JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 20, Number 1, Spring 2008

FAST COLLOCATION METHODS FOR HIGH-DIMENSIONAL WEAKLY SINGULAR INTEGRAL EQUATIONS

ZHONGYING CHEN, BIN WU AND YUESHENG XU

Communicated by Kendall Atkinson

Dedicated to Professor Qun Lin on the occasion
of his 70th birthday with friendship and esteem

ABSTRACT. We realize fast collocation methods for solving Fredholm integral equations of the second kind with weakly singular kernels on a polyhedral domain in $\mathbf{R}^d$ with $d \geq 3$. A polyhedral domain is subdivided into a finite number of simplices. We construct a uniform self-similar partition of a simplex for the purpose of constructing multi-scale bases and their corresponding collocation functionals. The multi-scale bases and the collocation functionals lead to a compression of the matrix representation of the weakly singular integral operator and thus to a fast collocation scheme for solving the integral equation. We develop a quadrature rule for computing the weakly singular integrals appearing in the matrix.

2000 AMS *Mathematics subject classification.* Primary 65B05, 45L10.
Keywords and phrases. Fredholm integral equations of the second kind, high dimension, fast collocation methods, multi-scale methods.
The first author was supported in part by the Natural Science Foundation of China under grant 10771224, the Foundation of Doctoral Program of National Higher Education of China under grant 20030558008, Guangdong Provincial Natural Science Foundation of China under grant 05003308 and the Foundation of Sun Yat-sen University Advanced Research Center. The second author was supported in part by the Natural Science Foundation of China under grants 10771224 and 10601070, by the MOE Project of Key Research Institute of Humanities and Social Sciences at Universities and by China-France-Russia mathematics collaboration grant 34000-3275100. The third author was Supported in part by the US National Science Foundation under grant CCR-0407476, by National Aeronautics and Space Administration under Cooperative Agreement NNX07AC37A, by the Natural Science Foundation of China under grants 10371122 and 10631080 and by the Education Ministry of the People's Republic of China under the Changjiang Scholar Chair Professorship Program through Sun Yat-sen University. The third author is the corresponding author.
Received by the editors on October 9, 2006, and in revised form on February 2, 2007.
DOI:10.1216/JIE-2008-20-1-49

We propose an error control strategy for the numerical integration so that the nearly optimal convergence order for the discrete fast collocation method is obtained. Finally, a numerical experiment of solving the three-dimensional equation is presented to confirm the theoretical estimates.

1. Introduction. We consider in this paper solving Fredholm integral equations of the second kind with weakly singular kernels on a polyhedral domain in $\mathbf{R}^d$ for $d \geq 3$ by using fast collocation methods. Equations of this type are of importance in many engineering application areas [**1, 24**]. Collocation methods are widely used in solving the equations due to their significant computational efficiency and attractive convergence properties, cf. [**1, 2, 4**]. The coefficient matrix for the linear system obtained from the standard collocation method of the integral equation is a full matrix. Generating the full matrix requires computing N^2 integrals among which there are $\mathcal{O}(N)$ singular integrals, where N is the size of the matrix. When N is large, the standard collocation method is too costly to be used in practice.

Aiming at designing fast collocation methods, a general setting of the fast collocation method for solving the equation was developed in [**11**] and appropriate basis functions and collocation functionals are constructed in the paper. By appropriately choosing basis functions and collocation functionals so that they have multi-scale structures and certain order of vanishing moments, the coefficient matrix can be approximated by a sparse matrix having only $\mathcal{O}(N \log N)$ number of nonzero entries. The optimal order (up to a logarithmic factor) of convergence for the approximate solution resulting from the compressed sparse coefficient matrix was proved in the paper. The quasi-linear order of the computational complexity for the method was estimated. It was also shown that the condition number of the compressed sparse coefficient matrix is in order of $\log^2 N$. Realizations of the method for one-dimensional equations and that for two-dimensional equations was presented in [**14, 38**], respectively. See also [**13**] for the control of numerical quadratures for the one-dimensional equations. A multilevel augmentation method and a multi-level iteration method were proposed respectively in [**12, 17**], to efficiently solving the linear system resulting in the fast collocation method for integral equations. The fast collocation method was used in [**16**] to solve an inverse boundary value problem. Collocation methods based on wavelets were proposed in [**20, 33, 35**].

In this paper, as in our previous work [**6, 11**], we assume that the solution of the weakly singular integral equation is in certain Sobolev spaces. Such an assumption is normally not met if the boundary has corners and edges. Regularity analysis for solutions of Fredholm integral equations are found in [**22, 32, 36, 37**]. However, there are many cases when the solution is in a Sobolev space. For example, solutions of some boundary integral equations on a smooth boundary are smooth even though the kernels have singularities [**1**]. The main purpose of this paper and our previously published papers in the same context is to understand how we compress the coefficient matrix resulting from the collocation method when the kernel has a weak singularity so that the fast solutions give a nearly optimal convergence order. The assumption that the solution is in a Sobolev space helps us isolate the difficulty caused by the singular kernel from the difficulty caused by the singular solution. A study on the treatment of using wavelets for solutions that are not in a Sobolev space is a future research topic.

Realization of the fast collocation method for integral equations on a polyhedral domain in $\mathbf{R}^d$ with $d \geq 3$ is a challenging task. It requires the availability of multi-scale basis functions and the corresponding multi-scale collocation functionals and efficient numerical integration methods for computing high-dimensional weakly singular integrals. It is the purpose of this paper to study these issues. Noting that a d-dimensional polyhedral domain can be decomposed as the union of a finite number of d-dimensional simplices, for convenience of presentation, we will only present our method for integral equations on a simplex. It is straightforward to extend it to an arbitrary polyhedral domain. Remarks on the extension will be given.

This paper is organized into six sections. In Section 2, we present a uniform self-similar partition for the standard d-dimensional simplex. Based on such a partition, we describe in Section 3 the construction of multi-scale bases and the corresponding collocation functionals on a simplex. Specific constructions for several examples of important applications are presented. The fast collocation scheme using these multi-scale bases and collocation functionals is described in Section 4. We also improve the analysis of convergence and computational complexity for the fast collocation method presented in [**11**] by removing a hypothesis.

The entries of the coefficient matrix resulted from the fast collocation method are all nearly weakly singular. In Section 5, a quadrature rule for computing the weakly singular integral is developed. An error control strategy for the numerical integration is designed to preserve the nearly optimal order of convergence and computational complexity for the discrete fast collocation method. A numerical experiment on the implementation of a three-dimensional equation is presented in Section 6 to confirm the theoretical estimates.

2. A uniform partition of the simplex. The realization of the fast collocation method requires the availability of multi-scale bases and collocation functionals on the simplex having a multi-scale structure. The construction of these bases and functionals demands a uniform self-similar partition of the simplex. For general self-similar sets, see [**21**] and also see [**7, 26, 27, 28**] for wavelet constructions on fractal sets. In this section we construct such a partition for the d-dimensional simplex. We first describe the partition strategy of the unit simplex and prove that the partition has some uniformness property. We then extend the partition strategy to a general simplex through affine mappings.

For a vector $x \in \mathbf{R}^d$, we write $x = [x_j \in \mathbf{R} : j \in \mathbf{Z}_d]$, where $\mathbf{Z}_d := \{0, 1, \ldots, d-1\}$. The unit simplex S in $\mathbf{R}^d$ is the subset $S := \{x \in \mathbf{R}^d : 0 \le x_0 \le x_1 \cdots \le x_{d-1} \le 1\}$. In order to partition S, for a positive integer μ, we define a family of *counting functions* $\chi_j : \mathbf{Z}_\mu^d \to \mathbf{Z}_{d+1}$, $j \in \mathbf{Z}_\mu$ for $\mathbf{e} := (e_0, e_1, \ldots, e_{d-1}) \in \mathbf{Z}_\mu^d$ by

$$\chi_j(\mathbf{e}) = \sum_{i \in \mathbf{Z}_d} \delta_j(e_i), \tag{2.1}$$

where $\delta_j(k) = 1$ when $j = k$ and otherwise $\delta_j(k) = 0$. The value of $\chi_j(\mathbf{e})$ is exactly the number of components of $\mathbf{e}$ that equals to j. Given $\mathbf{e} \in \mathbf{Z}_\mu^d$, we identify a vector $\mathbf{c}(\mathbf{e}) := [c_j : j \in \mathbf{Z}_{\mu+1}] \in \mathbf{Z}_{d+1}^{\mu+1}$ by

$$c_0 = 0, \quad c_j = \sum_{i \in \mathbf{Z}_j} \chi_i(\mathbf{e}), \;\; j = 1, 2, \ldots, \mu. \tag{2.2}$$

We remark that $\mathbf{c}(\mathbf{e})$ is always nondecreasing since each χ_j takes nonnegative value, and c_μ is always equal to d. For $e \in \mathbf{Z}_\mu^d$ and $j < k$, we define the index set $\Psi_j^k := \{e_l : j \le l < k, e_l = e_k\}$. Then we define the *permutation vector* $\mathbf{I_e} = [i_k : k \in \mathbf{Z}_d] \in \mathbf{Z}_d^d$ of $\mathbf{e}$ by

$$i_k = c_{e_k} + |\Psi_0^k|, \tag{2.3}$$

where $|A|$ denotes the cardinality of set A, and we assume $\mathrm{card}\,(\varnothing) = 0$. We have the following lemma about $\mathbf{I_e}$.

Lemma 2.1. *For any $\mathbf{e} \in \mathbf{Z}_\mu^d$, the permutation vector $\mathbf{I_e}$ has the following properties.*

(1) *For $k \in \mathbf{Z}_d$, $c_m \le i_k < c_{m+1}$ if and only if $m = e_k$.*

(2) *For any $j, k \in \mathbf{Z}_d$, $i_j < i_k$ if and only if $e_j < e_k$ or $e_j = e_k$ with $j < k$.*

(3) *The equality $i_j = i_k$ holds if and only if $j = k$.*

(4) *The vector $\mathbf{I_e}$ is a permutation of $\mathbf{v}_d := [j : j \in \mathbf{Z}_d]$.*

Proof. According to the definition of $\mathbf{I_e}$, we have for any $k \in \mathbf{Z}_k$ that

$$c_{e_k} \le i_k < c_{e_k} + |\{e_j : j \in \mathbf{Z}_d, e_j = e_k\}| = c_{e_k+1}. \tag{2.4}$$

This implies that if $m = e_k$, $c_m \le i_k < c_{m+1}$. On the other hand, if there is an m such that $c_m \le i_k < c_{m+1}$, it is unique because the components of $\mathbf{c}(\mathbf{e})$ are nondecreasing. It follows from the uniqueness of m and (2.4) that $m = e_k$. Thus, property (1) is proved.

We now turn to proving property (2). If $e_j < e_k$, then $e_j + 1 \le e_k$ and hence $c_{e_j+1} \le c_{e_k}$ since the component of $\mathbf{c}(\mathbf{e})$ is a nondecreasing sequence. By (2.4) we conclude that $i_j < c_{e_j+1} \le c_{e_k} \le i_k$. If $e_j = e_k$ with $j < k$, then $i_k - i_j = |\Psi_j^k| \ge 1$, hence $i_j < i_k$. It remains to prove that if $i_j < i_k$ then $e_j < e_k$ or $e_j = e_k$, $j < k$. Since in general for $j, k \in \mathbf{Z}_d$ one of the following cases holds: $e_j < e_k$, $e_j = e_k$ with $j < k$, $e_j = e_k$ with $j \ge k$, or $e_j > e_k$, it suffices to show that if $e_j > e_k$ or $e_j = e_k$ with $j \ge k$, then $i_j \ge i_k$. If $e_j > e_k$, by the proof we showed earlier in this paragraph, we conclude that $i_j > i_k$. If $e_j = e_k$ with $j \ge k$, we have that $i_j - i_k = |\Psi_k^j| \ge 0$, that is, $i_j \ge i_k$. Thus, we complete a proof for property (2).

The above analysis also implies that the only possibility to have $i_j = i_k$ is $j = k$. This proves property (3).

Noticing that $e_k \in \mathbf{Z}_\mu$ for $k \in \mathbf{Z}_d$ and $0 \le c_{e_k} \le i_k < c_{e_k+1} \le d$, we conclude that $\mathbf{I_e}$ is a permutation of $\mathbf{v}_d$. $\square$

We next partition the unit simplex S. Associated with each $\mathbf{e} \in \mathbf{Z}_\mu^d$, we define a set in $\mathbf{R}^d$ by

$$S_{\mathbf{e}} := \left\{ \mathbf{x} \in \mathbf{R}^d : 0 \le x_{i_0} - \frac{e_0}{\mu} \le x_{i_1} - \frac{e_1}{\mu} \le \cdots \le x_{i_{d-1}} - \frac{e_{d-1}}{\mu} \le \frac{1}{\mu} \right\}, \tag{2.5}$$

where i_k, $k \in \mathbf{Z}_d$, are the components of the permutation vector $\mathbf{I_e}$ of $\mathbf{e}$. Since $\mathbf{I_e}$ is a permutation of $\mathbf{v}_d$, $S_{\mathbf{e}}$ is a simplex in $\mathbf{R}^d$. In the following lemma we present properties of the family of simplices $S_{\mathbf{e}}$, $\mathbf{e} \in \mathbf{Z}_\mu^d$.

Lemma 2.2. *The simplices $S_{\mathbf{e}}$, $\mathbf{e} \in \mathbf{Z}_\mu^d$ have the properties*:

(1) *For any $\mathbf{x} \in S_{\mathbf{e}}$, there holds*

$$\frac{k}{\mu} \le x_{c_k} \le x_{c_k+1} \le \cdots \le x_{c_{k+1}-1} \le \frac{k+1}{\mu}, \quad k \in \mathbf{Z}_\mu. \tag{2.6}$$

(2) *For any $\mathbf{e} \in \mathbf{Z}_\mu^d$, $S_{\mathbf{e}} \subset S$.*

(3) *If $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{Z}_\mu^d$ with $\mathbf{e}_1 \ne \mathbf{e}_2$, then* $\operatorname{int}(S_{\mathbf{e_1}}) \cap \operatorname{int}(S_{\mathbf{e_2}}) = \varnothing$.

