_0 < \|u^1\|_{\phi,T} \le \beta_1 \le \|u^2\|_{\phi,T} \\ \le \alpha_1 \le \cdots \le \beta_m \le \|u^{2m}\|_{\phi,T} \le \alpha_m.$$

Proof. In (a) and (b), we just apply Theorem 4.5 repeatedly. In (c) and (d), Theorem 4.3 (and Remark 4.5) is used to obtain the existence of $u^0 \in \left(E_\phi([0,T], \mathbb{R}^N)\right)^n$ with $0 \le \|u^0\|_{\phi,T} < \alpha_0$, the results then follow by repeated use of Theorem 4.5. □

Theorem 4.7. *Let* P *and* Q *be complementary* N*-functions. Assume* (H1)–(H4) *and* (H9)–(H12) *hold. Let* (H8)′ *be satisfied for* $\alpha = \alpha_0$, (H13) *be satisfied for* $\alpha = \alpha_\ell$, $\ell = 1, 2, \cdots, k$, *and* (H14) *be satisfied for* $\beta = \beta_\ell$, $\ell = 1, 2, \cdots, m$. *Then, the conclusions* (a)–(d) *of Theorem* 4.6 *hold.*

Proof. The proof is similar to that of Theorem 4.6, with an application of Theorem 4.4 instead of Theorem 4.3. □

Remark 4.7. Suppose in Theorems 4.6 and 4.7 we have some *strict* inequalities in (H13) and (H14), say, involving α_i and β_j for some $i \in \{1, 2, \cdots, k\}$ and some $j \in \{1, 2, \cdots, m\}$. Then, noting Remark 4.6, those inequalities in the conclusion involving α_i and β_j will also be *strict*.

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Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, Florida 32901-6975
Email address: agarwal@fit.edu

Department of Mathematics, National University of Ireland, Galway, Ireland
Email address: donal.oregan@nuigalway.ie

School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore
Email address: ejywong@ntu.edu.sg