(4) *For any $\mathbf{e} \in \mathbf{Z}_\mu^d$,* $\operatorname{meas}(S_{\mathbf{e}}) = 1/(\mu^d \cdot d!)$, *where* $\operatorname{meas}(\Omega)$ *denotes the Lebesgue measure of set Ω.*

Proof. In order to prove (2.6), it suffices to show

$$0 \le x_{c_k} - \frac{k}{\mu} \le x_{c_k+1} - \frac{k}{\mu} \le \cdots \le x_{c_{k+1}-1} - \frac{k}{\mu} \le \frac{1}{\mu}, \quad k \in \mathbf{Z}_\mu, \tag{2.7}$$

or equivalently,

$$0 \le x_p - \frac{k}{\mu} \le x_q - \frac{k}{\mu} \le \frac{1}{\mu}$$

for any $c_k \le p < q < c_{k+1}$. In fact, since $\mathbf{I_e}$ is a permutation of $\mathbf{v}_d$, for any integers $c_k \le p < q < c_{k+1}$, there exists a unique pair $p', q' \in \mathbf{Z}_d$ such that $i_{p'} = p$, $i_{q'} = q$. It follows from Lemma 2.1 that $e_{p'} = e_{q'} = k$ and $p' < q'$. Thus (2.5) states that

$$0 \le x_p - \frac{k}{\mu} = x_{i_{p'}} - \frac{e_{p'}}{\mu} \le x_{i_{q'}} - \frac{e_{q'}}{\mu} = x_q - \frac{k}{\mu} \le \frac{1}{\mu},$$

which concludes property (1).

Property (2) is a direct consequence of (1) and the definition of S.

For the proof of (3), we first notice that

$$\text{(2.8)} \quad \operatorname{int}(S_{\mathbf{e}}) = \left\{\mathbf{x} \in \mathbf{R}^d : 0 < x_{i_0} - \frac{e_0}{\mu} < x_{i_1} - \frac{e_1}{\mu} < \cdots < x_{i_{d-1}} - \frac{e_{d-1}}{\mu} < \frac{1}{\mu}\right\}.$$

Moreover, by a proof similar to the one for (2.6), we utilize (2.8) to conclude for any $\mathbf{x} \in \operatorname{int}(S_{\mathbf{e}})$ that

$$\text{(2.9)} \qquad \frac{k}{\mu} < x_{c_k} < x_{c_k+1} < \cdots < x_{c_{k+1}-1} < \frac{k+1}{\mu}, \quad k \in \mathbf{Z}_\mu.$$

For $j = 1, 2$ we denote $\mathbf{e}_j = (e_0^j, \ldots, e_{d-1}^j)$, $\mathbf{I}_{\mathbf{e}_j} = (i_0^j, \ldots, i_{d-1}^j)$, $\mathbf{c}(\mathbf{e}_j) = (c_0^j, \ldots, c_\mu^j)$.

Assume to the contrary that $\operatorname{int}(S_{\mathbf{e}_1}) \cap \operatorname{int}(S_{\mathbf{e}_2})$ is not empty. We consider two cases. In Case 1 if $\mathbf{c}(\mathbf{e}_1) \neq \mathbf{c}(\mathbf{e}_2)$, we let k be the smallest integer such that $c_k^1 \neq c_k^2$ and assume $c_k^1 < c_k^2$ without loss of generality. For any $\mathbf{x} \in \operatorname{int}(S_{\mathbf{e}_1}) \cap \operatorname{int}(S_{\mathbf{e}_2})$, by (2.9), we have $x_{c_k^1} > k/\mu$ and $x_{c_k^2-1} < k/\mu$. On the other hand, because $\mathbf{x} \in S$, we have that $x_{c_k^1} \leq x_{c_k^2-1}$, a contradiction. In Case 2 if $\mathbf{c}(\mathbf{e}_1) = \mathbf{c}(\mathbf{e}_2)$, since $\mathbf{e}_1 \neq \mathbf{e}_2$, we let k be the smallest integer such that $e_k^1 \neq e_k^2$. Hence, $e_j^1 = e_j^2$ for $j < k$, and we assume that $e_k^1 < e_k^2$ without loss of generality. Thus, we have that $i_k^1 < c^1_{e_k^1+1} \leq c^2_{e_k^2} \leq i_k^2$. There exists a unique $p \in \mathbf{Z}_d$ such that $i_p^1 = i_k^2$ since $\mathbf{I}_{\mathbf{e}_1}$ is a permutation, and $p \geq k$ because $i_j^1 = i_j^2 \neq i_k^2$ for all $j < k$. Furthermore, it follows from Lemma 2.1, $\mathbf{c}(\mathbf{e}_1) = \mathbf{c}(\mathbf{e}_2)$ and $i_p^1 = i_k^2$ that $e_p^1 = e_k^2 \neq e_k^1$, which implies $p \neq k$. Therefore, for any $\mathbf{x} \in \operatorname{int}(S_{\mathbf{e}_1})$, there holds

$$x_{i_k^1} - \frac{e_k^1}{\mu} < x_{i_p^1} - \frac{e_p^1}{\mu} = x_{i_k^2} - \frac{e_k^2}{\mu}.$$

On the other hand, there is a unique $q \in \mathbf{Z}_d$ such that $q > k$, $i_q^2 = i_k^1$, and for any $\mathbf{x} \in \operatorname{int}(S_{\mathbf{e}_2})$,

$$x_{i_k^2} - \frac{e_k^2}{\mu} < x_{i_q^2} - \frac{e_q^2}{\mu} = x_{i_k^1} - \frac{e_k^1}{\mu},$$

again, a contradiction. This completes the proof of property (3).

For property (4), we find by a direct computation that $\operatorname{meas}(S'_{\mathbf{e}}) = 1/(\mu^d d!)$, where

$$S'_{\mathbf{e}} := \left\{\mathbf{x} \in \mathbf{R}^d : 0 \le x_{i_0} \le x_{i_1} \le \cdots \le x_{i_{d-1}} \le \frac{1}{\mu}\right\}.$$

Notice that $S_{\mathbf{e}}$ is the translation of simplex $S'_{\mathbf{e}}$ through the vector $\mathbf{e}/\mu$. Since the Lebesgue measure of a set is invariant under translation, we thus conclude property (4). □

Theorem 2.3. *The family* $S(\mathbf{Z}_\mu^d) := \{S_{\mathbf{e}} : \mathbf{e} \in \mathbf{Z}_\mu^d\}$ *is an equi-volume partition of the unit simplex* S.

Proof. By Lemma 2.2, we see that for any $\mathbf{e} \in \mathbf{Z}_\mu^d$, $S_{\mathbf{e}} \subset S$, and for $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{Z}_\mu^d$ with $\mathbf{e}_1 \neq \mathbf{e}_2$, $\operatorname{int}(S_{\mathbf{e}_1}) \cap \operatorname{int}(S_{\mathbf{e}_2}) = \varnothing$ and $\operatorname{meas}(S_{\mathbf{e}_1}) = \operatorname{meas}(S_{\mathbf{e}_2})$. It remains to prove that $S \subseteq \cup_{\mathbf{e}\in\mathbf{Z}_\mu^d} S_{\mathbf{e}}$.

To this end, for each $\mathbf{x} \in S$ we will find $\mathbf{e} \in \mathbf{Z}_\mu^d$ such that $\mathbf{x} \in S_{\mathbf{e}}$. Note that for each $\mathbf{x} \in S$ we have $0 \le x_0 \le x_1 \le \cdots \le x_{d-1} \le 1$. For each $k \in \mathbf{Z}_\mu$ we denote by c_k the subscript of the smallest component x_j greater than or equal to k/μ. We order the elements in set $\{x_j : j \in \mathbf{Z}_d\} \cup \{k/\mu : k \in \mathbf{Z}_{\mu+1}\}$ in increasing order. We then obtain that

$$\begin{aligned} 0 &\le x_0 \le \cdots \le x_{c_1-1} \\ &\le \frac{1}{\mu} \le x_{c_1} \le \cdots \le x_{c_{\mu-1}-1} \\ &\le \frac{\mu-1}{\mu} \le x_{c_{\mu-1}} \le \cdots \le x_{c_\mu - 1} \\ &= x_{d-1} \le 1. \end{aligned}$$

In other words, we have that

$$0 \le x_{c_k} - \frac{k}{\mu} \le x_{c_k+1} - \frac{k}{\mu} \le \cdots \le x_{c_{k+1}-1} - \frac{k}{\mu} \le \frac{1}{\mu}, \quad k \in \mathbf{Z}_\mu. \tag{2.10}$$

Denote $p_j := \max\{k : c_k \le j\}$. It follows from (2.10) that the set $\{x_j - (p_j/\mu) : j \in \mathbf{Z}_d\} \subset [0, (1/\mu)]$. We sort the elements of this set into

$$0 \le x_{i_0} - \frac{p_{i_0}}{\mu} \le x_{i_1} - \frac{p_{i_1}}{\mu} \le \cdots \le x_{i_{d-1}} - \frac{p_{i_{d-1}}}{\mu} \le \frac{1}{\mu}. \tag{2.11}$$

Notice that the vector $\mathbf{I} := (i_0, i_1, \ldots, i_{d-1})$ is a permutation of $\mathbf{v}_d$. Let $\mathbf{e} := (e_0, \ldots, e_{d-1})$ be a vector such that $e_j = p_{i_j}$. It is easy to verify $i_j = c_{e_j} + |\Psi_0^j|$. Hence, $\mathbf{I} = \mathbf{I}_\mathbf{e}$, which together with (2.11) shows $\mathbf{x} \in S_\mathbf{e}$. $\square$

In the rest of this section, we consider the important affine mappings from S to $S_\mathbf{e}$, which are utilized to define linear operators for the recursive construction of the multi-scale basis functions and functionals from a lower level to higher levels. A permutation matrix has exactly one entry in each row and column equal to 1 and all other entries being zero, cf. [**19**]. Hence, a permutation matrix is an orthogonal matrix. For any permutation $\mathbf{I}_\mathbf{e}$ of $\mathbf{v}_d$, there is a unique permutation matrix $P_\mathbf{e}$ such that $\mathbf{I}_\mathbf{e} = P_\mathbf{e}\mathbf{v}_d$. We call the vector

$$\mathbf{I}_\mathbf{e}^* = (i_0^*, \ldots, i_{d-1}^*) := P_\mathbf{e}^T \mathbf{v}_d$$

the *conjugate permutation* of $\mathbf{I}_\mathbf{e}$. Thus, $\mathbf{I}_\mathbf{e}^*$ itself is also a permutation of $\mathbf{v}_d$. It follows from the definition above that for $l \in \mathbf{Z}_d$, $i_l^* = k$ if and only if $i_k = l$. We define the *conjugate vector* $\mathbf{e}^* := (e_0^*, e_1^*, \ldots, e_{d-1}^*)$ of $\mathbf{e}$ by setting $e_l^* = e_{i_l^*}$, $l \in \mathbf{Z}_d$. Utilizing the above notations, we define the mapping $\mathcal{G}_\mathbf{e}$ by

$$\mathcal{G}_\mathbf{e}(\mathbf{x}) := \tilde{\mathbf{x}} = \left[\tilde{x}_l = \frac{x_{i_l^*} + e_l^*}{\mu} : l \in \mathbf{Z}_d\right], \quad \mathbf{x} \in S. \tag{2.12}$$

We intend to show $\mathcal{G}_\mathbf{e}(S) = S_\mathbf{e}$. Indeed, for $k \in \mathbf{Z}_d$, we let $l = i_k$ and observe by definition that $i_l^* = k$, $e_l^* = e_k$. Thus $\tilde{x}_l = (x_k + e_k)/\mu$, or

$$x_k = \mu\tilde{x}_l - e_k = \mu\tilde{x}_{i_k} - e_k. \tag{2.13}$$

If $\mathbf{x} \in S$, then $0 \le x_0 \le x_1 \le \cdots \le x_{d-1} \le 1$, which implies that

$$0 \le \mu\tilde{x}_{i_0} - e_0 \le \mu\tilde{x}_{i_1} - e_1 \le \cdots \le \mu\tilde{x}_{i_{d-1}} - e_{d-1} \le 1,$$

or

$$0 \le \tilde{x}_{i_0} - \frac{e_0}{\mu} \le \tilde{x}_{i_1} - \frac{e_1}{\mu} \le \cdots \le \tilde{x}_{i_{d-1}} - \frac{e_{d-1}}{\mu} \le \frac{1}{\mu},$$

so that $\tilde{\mathbf{x}} \in S_\mathbf{e}$. On the other hand, given $\tilde{\mathbf{x}} \in S_\mathbf{e}$, we define $\mathbf{x} := [x_k : k \in \mathbf{Z}_d]$ by equation (2.13). Thus, $\mathbf{x} \in S$ and $\tilde{\mathbf{x}} = \mathcal{G}_\mathbf{e}(\mathbf{x})$. Therefore, $\mathcal{G}_\mathbf{e}(S) = S_\mathbf{e}$.

The expression of the inverse mapping $\mathcal{G}_{\mathbf{e}}^{-1}$ has been given by equation (2.13), which is written formally as

$$\mathbf{x} := \mathcal{G}_{\mathbf{e}}^{-1}(\tilde{\mathbf{x}}) = [x_k = \mu\tilde{x}_{i_k} - e_k : k \in \mathbf{Z}_d], \quad \tilde{\mathbf{x}} \in S_{\mathbf{e}}. \tag{2.14}$$

For any $\mathbf{e} \in \mathbf{Z}_\mu^d$ and $\mathbf{x}', \mathbf{x}'' \in \mathbf{R}^d$, there holds

$$\|\mathcal{G}_{\mathbf{e}}(\mathbf{x}') - \mathcal{G}_{\mathbf{e}}(\mathbf{x}'')\|_p = \frac{1}{\mu}\|\mathbf{x}' - \mathbf{x}''\|_p, \tag{2.15}$$

$$\|\mathcal{G}_{\mathbf{e}}^{-1}(\mathbf{x}') - \mathcal{G}_{\mathbf{e}}^{-1}(\mathbf{x}'')\|_p = \mu\|\mathbf{x}' - \mathbf{x}''\|_p, \tag{2.16}$$

where $\|\cdot\|_p$ is the standard ℓ^p norm on $\mathbf{R}^d$ for $1 \le p \le \infty$.

Proposition 2.4. *The family $S(\mathbf{Z}_\mu^d)$ is a uniform partition of the unit simplex S in the sense that all elements of $S(\mathbf{Z}_\mu^d)$ have an identical diameter.*

Proof. We denote $\Delta := \max_{\mathbf{x}', \mathbf{x}'' \in S} \|\mathbf{x}' - \mathbf{x}''\|_p$. It suffices to prove that for any $\mathbf{e} \in \mathbf{Z}_\mu^d$,

$$\max_{\mathbf{x}'_{\mathbf{e}}, \mathbf{x}''_{\mathbf{e}} \in S_{\mathbf{e}}} \|\mathbf{x}'_{\mathbf{e}} - \mathbf{x}''_{\mathbf{e}}\|_p = \frac{\Delta}{\mu}.$$

It follows from formula (2.16) that for any $\mathbf{x}'_{\mathbf{e}}, \mathbf{x}''_{\mathbf{e}} \in S_{\mathbf{e}}$,

$$\mu\|\mathbf{x}'_{\mathbf{e}} - \mathbf{x}''_{\mathbf{e}}\|_p = \|\mathcal{G}_{\mathbf{e}}^{-1}(\mathbf{x}'_{\mathbf{e}}) - \mathcal{G}_{\mathbf{e}}^{-1}(\mathbf{x}''_{\mathbf{e}})\|_p \le \Delta.$$

On the other hand, suppose that $\bar{\mathbf{x}}', \bar{\mathbf{x}}'' \in S$ such that $\|\bar{\mathbf{x}}' - \bar{\mathbf{x}}''\|_p = \Delta$ and let $\bar{\mathbf{x}}'_{\mathbf{e}} := \mathcal{G}_{\mathbf{e}}(\bar{\mathbf{x}}')$, and $\bar{\mathbf{x}}''_{\mathbf{e}} := \mathcal{G}_{\mathbf{e}}(\bar{\mathbf{x}}'')$. By (2.15), we have that

$$\|\bar{\mathbf{x}}'_{\mathbf{e}} - \bar{\mathbf{x}}''_{\mathbf{e}}\|_p = \frac{1}{\mu}\|\bar{\mathbf{x}}' - \bar{\mathbf{x}}''\|_p = \frac{\Delta}{\mu},$$

which completes the proof. ◻

When a partition of the unit simplex has been established, it is not difficult to obtain a corresponding partition of a general simplex in $\mathbf{R}^d$. For a nondegenerate simplex S' in $\mathbf{R}^d$, in the sense $\mathrm{Vol}\,(S') \ne 0$, there exists an affine mapping $\mathcal{F} : \mathbf{R}^d \to \mathbf{R}^d$ such that $\mathcal{F}(S') = S$. It can be

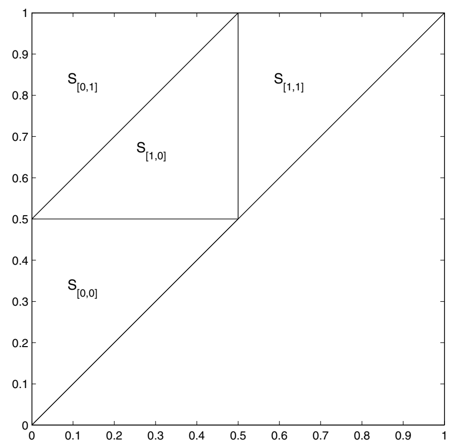

FIGURE 1. The distribution of $S_{\mathrm{e}}, \mathrm{e} \in \mathbf{Z}_2^2$ in S.

shown that for $1 \leq p \leq \infty$ there are two positive constants c_1 and c_2 such that

$$c_1\|\mathbf{x}' - \mathbf{x}''\|_p \leq \|\mathcal{F}(\mathbf{x}') - \mathcal{F}(\mathbf{x}'')\|_p \leq c_2\|\mathbf{x}' - \mathbf{x}''\|_p, \tag{2.17}$$

for any $\mathbf{x}', \mathbf{x}'' \in S'$. For any $\mathbf{e} \in \mathbf{Z}_\mu^d$, we define $\mathcal{G}'_{\mathbf{e}} := \mathcal{F}^{-1} \circ \mathcal{G}_{\mathbf{e}} \circ \mathcal{F}$. Thus, the family of simplices $\{\mathcal{G}'_{\mathbf{e}}(S') : \mathbf{e} \in \mathbf{Z}_\mu^d\}$ is a partition of S'. Furthermore, for any $\mathbf{x}', \mathbf{x}'' \in \mathbf{R}^d$ and $\mathbf{e} \in \mathbf{Z}_\mu^d$, there holds

$$\frac{c_1}{c_2\mu}\|\mathbf{x}' - \mathbf{x}''\|_p \leq \|\mathcal{G}'_{\mathbf{e}}(\mathbf{x}') - \mathcal{G}'_{\mathbf{e}}(\mathbf{x}'')\|_p \leq \frac{c_2}{c_1\mu}\|\mathbf{x}' - \mathbf{x}''\|_p.$$

For $\mathbf{E} := [\mathbf{e}_j : j \in \mathbf{Z}_m] \in (\mathbf{Z}_\mu^d)^m$, we define the composite mappings

$$\mathcal{G}_{\mathbf{E}} := \mathcal{G}_{\mathbf{e_0}} \circ \cdots \circ \mathcal{G}_{\mathbf{e_{m-1}}} \quad \text{and} \quad \mathcal{G}'_{\mathbf{E}} := \mathcal{G}'_{\mathbf{e_0}} \circ \cdots \circ \mathcal{G}'_{\mathbf{e_{m-1}}}$$

and observe that $\mathcal{G}'_{\mathbf{E}} = \mathcal{F}^{-1} \circ \mathcal{G}_{\mathbf{E}} \circ \mathcal{F}$. In the next theorem we show that the partition $\{\mathcal{G}'_{\mathbf{e}}(S') : \mathbf{e} \in \mathbf{Z}_\mu^d\}$ of S' is uniform, and it meets the requirement of the fast collocation method. To this end, we let $S_{\mathbf{E}} := \mathcal{G}_{\mathbf{E}}(S)$ and $S'_{\mathbf{E}} := \mathcal{G}_{\mathbf{E}}(S')$. Also, we use diam_p to denote the diameter of a domain in $\mathbf{R}^d$ with respect to the ℓ^p norm.

Theorem 2.5. *For any* $\mathbf{x}', \mathbf{x}'' \in \mathbf{R}^d$ *and* $\mathbf{E} \in (\mathbf{Z}_\mu^d)^m$, *there hold*

$$\frac{c_1}{c_2}\left(\frac{1}{\mu}\right)^m \|\mathbf{x}' - \mathbf{x}''\|_p \le \|\mathcal{G}'_{\mathbf{E}}(\mathbf{x}') - \mathcal{G}'_{\mathbf{E}}(\mathbf{x}'')\|_p \le \frac{c_2}{c_1}\left(\frac{1}{\mu}\right)^m \|\mathbf{x}' - \mathbf{x}''\|_p,$$

$$\operatorname{diam}_p(S_{\mathbf{E}}) = \left(\frac{1}{\mu}\right)^m \operatorname{diam}_p(S),$$

and

$$\frac{c_1}{c_2}\left(\frac{1}{\mu}\right)^m \operatorname{diam}_p(S') \le \operatorname{diam}_p(S'_{\mathbf{E}}) \le \frac{c_2}{c_1}\left(\frac{1}{\mu}\right)^m \operatorname{diam}_p(S').$$

At the end of this section, we give partitions for important cases in practice. For $d = 1$, the unit simplex is just the unit interval $[0, 1]$. We obtain from (2.13) and (2.14) that

$$\mathcal{G}_j(x) = \frac{x+j}{\mu}, \quad \mathcal{G}_j^{-1}(x) = \mu x - j, \quad j \in \mathbf{Z}_\mu,$$

and the subintervals obtained from the mappings are $S_j = [(j/\mu), ((j+1)/\mu)]$, $j \in \mathbf{Z}_\mu$. For $d = 2$, the unit simplex is the unit triangle $S = \{(x_0, x_1) : 0 \le x_0 \le x_1 \le 1\}$. We only consider the case $\mu = 2$. The expressions of the contractive mappings are listed in Table 1, and the mapped triangles are $S_{\mathbf{e}}$, $\mathbf{e} \in \mathbf{Z}_2^2$. We illustrate their position in S in Figure 1. For $d = 3$, the unit simplex is given by $S = \{(x_0, x_1, x_2) : 0 \le x_0 \le x_1 \le x_2 \le 1\}$. We also restrict ourselves to the case $\mu = 2$. In Table 2, we list the expressions of the eight affine mappings $\mathcal{G}_{\mathbf{e}}, \mathbf{e} \in \mathbf{Z}_2^3$ as well as their inverse mappings. The sub-simplices $S_{\mathbf{e}}$, $\mathbf{e} \in \mathbf{Z}_2^3$ are shown in Figure 2.

TABLE 1. The expressions of the mappings $\mathcal{G}_{\mathbf{e}}$ for $\mu = 2, d = 2$.

e	$\mathrm{C_e}$	$\mathrm{I_e}$	$\mathrm{I_e^*}$	e^*	$\mathcal{G}_{\mathrm{e}}(x_0, x_1)$	$\mathcal{G}_{\mathrm{e}}^{-1}(\tilde{x}_0, \tilde{x}_1)$
$(0,0)$	$(0,2,2)$	$(0,1)$	$(0,1)$	$(0,0)$	$((x_0/2),(x_1/2))$	$(2\tilde{x}_0, 2\tilde{x}_1)$
$(0,1)$	$(0,1,2)$	$(0,1)$	$(0,1)$	$(0,1)$	$((x_0/2),(x_1+1/2))$	$(2\tilde{x}_0, 2\tilde{x}_1 - 1)$
$(1,0)$	$(0,1,2)$	$(1,0)$	$(1,0)$	$(0,1)$	$((x_1/2),(x_0+1/2))$	$(2\tilde{x}_1 - 1, 2\tilde{x}_0)$
$(1,1)$	$(0,0,2)$	$(0,1)$	$(0,1)$	$(1,1)$	$((x_0+1/2),(x_1+1/2))$	$(2\tilde{x}_0 - 1, 2\tilde{x}_1 - 1)$

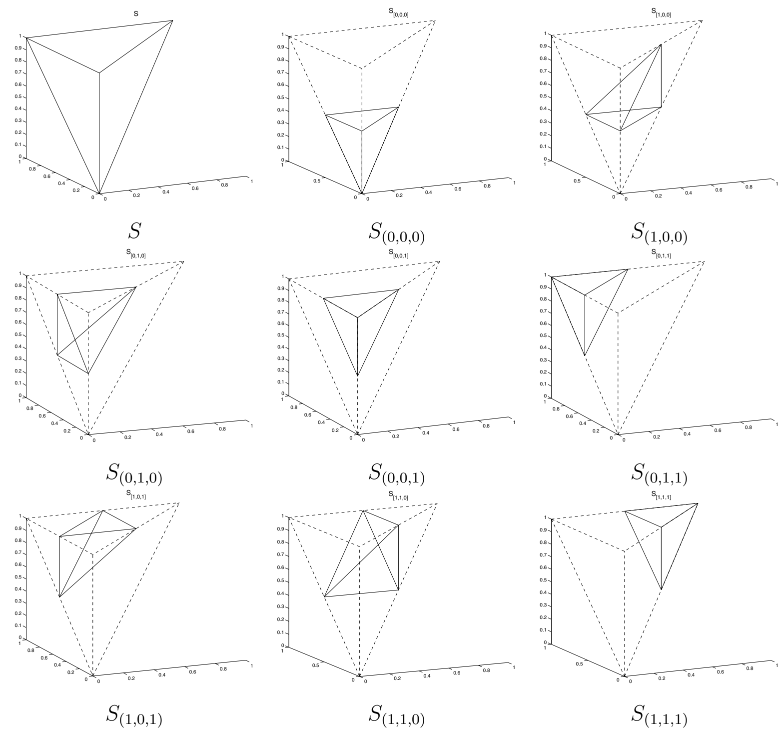

FIGURE 2. A uniform partition of the three-dimensional unit simplex.

3. The multi-scale bases and collocation functionals. In this section, we specialize the general construction of multi-scale bases and their corresponding collocation functionals described in [**11**] to the simplex S in $\mathbf{R}^d$, by using the affine contractive mappings constructed for the simplex in the previous section. For practical use, we also present concrete multi-scale bases and collocation functionals for the cases $d = 1, 2, 3$. The basis functions and collocation functionals on the simplex are used to construct those on a polyhedral domain by subdividing it into a fixed number of simplices. For more information on construction of multi-scale bases and functionals, see [**7, 8, 25, 27, 28**].

We begin with the space $\mathbf{X}_0$ of polynomials of order k, i.e., polynomials of total degree less than k, on S and the family $\Phi := \{\phi_e : e \in \mathbf{Z}_\nu\}$

of affine contractive mappings of cardinality $\nu := \mu^d$, constructed in the previous section, which satisfy

$$S = \Phi(S) := \bigcup_{e \in \mathbf{Z}_\nu} \phi_e(S) \text{ and } \phi_e(S) \cap \phi_{e'}(S) = \varnothing,$$
$$\text{for } e, e' \in \mathbf{Z}_\nu \text{ with } e \neq e'.$$

The dimension of $\mathbf{X}_0$ is $w(0) := \binom{k-1+d}{d}$. A set $G_0 \subseteq \mathbf{R}^d$ is said to be refinable with respect to $\mathbf{\Phi}$ if $G_0 \subseteq \mathbf{\Phi}(G_0)$. The notion of refinable sets was introduced in [**7**] and a concrete construction of G_0 which admits a unique Lagrange interpolatory polynomial of a prescribed order was described in [**25**]. Associated with the space $\mathbf{X}_0$, we construct a refinable set $G_0 := \{s_j : j \in \mathbf{Z}_{w(0)}\}$ with respect to $\mathbf{\Phi}$ of cardinality $|G_0| = w(0)$, which admits a unique Lagrange interpolatory polynomial of order k. For any $j \in \mathbf{Z}_{w(0)}$, we let $w_{0,j}$ be the Lagrange interpolating polynomial with respect to the interpolation set G_0 at the point s_j and define the point evaluation functional at s_j by $\ell_{0,j} := \delta_{s_j}$. This construction ensures that $\langle \ell_{0,i}, w_{0,j} \rangle = \delta_{ij}$, for $i, j \in \mathbf{Z}_{w(0)}$, where $\langle \cdot, \cdot \rangle$ denotes the functional application as in [**11**].

To construct higher level basis functions, for each $e \in \mathbf{Z}_\nu$, we introduce a linear operator $\mathcal{T}_e : L^\infty(S) \to L^\infty(S)$ defined for $f \in L^\infty(S)$ by

$$\mathcal{T}_e f := f \circ \phi_e^{-1} \chi_{\phi_e(S)}, \tag{3.1}$$

where χ_A denotes the characteristic function of set A. For $n \in \mathbf{N}$, we define $\mathbf{X}_n$ recursively by

$$\mathbf{X}_n := \bigcup_{e \in \mathbf{Z}_\nu} \mathcal{T}_e \mathbf{X}_{n-1}.$$

Hence, $\mathbf{X}_n$ is the space of piecewise polynomials of order k with respect to the partition $\mathbf{\Phi}^n(S)$ and a basis for $\mathbf{X}_n$ is recursively constructed by a basis for $\mathbf{X}_1$ with operator applications of $\mathcal{T}_e$. We next describe a construction of a basis for $\mathbf{X}_1$. Clearly, the dimension of $\mathbf{X}_1$ is given by $\nu w(0)$ and we have the nestedness property that $\mathbf{X}_0 \subset \mathbf{X}_1$. We let $G_1 := \mathbf{\Phi}(G_0)$ and observe that the cardinality of G_1 is $\nu w(0)$. Let $\mathbf{L}_1$ denote the space of the point evaluation functionals ℓ of the form $\ell = \sum_{s \in G_1} c_s \delta_s$ which satisfies $\langle \ell, f \rangle = 0$, for $f \in \mathbf{X}_0$. We denote by $\{\ell_{1,j} : j \in \mathbf{Z}_{w(1)}\}$ a basis of $\mathbf{L}_1$, where $w(1) := (\nu - 1)w(0)$. For $j \in \mathbf{Z}_{w(1)}$, let $w_{1,j} \in \mathbf{X}_1$ satisfy

$$(w_{0,j'}, w_{1,j}) = 0, \quad j' \in \mathbf{Z}_{w(0)}, \quad \text{and} \qquad (3.2)$$
$$\langle \ell_{1,j'}, w_{1,j} \rangle = \delta_{j',j}, \quad j' \in \mathbf{Z}_{w(1)}.$$

We remark that each $w_{1,j}$ is uniquely determined by (3.2).

The basis functions $w_{i,j}$, $j \in \mathbf{Z}_{w(i)}$, $i > 1$, where $w(i) := \nu^{i-1}w(1)$, are constructed by recursions as described in [**11**]. Specifically, for $i > 1$ and $\mathbf{e} := (e_0, e_1, \ldots, e_{i-1}) \in \mathbf{Z}_\nu^i$, we introduce the composite operator $\mathcal{T}_\mathbf{e} := \mathcal{T}_{e_0} \circ \cdots \circ \mathcal{T}_{e_{i-1}}$ and define the number $\mu(\mathbf{e}) := \nu^{i-1}e_0 + \cdots + \nu e_{i-2} + e_{i-1}$. We construct $w_{i,j}$, $i > 1$, $j \in \mathbf{Z}_{w(i)}$ from $w_{1,l}$, $l \in \mathbf{Z}_{w(1)}$ in the way

$$w_{i,j} := \mathcal{T}_\mathbf{e} w_{1,l}, \quad j = \mu(\mathbf{e})w(1) + l, \quad \mathbf{e} \in \mathbf{Z}_\nu^{i-1}, \quad l \in \mathbf{Z}_{w(1)}. \qquad (3.3)$$

The collocation functionals $\ell_{i,j}$, $j \in \mathbf{Z}_{w(i)}$, $i > 1$, are constructed in a similar manner. For $e \in \mathbf{Z}_\nu$, we define a linear operator $\mathcal{L}_e : L^\infty(S)^* \to L^\infty(S)^*$ by the equation

$$\langle \mathcal{L}_e \ell, f \rangle = \langle \ell, f \circ \phi_e \rangle, \quad f \in L^\infty(S), \quad \ell \in L^\infty(S)^*,$$

and for $\mathbf{e} := (e_0, \ldots, e_{i-1}) \in \mathbf{Z}_\nu^i$, we define the composition operator $\mathcal{L}_\mathbf{e} := \mathcal{L}_{e_0} \circ \cdots \circ \mathcal{L}_{e_{i-1}}$. For $i = 2, 3, \ldots, n$, we generate the functionals

$$\ell_{i,j} := \mathcal{L}_\mathbf{e} \ell_{1,l}, \quad j = \mu(\mathbf{e})w(1) + l, \ \mathbf{e} \in \mathbf{Z}_\nu^{i-1}, \ l \in \mathbf{Z}_{w(1)}. \qquad (3.4)$$

The multi-scale bases $\{w_{i,j} : i \in \mathbf{Z}_{n+1}, j \in \mathbf{Z}_{w(i)}\}$ and collocation functionals $\{\ell_{i,j} : i \in \mathbf{Z}_{n+1}, j \in \mathbf{Z}_{w(i)}\}$ will lead to fast collocation methods which will be described in the following section.

In the rest of this section, we follow the construction described above to generate the multi-scale bases and collocation functionals of several cases of practical importance. Since basis functions and collocation functionals of levels higher than 1 are constructed by (3.3) and (3.4), respectively, we present only those of levels 0 and 1.

3.1. *The one-dimensional case.* In the one-dimensional case, $S = [0, 1]$ and we choose $\mathbf{\Phi} = \{\phi_j : j \in \mathbf{Z}_2\}$ with

$$\phi_0(t) := \frac{t}{2}, \quad \phi_0(t) := \frac{t+1}{2}, \ \ t \in [0, 1].$$

The linear basis. In this case, $k = 2$, and thus, $\mathbf{X}_0$ is the space of polynomials of order 2 and $\dim(\mathbf{X}_0) = 2$. A refinable set of cardinality 2 with respect to $\mathbf{\Phi}$ is $G_0 = \{(1/3), (2/3)\}$. Hence, at level 0, we have two basis functions

$$w_{0,0}(t) = 2 - 3t, \quad w_{0,1}(t) = -1 + 3t,$$

and two collocation functionals

$$\ell_{0,0} = \delta_{1/3}, \quad \ell_{0,1} = \delta_{2/3}.$$

At level 1, we have two basis functions

$$w_{1,0}(t) = \begin{cases} 1 - (9/2)t & t \in [0, (1/2)], \\ -1 + (3/2)t & t \in ((1/2), 1] \end{cases}$$
$$w_{1,1}(t) = \begin{cases} (1/2) - (3/2)t & t \in [0, (1/2)], \\ -(7/2) + (9/2)t & t \in ((1/2), 1] \end{cases}$$

and two collocation functionals

$$\ell_{1,0} = \delta_{1/6} - \frac{3}{2}\delta_{1/3} + \frac{1}{2}\delta_{2/3}, \qquad \ell_{1,1} = \frac{1}{2}\delta_{1/3} - \frac{3}{2}\delta_{2/3} + \delta_{5/6}.$$

The cubic basis. In this case $k = 4$ and correspondingly, $\mathbf{X}_0$ is the space of polynomials of order 4 and $\dim(\mathbf{X}_0) = 4$. A refinable set of cardinality 4 with respect to $\mathbf{\Phi}$ is $G_0 = \{(1/5), (2/5), (3/5), (4/5)\}$. At level 0, we have four basis functions

$$w_{0,0}(t) = -\frac{1}{6}(5t-2)(5t-3)(5t-4),$$
$$w_{0,1}(t) = \frac{1}{2}(5t-1)(5t-3)(5t-4),$$
$$w_{0,2}(t) = -\frac{1}{2}(5t-1)(5t-2)(5t-4),$$
$$w_{0,3}(t) = \frac{1}{6}(5t-1)(5t-2)(5t-3)$$

and four collocation functionals

$$\ell_{0,j} = \delta_{(j+1)/5}, \ \ j \in \mathbf{Z}_4.$$

At level 1 we have four basis functions

$$w_{1,0}(t) = \begin{cases} (85/32) - (725/12)t + (575/2)t^2 - (1475/4)t^3 \\ \qquad t \in [0, (1/2)], \\ -(235/32) + (575/12)t - (175/2)t^2 + (575/12)t^3 \\ \qquad t \in ((1/2), 1], \end{cases}$$

$$w_{1,1}(t) = \begin{cases} (1145/288) - (1775/24)t + (1675/6)t^2 - (4975/18)t^3 \\ \qquad t \in [0, (1/2)], \\ -(7495/288) + (3625/24)t - (525/2)t^2 + (2525/18)t^3 \\ \qquad t \in ((1/2), 1], \end{cases}$$

$$w_{1,2}(t) = \begin{cases} (805/288) - (375/8)t + (475/3)t^2 - (2525/18)t^3 \\ \qquad t \in [0, (1/2)], \\ -(19355/288) + (8275/24)t - 550t^2 + (4975/18)t^3 \\ \qquad t \in ((1/2), 1], \end{cases}$$

$$w_{1,3}(t) = \begin{cases} (95/96) - (50/3)t + (225/4)t^2 - (575/12)t^3 \\ \qquad t \in [0, (1/2)], \\ -(13345/96) + (1775/3)t - (3275/4)t^2 + (1475/4)t^3 \\ \qquad t \in ((1/2), 1] \end{cases}$$

and four collocation functionals

$$\begin{aligned}
\ell_{1,0} &= \frac{2}{5}\delta_{1/10} - \frac{3}{2}\delta_{2/10} + 2\delta_{3/10} - \delta_{4/10} + \frac{1}{10}\delta_{6/10}, \\
\ell_{1,1} &= \frac{3}{10}\delta_{2/10} - \delta_{3/10} + \delta_{4/10} - \frac{1}{2}\delta_{6/10} + \frac{1}{5}\delta_{7/10}, \\
\ell_{1,2} &= \frac{1}{5}\delta_{3/10} - \frac{1}{2}\delta_{4/10} + \delta_{6/10} - \delta_{7/10} + \frac{3}{10}\delta_{8/10}, \\
\ell_{1,3} &= \frac{1}{10}\delta_{4/10} - \delta_{6/10} + 2\delta_{7/10} - \frac{3}{2}\delta_{8/10} + \frac{2}{5}\delta_{9/10}.
\end{aligned}$$

3.2. *The two-dimensional case.* In the two-dimensional case, $S = \{(x_0, x_1) \in \mathbf{R}^2 : 0 \le x_0 \le x_1 \le 1\}$, and we choose the contractive

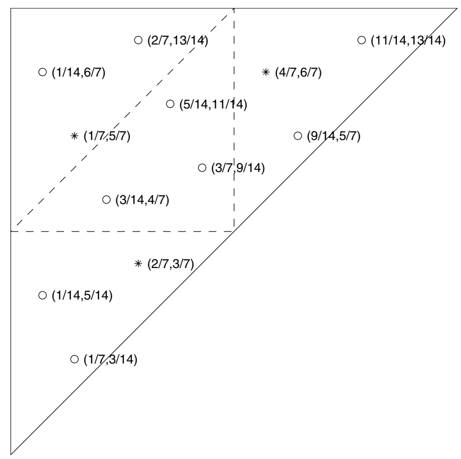

FIGURE 3. The refinable set G_1 for two-dimensional linear basis.

mappings as $\mathbf{\Phi} = \{\phi_e : e \in \mathbf{Z}_4\}$ with

$$\phi_0(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right), \qquad \phi_1(x,y) = \left(\frac{x}{2}, \frac{y+1}{2}\right),$$
$$\phi_2(x,y) = \left(\frac{1-x}{2}, 1-\frac{y}{2}\right), \quad \phi_3(x,y) = \left(\frac{x+1}{2}, \frac{y+1}{2}\right).$$

We choose $k = 2$ and thus $\mathbf{X}_0$ is the space of linear polynomials on S, and $\dim(\mathbf{X}_0) = 3$. A refinable set of cardinality 3 with respect to $\mathbf{\Phi}$ is given by

$$G_0 = \left\{ \left(\frac{2}{7}, \frac{3}{7}\right), \left(\frac{1}{7}, \frac{5}{7}\right), \left(\frac{4}{7}, \frac{6}{7}\right) \right\}.$$

The basis functions of level 0 are

$$w_{0,0}(t) = -3x + 2y, \quad w_{0,1}(t) = x - 3y + 2, \quad w_{0,2}(t) = 2x + y - 1,$$

and collocation functionals of level 0 are

$$\ell_{00} := \delta_{((2/7),(3/7))}, \quad \ell_{01} := \delta_{((1/7),(5/7))}, \quad \ell_{02} := \delta_{((4/7),(6/7))}.$$

The set

$$G_1 = \left\{ \left(\frac{1}{7}, \frac{5}{7}\right), \left(\frac{2}{7}, \frac{3}{7}\right), \left(\frac{4}{7}, \frac{6}{7}\right), \left(\frac{1}{14}, \frac{6}{7}\right), \left(\frac{1}{14}, \frac{5}{14}\right), \left(\frac{3}{7}, \frac{9}{14}\right), \left(\frac{9}{14}, \frac{5}{7}\right), \left(\frac{1}{7}, \frac{3}{14}\right), \left(\frac{5}{14}, \frac{11}{14}\right), \left(\frac{2}{7}, \frac{13}{14}\right), \left(\frac{11}{14}, \frac{13}{14}\right), \left(\frac{3}{14}, \frac{4}{7}\right) \right\}.$$

We plot the points of G_1 in Figure 3, in which the points marked with '$*$' are those of the subset G_0.

The nine basis functions at level 1 are given by

$$w_{1,0}(x,y) = \begin{cases} -(11/8) - (15/8)x + (41/8)y & (x,y) \in S_0, \\ 5/8 + (1/8)x - (7/8)y & (x,y) \in S \setminus S_0, \end{cases}$$
$$w_{1,1}(x,y) = \begin{cases} 1 - (15/4)x - (7/8)y & (x,y) \in S_0, \\ -1 + (1/4)x + (9/8)y & (x,y) \in S \setminus S_0, \end{cases}$$
$$w_{1,2}(x,y) = \begin{cases} 9/8 + (15/8)x - (29/8)y & (x,y) \in S_0, \\ -(15/8) - (1/8)x + (19/8)y & (x,y) \in S \setminus S_0, \end{cases}$$
$$w_{1,3}(x,y) = \begin{cases} -(15/8) - (41/8)x + (13/4)y & (x,y) \in S_1, \\ 1/8 + (7/8)x - (3/4)y & (x,y) \in S \setminus S_1, \end{cases}$$
$$w_{1,4}(x,y) = \begin{cases} (29/8) + (7/8)x - (37/8)y & (x,y) \in S_1, \\ -3/8 - (9/8)x + (11/8)y & (x,y) \in S \setminus S_1, \end{cases}$$
$$w_{1,5}(x,y) = \begin{cases} -5/8 - (29/8)x + (7/4)y & (x,y) \in S_1, \\ 3/8 + (19/8)x - (9/4)y & (x,y) \in S \setminus S_1, \end{cases}$$
$$w_{1,6}(x,y) = \begin{cases} (15/4) - (13/4)x - (15/8)y & (x,y) \in S_3, \\ -1/4 + (3/4)x + (1/8)y & (x,y) \in S \setminus S_3, \end{cases}$$
$$w_{1,7}(x,y) = \begin{cases} -(1/8) - (37/8)x + (15/4)y & (x,y) \in S_3, \\ -(1/8) + (11/8)x - (1/4)y & (x,y) \in S \setminus S_3, \end{cases}$$
$$w_{1,8}(x,y) = \begin{cases} -(5/2) + (7/4)x + (15/8)y & (x,y) \in S_3, \\ 1/2 - (9/4)x - (1/8)y & (x,y) \in S \setminus S_3, \end{cases}$$

where $S_e = \phi_e(S)$, $e \in \mathbf{Z}_4$. Correspondingly, the nine collocation functionals are given by

$$\ell_{1,0} = \delta_{((1/14),(5/14))} - \delta_{((1/7),(3/14))} + \delta_{((2/7),(3/7))} - \delta_{((3/14),(4/7))}$$

$$\ell_{1,1} = \delta_{((1/14),(5/14))} - \delta_{((2/7),(3/7))} + \delta_{((3/7),(9/14))} - \delta_{((3/14),(4/7))},$$
$$\ell_{1,2} = \delta_{((5/14),(11/14))} - \delta_{((3/7),(9/14))} + \delta_{((2/7),(3/7))} - \delta_{((3/14),(4/7))}$$
$$\ell_{1,3} = \delta_{((1/14),(6/7))} - \delta_{((1/7),(5/7))} + \delta_{((5/14),(11/14))} - \delta_{((2/7),(13/14))},$$
$$\ell_{1,4} = \delta_{((1/7),(5/7))} - \delta_{((2/7),(13/14))} + \delta_{((5/14),(11/14))} - \delta_{((3/14),(4/7))}$$
$$\ell_{1,5} = \delta_{((3/7),(9/14))} - \delta_{((5/14),(11/14))} + \delta_{((1/7),(5/7))} - \delta_{((3/14),(4/7))},$$
$$\ell_{1,6} = \delta_{((4/7),(6/7))} - \delta_{((11/14),(13/14))} + \delta_{((9/14),(5/7))} - \delta_{((3/7),(9/14))}$$
$$\ell_{1,7} = \delta_{((4/7),(6/7))} - \delta_{((9/14),(5/7))} + \delta_{((3/7),(9/14))} - \delta_{((5/14),(11/14))},$$
$$\ell_{1,8} = \delta_{((3/14),(4/7))} - \delta_{((3/7),(9/14))} + \delta_{((4/7),(6/7))} - \delta_{((5/14),(11/14))}.$$

Note that all functionals listed above are linear combinations of four point evaluation functionals, which use the least number of points to construct a functional that annihilates all linear polynomials. We also note that all weights in the linear combinations are either 1 or -1. In fact, these functionals are *divided difference* functionals in some sense. We plot the positions of the point sets of the nine functionals in Figure 4. The points marked with '+' have weights 1 while those marked with '$\circ$' have weights -1.

3.3. *The three-dimensional case.* In the three-dimensional case

$$S = \{(x_0, x_1, x_2) : \quad 0 \le x_0 \le x_1 \le x_2 \le 1\},$$

and we choose the eight contractive mappings $\{\mathcal{G}_{\mathbf{e}} : \mathbf{e} \in \mathbf{Z}_2^3\}$ listed in Table 2. We construct a linear basis and its corresponding collocation functionals. Hence, $k = 2$ and $\mathbf{X}_0$ is the space of linear polynomials on S, and $\dim(\mathbf{X}_0) = 4$. A basis for $\mathbf{X}_0$ has the form

$$w_{0,0}(x,y,z) = 2 + y - 3z, \qquad w_{0,1}(x,y,z) = x - 3y + 2z,$$
$$w_{0,2}(x,y,z) = -3x + 2y, \qquad w_{0,3}(x,y,z) = -1 + 2x + z.$$

An appropriate refinable set of cardinality 4 is given by

$$G_0 := \left\{ \left(\frac{8}{15}, \frac{4}{5}, \frac{14}{15}\right), \left(\frac{4}{15}, \frac{2}{5}, \frac{7}{15}\right), \left(\frac{2}{15}, \frac{1}{5}, \frac{11}{15}\right), \left(\frac{1}{15}, \frac{3}{5}, \frac{13}{15}\right) \right\}.$$

The elements of G_0 define the collocation functions ℓ_{0j} of level 0, for $j = 0, 1, 2, 3$.

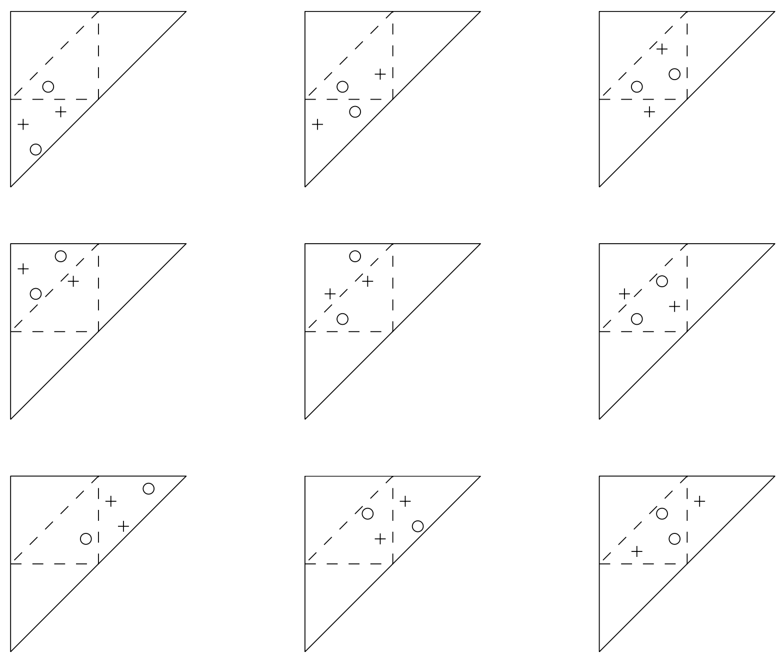

FIGURE 4. The functionals ℓ_{1j}, $j \in \mathbf{Z}_9$ for two-dimensional linear basis.

Let $S_{\mathbf{e}} = \mathcal{G}_{\mathbf{e}}(S)$ for $\mathbf{e} \in \mathbf{Z}_2^3$. The 28 basis functions $w_{1,j}$, $j \in \mathbf{Z}_{28}$, of level 1 are given by

$$w_{1,0}(x,y,z) = \begin{cases} (55/16) - (65/8)z & (x,y,z) \in S_{(0,0,0)}, \\ -(25/16) + (15/8)z & (x,y,z) \in S \setminus S_{(0,0,0)}, \end{cases}$$

$$w_{1,1}(x,y,z) = \begin{cases} -(75/16) - (65/8)y + (65/8)z & (x,y,z) \in S_{(0,0,1)}, \\ (5/16) + (15/8)y - (15/8)z & (x,y,z) \in S \setminus S_{(0,0,1)}, \end{cases}$$

$$w_{1,2}(x,y,z) = \begin{cases} -(75/16) - (65/8)x + (65/8)y & (x,y,z) \in S_{(0,1,1)}, \\ (5/16) + (15/8)x - (15/8)y & (x,y,z) \in S \setminus S_{(0,1,1)}, \end{cases}$$

$$w_{1,3}(x,y,z) = \begin{cases} -(75/16) + (65/8)x & (x,y,z) \in S_{(1,1,1)}, \\ (5/16) - (15/8)x & (x,y,z) \in S \setminus S_{(1,1,1)}, \end{cases}$$

$$w_{1,4}(x,y,z) = \begin{cases} -(29/16) + (31/16)y + (21/8)z & (x,y,z) \in S_{(0,0,0)}, \\ (19/16) - (1/16)y - (11/8)z & (x,y,z) \in S \setminus S_{(0,0,0)}, \end{cases}$$

$$w_{1,5}(x,y,z)=\begin{cases} -(5/2)+(31/16)x-(31/16)y+(99/16)z \\ \qquad (x,y,z)\in S_{(0,0,0)}, \\ (3/2)-(1/16)x+(1/16)y-(29/16)z \\ \qquad (x,y,z)\in S\setminus S_{(0,0,0)}, \end{cases}$$

$$w_{1,6}(x,y,z)=\begin{cases} -(25/32)-(31/16)x+(17/8)z & (x,y,z)\in S_{(0,0,0)}, \\ (23/32)+(1/16)x-(7/8)z & (x,y,z)\in S\setminus S_{(0,0,0)}, \end{cases}$$

$$w_{1,7}(x,y,z)=\begin{cases} (11/4)+(31/16)x+(21/8)y-(73/16)z \\ \qquad (x,y,z)\in S_{(0,0,1)}, \\ -(1/4)-(1/16)x-(11/8)y+(23/16)z \\ \qquad (x,y,z)\in S\setminus S_{(0,0,1)}, \end{cases}$$

$$w_{1,8}(x,y,z)=\begin{cases} (59/16)-(31/16)x+(99/16)y-(99/16)z \\ \qquad (x,y,z)\in S_{(0,0,1)}, \\ -(5/16)+(1/16)x-(29/16)y+(29/16)z \\ \qquad (x,y,z)\in S\setminus S_{(0,0,1)}, \end{cases}$$

$$w_{1,9}(x,y,z)=\begin{cases} -(19/32)+(17/8)y-(3/16)z & (x,y,z)\in S_{(0,0,1)}, \\ -(3/32)-(7/8)y+(13/16)z & (x,y,z)\in S\setminus S_{(0,0,1)} \end{cases}$$

$$w_{1,10}(x,y,z)=\begin{cases} (11/4)+(21/8)x-(73/16)y & (x,y,z)\in S_{(0,1,1)}, \\ -(1/4)-(11/8)x+(23/16)y & (x,y,z)\in S\setminus S_{(0,1,1)}, \end{cases}$$

$$w_{1,11}(x,y,z)=\begin{cases} (7/4)+(99/16)x-(99/16)y+(31/16)z \\ \qquad (x,y,z)\in S_{(0,1,1)}, \\ -(1/4)-(29/16)x+(29/16)y-(1/16)z \\ \qquad (x,y,z)\in S\setminus S_{(0,1,1)}, \end{cases}$$

$$w_{1,12}(x,y,z)=\begin{cases} (43/32)+(17/8)x-(3/16)y-(31/16)z \\ \qquad (x,y,z)\in S_{(0,1,1)}, \\ -(5/32)-(7/8)x+(13/16)y+(1/16)z \\ \qquad (x,y,z)\in S\setminus S_{(0,1,1)}, \end{cases}$$

$$w_{1,13}(x,y,z)=\begin{cases} (13/16)-(73/16)x+(31/16)z & (x,y,z)\in S_{(1,1,1)}, \\ -(3/16)+(23/16)x-(1/16)z & (x,y,z)\in S\setminus S_{(1,1,1)}, \end{cases}$$

$$w_{1,14}(x,y,z)=\begin{cases} (59/16)-(99/16)x+(31/16)y-(31/16)z \\ \qquad (x,y,z)\in S_{(1,1,1)}, \\ -(5/16)+(29/16)x-(1/16)y+(1/16)z \\ \qquad (x,y,z)\in S\setminus S_{(1,1,1)}, \end{cases}$$

$$w_{1,15}(x,y,z) = \begin{cases} (43/32) - (3/16)x - (31/16)y & (x,y,z) \in S_{(1,1,1)}, \\ -(5/32) + (13/16)x + (1/16)y & (x,y,z) \in S \setminus S_{(1,1,1)}, \end{cases}$$

$$w_{1,16}(x,y,z) = \begin{cases} -(1/4) + (79/240)x + (451/240)y - (229/240)z \\ \qquad (x,y,z) \in S_{(0,0,0)} \cup S_{(1,0,0)}, \\ (29/60) - (49/240)x - (29/240)y - (101/240)z \\ \qquad (x,y,z) \in S_{(0,1,0)} \cup S_{(0,0,1)}, \\ (11/20) - (49/240)x - (61/240)y - (101/240)z \\ \qquad (x,y,z) \in S_{(0,1,1)} \cup S_{(1,0,1)}, \\ (49/60) + (79/240)x - (61/240)y - (229/240)z \\ \qquad (x,y,z) \in S_{(1,1,0)} \cup S_{(1,1,1)}, \end{cases}$$

$$w_{1,17}(x,y,z) = \begin{cases} (25/464) + (727/696)x - (2857/1392)y + (461/696)z \\ \qquad (x,y,z) \in S_{(0,0,0)} \cup S_{(1,0,0)}, \\ (47/48) + (119/696)x - (73/1392)y - (827/696)z \\ \qquad (x,y,z) \in S_{(0,1,0)} \cup S_{(0,0,1)}, \\ (185/464) + (119/696)x + (679/1392)y - (611/696)z \\ \qquad (x,y,z) \in S_{(0,1,1)} \cup S_{(1,0,1)}, \\ -(13/1392) - (449/696)x - (329/1392)y + (461/696)z \\ \qquad (x,y,z) \in S_{(1,1,0)} \cup S_{(1,1,1)}, \end{cases}$$

$$w_{1,18}(x,y,z) = \begin{cases} -(1/16) - (179/240)x + (49/240)y + (29/240)z \\ \qquad (x,y,z) \in S_{(0,0,0)} \cup S_{(1,0,0)}, \\ -(31/240) - (211/240)x + (49/240)y + (61/240)z \\ \qquad (x,y,z) \in S_{(0,1,0)} \cup S_{(0,0,1)}, \\ (11/80) - (211/240)x - (79/240)y + (61/240)z \\ \qquad (x,y,z) \in S_{(0,1,1)} \cup S_{(1,0,1)}, \\ (289/240) + (301/240)x - (79/240)y - (451/240)z \\ \qquad (x,y,z) \in S_{(1,1,0)} \cup S_{(1,1,1)}, \end{cases}$$

$$w_{1,19}(x,y,z)=\begin{cases}(97/464)-(1489/1392)x-(119/696)y+(73/1392)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ (23/48)-(305/1392)x-(119/696)y-(679/1392)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ -(303/464)-(305/1392)x+(449/696)y+(329/1392)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ -(997/1392)-(481/1392)x-(727/696)y+(2857/1392)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)},\end{cases}$$

$$w_{1,20}(x,y,z)=\begin{cases}(1/16)+(61/240)x+(229/240)y-(211/240)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ -(241/240)-(451/240)x+(229/240)y+(301/240)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ (21/80)+(29/240)x+(101/240)y-(179/240)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ (79/240)+(61/240)x+(101/240)y-(211/240)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)}\end{cases}$$

$$w_{1,21}(x,y,z)=\begin{cases}(53/232)+(329/1392)x-(461/696)y-(305/1392)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ (7/24)+(2857/1392)x-(461/696)y-(481/1392)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ (21/232)+(73/1392)x+(827/696)y-(1489/1392)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ -(125/696)-(679/1392)x+(611/696)y-(305/1392)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)},\end{cases}$$

$$w_{1,22}(x,y,z)=\begin{cases}-(1/8)-(101/240)x+(211/240)y-(49/240)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ -(47/120)-(229/240)x+(211/240)y+(79/240)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ (27/40)-(229/240)x-(301/240)y+(79/240)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ -(7/120)-(101/240)x+(179/240)y-(49/240)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)}\end{cases}$$

$$w_{1,23}(x,y,z)=\begin{cases}(1/116)-(611/696)x+(305/1392)y+(119/696)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ (5/12)+(461/696)x+(305/1392)y-(449/696)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ -(155/116)+(461/696)x+(481/1392)y+(727/696)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ -(91/348)-(827/696)x+(1489/1392)y+(119/696)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)},\end{cases}$$

$$w_{1,24}(x,y,z)=\begin{cases}(345/464)+(1103/464)x+(12/29)y-(1263/464)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ -(27/16)-(593/464)x+(12/29)y+(993/464)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ -(231/464)-(593/464)x-(27/29)y+(753/464)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ (497/464)+(863/464)x+(8/29)y-(1263/464)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)},\end{cases}$$

$$w_{1,25}(x,y,z)=\begin{cases}(225/464)+(8/29)x-(1263/464)y+(17/29)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ (13/16)+(12/29)x-(1263/464)y-(2/29)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ -(191/464)+(12/29)x+(993/464)y-(37/29)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ -(503/464)-(27/29)x+(753/464)y+(17/29)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)},\end{cases}$$

$$w_{1,26}(x,y,z)=\begin{cases}(45/232)+(753/464)x+(17/29)y-(593/464)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ -(11/8)-(1263/464)x+(17/29)y+(863/464)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ -(363/232)-(1263/464)x-(2/29)y+(1103/464)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ (201/232)+(993/464)x-(37/29)y-(593/464)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)},\end{cases}$$

$$w_{1,27}(x,y,z)=\begin{cases}(105/232)-(37/29)x-(593/464)y+(12/29)z\\ \qquad (x,y,z)\in S_{(0,0,0)}\cup S_{(1,0,0)},\\ (9/8)+(17/29)x-(593/464)y-(27/29)z\\ \qquad (x,y,z)\in S_{(0,1,0)}\cup S_{(0,0,1)},\\ -(383/232)+(17/29)x+(863/464)y+(8/29)z\\ \qquad (x,y,z)\in S_{(0,1,1)}\cup S_{(1,0,1)},\\ -(459/232)-(2/29)x+(1103/464)y+(12/29)z\\ \qquad (x,y,z)\in S_{(1,1,0)}\cup S_{(1,1,1)}.\end{cases}$$

The set $G_1=\{t_{i,j}:i\in\mathbf{Z}_8, j\in\mathbf{Z}_4\}$ where

$$t_{0,0}:=\left(\frac{2}{15},\frac{1}{5},\frac{7}{30}\right),\qquad t_{0,1}:=\left(\frac{1}{15},\frac{1}{10},\frac{11}{30}\right),$$
$$t_{0,2}:=\left(\frac{1}{30},\frac{3}{10},\frac{1,3}{30}\right),\qquad t_{03}:=\left(\frac{4}{15},\frac{2}{5},\frac{7}{15}\right),$$
$$t_{1,0}:=\left(\frac{1}{15},\frac{1}{10},\frac{13}{15}\right),\qquad t_{1,1}:=\left(\frac{1}{30},\frac{3}{10},\frac{14}{15}\right),$$
$$t_{1,2}:=\left(\frac{4}{15},\frac{2}{5},\frac{29}{30}\right),\qquad t_{1,3}:=\left(\frac{2}{15},\frac{1}{5},\frac{11}{15}\right),$$
$$t_{2,0}:=\left(\frac{1}{30},\frac{4}{5},\frac{14}{15}\right),\qquad t_{2,1}:=\left(\frac{4}{15},\frac{9}{10},\frac{29}{30}\right),$$
$$t_{2,2}:=\left(\frac{2}{15},\frac{7}{10},\frac{11}{15}\right),\qquad t_{2,3}:=\left(\frac{1}{15},\frac{3}{5},\frac{13}{15}\right),$$
$$t_{3,0}:=\left(\frac{23}{30},\frac{9}{10},\frac{29}{30}\right),\qquad t_{3,1}:=\left(\frac{19}{30},\frac{7}{10},\frac{11}{15}\right),$$
$$t_{3,2}:=\left(\frac{17}{30},\frac{3}{5},\frac{13}{15}\right),\qquad t_{3,3}:=\left(\frac{8}{15},\frac{4}{5},\frac{14}{15}\right),$$
$$t_{4,0}:=\left(\frac{2}{5},\frac{7}{15},\frac{23}{30}\right),\qquad t_{4,1}:=\left(\frac{1}{5},\frac{7}{30},\frac{19}{30}\right),$$
$$t_{4,2}:=\left(\frac{1}{10},\frac{11}{30},\frac{17}{30}\right),\qquad t_{4,3}:=\left(\frac{3}{10},\frac{13}{30},\frac{8}{15}\right),$$
$$t_{5,0}:=\left(\frac{1}{15},\frac{11}{30},\frac{3}{5}\right),\qquad t_{5,1}:=\left(\frac{1}{30},\frac{13}{30},\frac{4}{5}\right),$$
$$t_{5,2}:=\left(\frac{4}{15},\frac{7}{15},\frac{9}{10}\right),\qquad t_{5,3}:=\left(\frac{2}{15},\frac{7}{30},\frac{7}{10}\right),$$

$$
\begin{aligned}
t_{6,0} &:= \left(\frac{3}{10}, \frac{8}{15}, \frac{14}{15}\right), \qquad & t_{6,1} &:= \left(\frac{2}{5}, \frac{23}{30}, \frac{29}{30}\right),\\
t_{6,2} &:= \left(\frac{1}{5}, \frac{19}{30}, \frac{11}{15}\right), \qquad & t_{6,3} &:= \left(\frac{1}{10}, \frac{17}{30}, \frac{13}{15}\right),\\
t_{7,0} &:= \left(\frac{7}{30}, \frac{19}{30}, \frac{7}{10}\right), \qquad & t_{7,1} &:= \left(\frac{11}{30}, \frac{17}{30}, \frac{3}{5}\right),\\
t_{7,2} &:= \left(\frac{13}{30}, \frac{8}{15}, \frac{4}{5}\right), \qquad & t_{7,3} &:= \left(\frac{7}{15}, \frac{23}{30}, \frac{9}{10}\right).
\end{aligned}
$$

Correspondingly, the collocation functionals of level 1 have the form

$$
\begin{aligned}
\ell_{1,0} &:= \delta_{t_{00}} - \frac{2}{5}\delta_{t_{01}} - \frac{13}{10}\delta_{t_{02}} - \frac{4}{5}\delta_{t_{03}} + \frac{3}{2}\delta_{t_{42}},\\
\ell_{1,1} &:= \delta_{t_{10}} - \frac{2}{5}\delta_{t_{11}} - \frac{13}{10}\delta_{t_{12}} - \frac{4}{5}\delta_{t_{13}} + \frac{3}{2}\delta_{t_{52}},\\
\ell_{1,2} &:= \delta_{t_{20}} - \frac{2}{5}\delta_{t_{21}} - \frac{13}{10}\delta_{t_{22}} - \frac{4}{5}\delta_{t_{23}} + \frac{3}{2}\delta_{t_{62}},\\
\ell_{1,3} &:= \delta_{t_{30}} - \frac{2}{5}\delta_{t_{31}} - \frac{13}{10}\delta_{t_{32}} - \frac{4}{5}\delta_{t_{33}} + \frac{3}{2}\delta_{t_{72}},
\end{aligned}
$$

$$
\begin{aligned}
\ell_{1,4} &:= \delta_{t_{00}} - 2\delta_{t_{01}} - \delta_{t_{40}} + 2\delta_{t_{41}},\\
\ell_{1,5} &:= \delta_{t_{01}} - 2\delta_{t_{02}} - \delta_{t_{41}} + 2\delta_{t_{42}},\\
\ell_{1,6} &:= \delta_{t_{02}} - 2\delta_{t_{03}} - \delta_{t_{42}} + 2\delta_{t_{43}},\\
\ell_{1,7} &:= \delta_{t_{10}} - 2\delta_{t_{11}} - \delta_{t_{50}} + 2\delta_{t_{51}},\\
\ell_{1,8} &:= \delta_{t_{11}} - 2\delta_{t_{12}} - \delta_{t_{51}} + 2\delta_{t_{52}},\\
\ell_{1,9} &:= \delta_{t_{12}} - 2\delta_{t_{13}} - \delta_{t_{52}} + 2\delta_{t_{53}},\\
\ell_{1,10} &:= \delta_{t_{20}} - 2\delta_{t_{21}} - \delta_{t_{60}} + 2\delta_{t_{61}},\\
\ell_{1,11} &:= \delta_{t_{21}} - 2\delta_{t_{22}} - \delta_{t_{61}} + 2\delta_{t_{62}},\\
\ell_{1,12} &:= \delta_{t_{22}} - 2\delta_{t_{23}} - \delta_{t_{62}} + 2\delta_{t_{63}},\\
\ell_{1,13} &:= \delta_{t_{30}} - 2\delta_{t_{31}} - \delta_{t_{70}} + 2\delta_{t_{71}},\\
\ell_{1,14} &:= \delta_{t_{31}} - 2\delta_{t_{32}} - \delta_{t_{71}} + 2\delta_{t_{72}},\\
\ell_{1,15} &:= \delta_{t_{32}} - 2\delta_{t_{33}} - \delta_{t_{72}} + 2\delta_{t_{73}},\\
\ell_{1,16} &:= \delta_{t_{40}} - 2\delta_{t_{41}} - \delta_{t_{52}} + 2\delta_{t_{53}},\\
\ell_{1,17} &:= \delta_{t_{41}} - 2\delta_{t_{42}} - \delta_{t_{53}} + 2\delta_{t_{50}},\\
\ell_{1,18} &:= \delta_{t_{42}} - 2\delta_{t_{43}} - \delta_{t_{70}} + 2\delta_{t_{71}},
\end{aligned}
$$

$$\ell_{1,19} := \delta_{t_{43}} - 2\delta_{t_{40}} - \delta_{t_{71}} + 2\delta_{t_{72}},$$
$$\ell_{1,20} := \delta_{t_{62}} - 2\delta_{t_{63}} - \delta_{t_{50}} + 2\delta_{t_{51}},$$
$$\ell_{1,21} := \delta_{t_{63}} - 2\delta_{t_{60}} - \delta_{t_{51}} + 2\delta_{t_{52}},$$
$$\ell_{1,22} := \delta_{t_{60}} - 2\delta_{t_{61}} - \delta_{t_{72}} + 2\delta_{t_{73}},$$
$$\ell_{1,23} := \delta_{t_{61}} - 2\delta_{t_{62}} - \delta_{t_{73}} + 2\delta_{t_{70}},$$
$$\ell_{1,24} := \delta_{t_{60}} - \frac{9}{5}\delta_{t_{40}} + \frac{3}{5}\delta_{t_{41}} - \frac{6}{5}\delta_{t_{42}} + \frac{7}{5}\delta_{t_{43}},$$
$$\ell_{1,25} := \delta_{t_{70}} - \frac{9}{5}\delta_{t_{50}} + \frac{3}{5}\delta_{t_{51}} - \frac{6}{5}\delta_{t_{52}} + \frac{7}{5}\delta_{t_{53}}.$$
$$\ell_{1,26} := \delta_{t_{40}} - \frac{9}{5}\delta_{t_{60}} + \frac{3}{5}\delta_{t_{61}} - \frac{6}{5}\delta_{t_{62}} + \frac{7}{5}\delta_{t_{63}},$$
$$\ell_{1,27} := \delta_{t_{50}} - \frac{9}{5}\delta_{t_{70}} + \frac{3}{5}\delta_{t_{71}} - \frac{6}{5}\delta_{t_{72}} + \frac{7}{5}\delta_{t_{73}}.$$

4. The fast collocation scheme. We describe in this section a matrix compression strategy which defines the fast collocation method. Matrix compression for singular integral operators by using Galerkin methods has been studied by many authors, cf. [**5, 6, 15, 29**]. The recent paper [**11**] established a general setting for the matrix compression of a weakly singular operator by using collocation methods. While we review the fast collocation method we will also improve the theoretical results in [**11**] by removing a technical hypothesis.

Let Ω be a compact domain in $\mathbf{R}^d$, and let the operator $\mathcal{K} : L^\infty(\Omega) \to L^\infty(\Omega)$ be defined by

$$(\mathcal{K}u)(s) := \int_\Omega K(s,t)u(t)\,dt, \quad s \in \Omega. \tag{4.1}$$

We consider the Fredholm integral equation of the second kind of the form

$$u - \mathcal{K}u = f, \tag{4.2}$$

where $f \in L^\infty(\Omega)$ is a given function, $u \in L^\infty(\Omega)$ is the unknown to be determined. For any d-dimensional vector $\alpha := [\alpha_i : i \in \mathbf{Z}_d] \in \mathbf{N}_0^d$, where $\mathbf{N}_0 := \{0, 1, 2, \dots\}$, we define $|\alpha| := \sum_{i \in \mathbf{Z}_d} \alpha_i$, and the partial derivative

$$D^\alpha = D_t^\alpha := \frac{\partial^{|\alpha|}}{\partial t_0^{\alpha_0} \cdots \partial t_{d-1}^{\alpha_{d-1}}},$$

where $t := [t_i : i \in \mathbf{Z}_d] \in \mathbf{R}^d$. We assume that for $s, t \in \Omega$, $s \neq t$, the kernel K has continuous derivatives $D_s^\alpha D_t^\beta K(s,t)$ for $|\alpha| < k$, $|\beta| < k$, and there exist positive constants σ and θ with $\sigma < d$ such that for $|\alpha| = |\beta| = k$, there holds

$$|D_s^\alpha D_t^\beta K(s,t)| \leq \frac{\theta}{|s-t|^{\sigma+|\alpha|+|\beta|}}. \tag{4.3}$$

In this case $\mathcal{K}$ is a compact operator on $L^\infty(\Omega)$. We also suppose that 1 is not an eigenvalue of $\mathcal{K}$ which ensures that the equation (4.2) has a unique solution in $L^\infty(\Omega)$.

We assume that Ω is the union of a finite number p of simplices. Multi-scale basis functions and functionals on Ω are constructed in terms of those described in the last section on the unit simplex. Specifically, Ω is subdivided into p simplices, and each simplex is mapped onto the unit simplex by the corresponding affine mapping. The basis functions and functionals on the unit simplex are transformed to those on the particular simplex which is a part of Ω, and they are extended to the entire domain Ω by setting to zero outside the simplex. In the following description of the collocation method, to avoid notational complication we restrict to the case $p = 1$. The extension to the general case is straightforward.

The collocation method for solving (4.2) is to find a vector $\mathbf{u}_n := [u_{i,j} : (i,j) \in \mathbf{U}_n]^T$, where $\mathbf{U}_n := \{(i,j) : j \in \mathbf{Z}_{w(i)}, i \in \mathbf{Z}_{n+1}\}$, such that the function $u_n := \sum_{(i,j)\in\mathbf{U}_n} u_{i,j} w_{i,j}$ satisfies the equation

$$\langle \ell_{i',j'}, u_n - \mathcal{K}u_n \rangle = \langle \ell_{i',j'}, f \rangle, \quad (i',j') \in \mathbf{U}_n. \tag{4.4}$$

Let $f(n) := \sum_{i\in\mathbf{Z}_{n+1}} w(i)$ and the dimension of the vector $\mathbf{u}_n$ is $f(n)$. By defining $f(n) \times f(n)$ matrices

$$\mathbf{E}_n := [\langle \ell_{i',j'}, w_{i,j} \rangle : (i',j'), (i,j) \in \mathbf{U}_n],$$
$$\mathbf{K}_n := [\langle \ell_{i',j'}, \mathcal{K}w_{i,j} \rangle : (i',j'), (i,j) \in \mathbf{U}_n],$$

and vector $\mathbf{f}_n := [\langle \ell_{i',j'}, f \rangle : (i',j') \in \mathbf{U}_n]^T$, equation (4.4) has the form

$$(\mathbf{E}_n - \mathbf{K}_n)\mathbf{u}_n = \mathbf{f}_n. \tag{4.5}$$

Note that the properties of the multi-scale basis and collocation functionals described in the previous section ensure that the matrix $\mathbf{E}_n$ is sparse and the full matrix $\mathbf{K}_n$ can be approximated by a sparse matrix $\widetilde{\mathbf{K}}_n$, which leads to a fast collocation method. Specifically, we partition $\mathbf{K}_n$ as

$$\mathbf{K}_n := [\mathbf{K}_{i'i} : i', i \in \mathbf{Z}_{n+1}]$$

where

$$\mathbf{K}_{i'i} := [K_{i'j',ij} : j' \in \mathbf{Z}_{w(i')},\ \ j \in \mathbf{Z}_{w(i)}]$$

with $K_{i'j',ij} := \langle \ell_{i',j'}, \mathcal{K} w_{i,j} \rangle$. Choosing a family of truncation parameters

$$\varepsilon^n_{i'i} := \max\{a\mu^{-n+b(n-i)+b'(n-i')}, \rho(\mu^{-i} + \mu^{-i'})\},\ \ i', i \in \mathbf{Z}_{n+1} \tag{4.6}$$

for some constants $a, b, b' > 0$ and $\rho > 1$, we define the truncated matrix

$$\widetilde{\mathbf{K}}_n := \left[\widetilde{\mathbf{K}}_{i'i} : i', i \in \mathbf{Z}_{n+1}\right],$$

where

$$\widetilde{\mathbf{K}}_{i'i} := \left[\widetilde{K}_{i'j',ij} : j' \in \mathbf{Z}_{w(i')}, j \in \mathbf{Z}_{w(i)}\right]$$

with

$$\widetilde{K}_{i'j',ij} := \begin{cases} K_{i'j',ij} & \mathrm{dist}\,(S_{i'j'}, S_{ij}) \le \varepsilon^n_{i'i}, \\ 0 & \text{otherwise.} \end{cases}$$

We solve the truncated linear system

$$(\mathbf{E}_n - \widetilde{\mathbf{K}}_n)\tilde{\mathbf{u}}_n = \mathbf{f}_n. \tag{4.7}$$

The truncation parameters are chosen so that (4.7) is a fast and accurate collocation method for solving equation (4.2).

Orders of convergence and computational complexity of this fast collocation method were proved in [**11**] under Hypotheses (I)–(X) described there. To close this section, we show that Hypothesis (VII) in [**11**] is not necessary. The only role of Hypothesis (VII) in [**11**] is to establish Lemma 4.1. We now prove the same conclusion of Lemma 4.1 without hypothesis (VII).

Proposition 4.1. *Suppose that the following conditions hold.*

(1) *For any* $i, i' \in \mathbf{N}_0$, $\langle \ell_{i',j'}, w_{i,j} \rangle = \delta_{i',i}\delta_{j',j}$, *for all* $(i,j), (i',j') \in \mathbf{U}_n$ *with* $n \in \mathbf{N}$, $i \le i'$.

(2) *There exists a positive constant c such that $\|\ell_{i,j}\| \leq c$, for all $(i,j) \in \mathbf{U}_n$ with $n \in \mathbf{N}$.*

(3) *There exists a positive constant c such that for $u \in W^{k,\infty}(\Omega)$*

$$\mathrm{dist}\,(u, \mathbf{X}_n) \leq cr^{-kn/d}\|u\|_{k,\infty},$$

where r denotes the number of contractive mappings used to subdivide Ω at the initial step. Let $\mathcal{P}_n$ be the interpolation projection from $L^\infty(\Omega)$ onto $\mathbf{X}_n$. Then for $v \in W^{k,\infty}(\Omega)$, and $\mathcal{P}_n v = \sum_{(i,j)\in\mathbf{U}_n} v_{i,j}w_{i,j}$, there exists a positive constant c such that

$$|v_{i,j}| \leq cr^{-ki/d}\|v\|_{k,\infty}, \quad (i,j) \in \mathbf{U}_n.$$

Proof. It follows from condition (1) that for $(i,j) \in \mathbf{U}_n$, $v_{i,j} = \langle \ell_{i,j}, \mathcal{P}_i v - \mathcal{P}_{i-1}v\rangle$. This with condition (2) yields

$$|v_{i,j}| \leq c\|\mathcal{P}_i v - \mathcal{P}_{i-1}v\|_\infty. \tag{4.8}$$

On the other hand, for $v \in W^{k,\infty}(\Omega)$, condition (3) leads to

$$\|\mathcal{P}_i v - \mathcal{P}_{i-1}v\|_\infty \leq \|\mathcal{P}_i v - v\|_\infty + \|v - \mathcal{P}_{i-1}v\|_\infty \leq cr^{-ki/d}\|v\|_{k,\infty}.$$

Combining the above inequality with (4.8) yields the estimate of this proposition. □

Note that the integer $r = \mu^d$ by our construction. We further remark that conditions (1) and (3) are Hypotheses (II) and (X) in [**11**], respectively, and condition (2) is a part of Hypothesis (IV) there. Note that the basis functions and collocation functionals constructed in the previous section satisfy Hypotheses (I)–(VI) and (VIII)–(X). Employing Proposition 4.1 to replace Lemma 4.1 in [**11**], we have the following theorem, which improves Theorems 4.4, 4.5, 4.6 in [**11**].

Theorem 4.2. *Let $u \in W^{k,\infty}(\Omega)$ be the solution of equation* (4.2), *and let $\tilde{u}_n \in \mathbf{X}_n$ be its approximate solution associated with the solution of* (4.7). *If the truncation parameter $\varepsilon^n_{i',i}$ is chosen as* (4.6) *with $b = 1$,*

$k/(2k-\sigma') < b' < 1$, then there exist a positive constant c and a positive integer N such that for all $n \geq N$

$$\|u - \tilde{u}_n\|_\infty = cf(n)^{-k/d} \log f(n) \|u\|_{k,\infty},$$
$$\mathcal{N}(\mathbf{E}_n - \widetilde{\mathbf{K}}_n) = cf(n) \log f(n)$$

and

$$\mathrm{cond}\,(\mathbf{E}_n - \widetilde{\mathbf{K}}_n) \leq c \log^2 f(n),$$

where for a matrix $\mathbf{A}$, $\mathcal{N}(\mathbf{A})$ and $\mathrm{cond}\,(\mathbf{A})$ denote the number of its nonzero entries and condition number, respectively.

5. Error control of the quadrature rule. The entries of matrix $\widetilde{\mathbf{K}}_n$ are all high-dimensional (nearly) weakly singular integrals due to the singularity of the kernel K. In practice, they all have to be computed numerically. In this section we derive a quadrature rule for computing such singular integrals. We prove that the quadrature rule has an optimal order of convergence. Based on this estimate, we develop a strategy to control the quadrature error so that the overhead error caused by the numerical integration is essentially in the same order as the approximation error presented in Theorem 4.2 while the computational cost to form the matrix $\widetilde{\mathbf{K}}_n$ is nearly in the same order of the number of nonzero entries of $\widetilde{\mathbf{K}}_n$. The design of the quadrature rule presented in this section is influenced by the work [**18, 23, 36, 39**].

We now describe the quadrature rule. Suppose that $\Omega \subset \mathbf{R}^d$ is a bounded domain and $s \in \Omega$ is a fixed point. We consider a class of functions f which have singular behavior near the point s. Specifically, a function f is said to be in the class $\mathcal{A}_s$ if there exists a disjoint decomposition of Ω

$$\Omega = \bigcup_{j \in \mathbf{Z}_M} \Omega_j, \quad \mathrm{int}\,(\Omega_i) \cap \mathrm{int}\,(\Omega_j) = \varnothing \quad \text{for} \qquad i \neq j,$$

such that for any $j \in \mathbf{Z}_M$, $f|_{\mathrm{int}(\Omega_j)} \in C^\infty(\mathrm{int}(\Omega_j)\backslash\{s\})$, and there exists a positive constant $\sigma \in [0, d)$ such that for any $t \in \Omega\backslash\{s\}$

$$|f(t)| \leq \theta' \begin{cases} 1/(|s-t|^\sigma) & 0 < \sigma < d, \\ \log(1/|s-t|) & \sigma = 0, \end{cases} \tag{5.1}$$

and

$$|D^\alpha f(t)| \leq \frac{|\alpha|!\theta'}{|s-t|^{\sigma+|\alpha|}}, \quad \alpha \in \mathbf{N}_0^d \tag{5.2}$$

for some positive constant θ'. Here the parameter σ measures the "degree" of the singularity of f at the point s. For a point $s \in \Omega$, we set $r_s := \sup\{|s-t| : t \in \Omega\}$. For any $\gamma \in (0,1)$, let $t_0 = 0$, $t_\iota = \gamma^{m-\iota}$, $\iota = 1, 2, \ldots, m$. We subdivide the domain Ω by

$$\Omega := \bigcup_{\iota \in \mathbf{Z}_m} \bigcup_{\tau \in \mathbf{Z}_{\mu(\iota)}} D_{\iota,\tau},$$

which satisfies the following conditions.

(I) The interiors of the subdomains $D_{\iota,\tau}$ are disjoint, i.e., $\mathrm{int}\,(D_{\iota,\tau}) \cap \mathrm{int}\,(D_{\iota',\tau'}) = \varnothing$ for $(\iota,\tau) \neq (\iota',\tau')$.

(II) For any $(\iota,\tau) \in \mathbf{V}_m := \{(\iota,\tau) : \tau \in \mathbf{Z}_{\mu(\iota)}, \iota \in \mathbf{Z}_m\}$, $D_{\iota,\tau} \subset \Omega_j$ for some $j \in \mathbf{Z}_M$.

(III) For any $(\iota,\tau) \in \mathbf{V}_m$, $c_1 r_s t_\iota \leq \mathrm{dist}\,(s, D_{\iota,\tau}) \leq c_1' r_s t_{\iota+1}$ for some positive constants c_1 and c_1', where

$$\mathrm{dist}\,(s, D) := \inf\{|s-t| : t \in D\}$$

for a domain $D \subset \mathbf{R}^d$.

(IV) There is a positive constant c_2 such that for $(\iota,\tau) \in \mathbf{V}_m$, $\mathrm{diam}\,(D_{\iota,\tau}) \leq c_2(t_{\iota+1} - t_\iota) r_s$, where $\mathrm{diam}\,(D) := \sup\{|t-t'| : t, t' \in D\}$.

(V) There exists a positive constant c_3 such that for $\iota \in \mathbf{Z}_m$, $\mu(\iota) \leq c_3 M(t_{\iota+1}^d - t_\iota^d)$.

A prototype of the above subdivision is the case that Ω is a ball centered at the point s and we subdivide Ω into a collection of rings centered at the same point s with radius $r_s t_{\iota+1}$, $\iota \in \mathbf{Z}_{m-1}$. Each ring is then decomposed into subdomains with the number of subdomains in each ring being proportional to t_ι. Through out this section, we assume that such a subdivision of Ω is available.

For $f \in \mathcal{A}_s$, to compute the integral of f on each subdomain $D_{\iota,\tau}$, we select a set of points $X_{\iota,\tau} := \{x_{\iota,\tau}^l : l \in \mathbf{Z}_{\nu(\iota,\tau)}\}$, where $\nu(\iota,\tau)$ is a positive integer depending on ι and τ. For each $\tau \in \mathbf{Z}_{\mu(0)}$, we choose the weights $a_{0,\tau}^l$ to be 0, that is, we omit the integral of f in $D_{0,\tau}$, $\tau \in \mathbf{Z}_{\mu(0)}$. For $\iota > 0$ and for a positive integer k_ι, we find the weights $A_{\iota,\tau} := \{a_{\iota,\tau}^l : l \in \mathbf{Z}_{\nu(\iota,\tau)}\}$, such that

$$\int_{D_{\iota,\tau}} f(t)\,dt = \sum_{l \in \mathbf{Z}_{\nu(\iota,\tau)}} a_{\iota,\tau}^l f(x_{\iota,\tau}^l) + E_{k_\iota}(D_{\iota,\tau}, A_{\iota,\tau}, X_{\iota,\tau})$$

with

$$|E_{k_\iota}(D_{\iota,\tau}, A_{\iota,\tau}, X_{\iota,\tau})| \le \frac{c_4}{k_\iota!} \sum_{|\alpha|=k_\iota} |(D^\alpha f)(\eta_\alpha)| [\mathrm{diam}\,(D_{\iota,\tau})]^{k_\iota} \mathrm{meas}\,(D_{\iota,\tau})$$

for some positive constant c_4 with $\eta_\alpha \in D_{\iota,\tau}$. Hence, by introducing the vector $\mathbf{k} := [k_\iota : \iota = 1, 2, \dots, m]$, we have the quadrature formula

$$Q_{m,\mathbf{k}}(f) := \sum_{(\iota,\tau) \in \mathbf{V}_m} \sum_{l \in \mathbf{Z}_{\nu(\iota,\tau)}} a_{\iota,\tau}^l f(x_{\iota,\tau}^l).$$

For $\iota \in \mathbf{Z}_m$, we set

$$\varepsilon_{\iota,\tau} := \int_{D_{\iota,\tau}} f(t)\,dt - \sum_{l \in \mathbf{Z}_{\nu(\iota,\tau)}} a_{\iota,\tau}^l f(x_{\iota,\tau}^l), \quad \tau \in \mathbf{Z}_{\mu(\iota)}$$

$$E_{m,\mathbf{k},\iota}(f) := \sum_{\tau \in \mathbf{Z}_{\mu(\iota)}} |\varepsilon_{\iota,\tau}|, \quad \text{and} \quad E_{m,\mathbf{k}}(f) := \sum_{\iota \in \mathbf{Z}_m} E_{m,\mathbf{k},\iota}(f).$$

Clearly, $E_{m,\mathbf{k}}(f)$ is an upper bound of the error between the integral of f on Ω and the quadrature $Q_{m,\mathbf{k}}(f)$. In the next lemma, we choose the vector $\mathbf{k}$ and estimate an upper bound of the corresponding error $E_{m,\mathbf{k}}(f)$.

Lemma 5.1. *Let $\epsilon > 0$ and choose*

$$k_\iota := \lceil \epsilon\iota \rceil, \quad \iota = 1, 2, \dots, m. \tag{5.3}$$

If $f \in \mathcal{A}_s$, then there exists $\gamma \in (0,1)$ such that

$$E_{m,\mathbf{k}}(f) \le c\gamma^{(d-\sigma)m} \tag{5.4}$$

for some positive constant c independent of m.

Proof. It follows from our quadrature rule that

$$E_{m,\mathbf{k},0}(f) \le \int_0^{c_1' r_s(t_1-t_0)} \theta' t^{-\sigma} t^{d-1}\, dt = \frac{\theta'[c_1' r_s]^{d-\sigma}}{(d-\sigma)\gamma^{d-\sigma}} \gamma^{(d-\sigma)m}.$$

For $(\iota,\tau) \in \mathbf{V}_m$ with $\iota > 0$,

$$\begin{aligned} |\varepsilon_{\iota,\tau}| &\le \frac{c_4}{k_\iota!}\binom{k_\iota+d-1}{d-1} k_\iota! \theta' (c_1 r_s t_\iota)^{-(\sigma+k_\iota)} [c_2 r_s (t_{\iota+1}-t_\iota)]^{k_\iota} \operatorname{meas}(D_{\iota,\tau}) \\ &= c_1^{-(\sigma+k_\iota)} c_2^{k_\iota} c_4 \theta' \binom{k_\iota+d-1}{d-1} r_s^{-\sigma} t_\iota^{-\sigma} \left(\frac{1}{\gamma}-1\right)^{k_\iota} \operatorname{meas}(D_{\iota,\tau}). \end{aligned}$$

It follows from Properties (I) and (III) that

$$\sum_{\tau \in \mathbf{Z}_{\mu(\iota)}} \operatorname{meas}(D_{\iota,\tau}) \le V_d[(c_1' r_s t_{\iota+1})^d - (c_1 r_s t_\iota)^d],$$

where V_d is the volume of the d-dimensional unit ball. Therefore,

$$\begin{aligned} E_{m,\mathbf{k},\iota}(f) &\le V_d c_1^{-(\sigma+k_\iota)} c_2^{k_\iota} c_4 \theta' \binom{k_\iota+d-1}{d-1} r_s^{-\sigma} t_\iota^{-\sigma} \left(\frac{1}{\gamma}-1\right)^{k_\iota} \\ &\quad \times [(c_1' r_s t_{\iota+1})^d - (c_1 r_s t_\iota)^d] \\ &= V_d c_1^{-(\sigma+k_\iota)} c_2^{k_\iota} c_4 \theta' \binom{k_\iota+d-1}{d-1} r_s^{d-\sigma} t_\iota^{d-\sigma} \left(\frac{1}{\gamma}-1\right)^{k_\iota} \\ &\quad \times \left[\left(\frac{c_1'}{\gamma}\right)^d - c_1^d\right] \\ &= V_d c_1^{-\sigma} c_4 \theta' r_s^{d-\sigma} \left[\left(\frac{c_1'}{\gamma}\right)^d - c_1^d\right] \binom{k_\iota+d-1}{d-1} \left(\frac{c_2}{c_1}\right)^{k_\iota} \\ &\quad \times \frac{(1-\gamma)^{k_\iota}}{\gamma^{k_\iota+\iota(d-\sigma)}} \gamma^{m(d-\sigma)} \\ &= c \left[\binom{k_\iota+d-1}{d-1} \gamma^{\sigma\iota/2}\right] \left[\left(\frac{c_2}{c_1}\right)^{k_\iota} \frac{(1-\gamma)^{k_\iota}}{\gamma^{k_\iota+\iota d}}\right] \gamma^{\sigma\iota/2} \gamma^{m(d-\sigma)} \end{aligned}$$

with

$$c := V_d c_1^{-\sigma} c_4 \theta' r_s^{d-\sigma} \left[\left(\frac{c_1'}{\gamma}\right)^d - c_1^d\right]$$

independent of m and ι. It is easily observed that there is $\gamma \in (0,1)$ such that $(1-\gamma)/(\gamma^{1+d/\varepsilon}) < (c_1/c_2)$. By the definition of k_ι, we observe that $k_\iota d/\varepsilon \geq \iota d$. It follows that

$$\left[\left(\frac{c_2}{c_1}\right)^{k_\iota} \frac{(1-\gamma)^{k_\iota}}{\gamma^{k_\iota+\iota d}}\right] \leq \left[\left(\frac{c_2}{c_1}\right)^{k_\iota} \frac{(1-\gamma)^{k_\iota}}{\gamma^{k_\iota+k_\iota d/\varepsilon}}\right] \leq 1$$

for any ι. When $k_\iota > d-1$, $\binom{k_\iota+d-1}{d-1} \leq k_\iota^{d-1}$. Hence, $\binom{k_\iota+d-1}{d-1}\gamma^{\sigma\iota/2}$ is bounded uniformly for all k_ι. Therefore,

$$E_{m,\mathbf{k},\iota}(f) \leq c\gamma^{\sigma\iota/2}\gamma^{(d-\sigma)m}.$$

Summing up the above inequalities with respect to ι concludes the lemma. $\square$

We next apply the quadrature rule to the computation of the entries of the compressed matrix $\widetilde{\mathbf{K}}_n$. We assume that the kernel $K(s,t)$ has the singularity property that for any $s \in \Omega$, $K(s,\cdot) \in \mathcal{A}_s$. Recall that $\widetilde{\mathbf{K}}_n$ is obtained from the full matrix $\mathbf{K}_n$ by the truncation strategy with the truncation parameters $\varepsilon^n_{i'i}$, $i', i \in \mathbf{Z}_{n+1}$. For given $\varepsilon^n_{i'i}$, we introduce an index set

$$\mathbf{Z}_{i'j',i} := \{j \in \mathbf{Z}_{w(i)} : \mathrm{dist}\,(S_{i'j'}, S_{ij}) \leq \varepsilon^n_{i'i}\}$$

for $(i',j') \in \mathbf{U}_n$ and for $\ell \in \mathbf{Z}_r$ define $\mathbf{Z}^\ell_{i'j',i} := \{j \in \mathbf{Z}_{i'j',i} : j = \mu(\mathbf{e})r + \ell\}$. Observing that $\mathbf{Z}^\ell_{i'j',i} \subset \mathbf{Z}_{i'j',i}$ and for $j_1, j_2 \in \mathbf{Z}^\ell_{i'j',i}$ with $j_1 \neq j_2$, $\mathrm{meas}\,(\mathrm{supp}\,(w_{i,j_1}) \cap \mathrm{supp}\,(w_{i,j_2})) = 0$, we define for $\ell \in \mathbf{Z}_r$ and $(i',j') \in \mathbf{U}_n$

$$\bar{w}_{i'j',i\ell}(t) := \begin{cases} w_{i,j}(t) & \text{if } t \in \mathrm{int}\,(\mathrm{supp}\,(w_{i,j})) \text{ for some } j \in \mathbf{Z}^\ell_{i'j',i}, \\ 0 & \text{otherwise}, \end{cases}$$

and let $\bar{h}_{i'j',i\ell}(s,t) := K(s,t)\bar{w}_{i'j',i\ell}(t)$. We next estimate the error of the quadrature for computing the integral of $\bar{h}_{i'j',i\ell}$.

Lemma 5.2. *If $\mathbf{k}$ is chosen as* (5.3)*, then there exists a positive constant c such that for all $i \in \mathbf{Z}_{n+1}$, $\ell \in \mathbf{Z}_r$, $(i',j') \in \mathbf{U}_n$ and $s \in \Omega$,*

$$|E_{m,\mathbf{k}}(\bar{h}_{i'j',i\ell})| \leq c\mu^{k(i-1)}\gamma^{(d-\sigma)m},$$

where k is the order of the piecewise polynomials in $\mathbf{X}_0$.

Proof. Let

$$\Delta := \max_{\substack{\beta\in\mathbf{N}_0^d,\\ |\beta|<k,\\ l\in\mathbf{Z}_r}} \text{ess sup}\{|D^\beta w_{1,l}(t)| : t\in\Omega\}.$$

Since $w_{1,l}$ is a piecewise polynomial of order k on Ω, Δ is finite. For $|\alpha| = k_\iota$ and $t \in \text{supp}\,(w_{i,j})$, a direct computation gives

$$\begin{aligned}
|D^\alpha \bar{h}_{i'j',i\ell}(t)| &= \left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta} D^{\alpha-\beta}K(s,t)D^\beta \bar{w}_{i'j',i\ell}(t)\right| \\
&\le \theta' \sum_{\beta\le\alpha}\binom{\alpha}{\beta}(k_\iota-|\beta|)!|s-t|^{-(\sigma+k_\iota-|\beta|)}\mu^{|\beta|(i-1)} \\
&\qquad\qquad \times |D^\beta w_{1,l}(\phi_{\mathbf{e}}^{-1}(t))| \\
&\le \theta'\Delta k_\iota!|s-t|^{-(\sigma+k_\iota)} \sum_{\substack{\beta\le\alpha\\ |\beta|\le k}}\binom{\alpha}{\beta}|s-t|^{|\beta|}\mu^{|\beta|(i-1)},
\end{aligned}$$

in which $\beta\le\alpha$ for two vectors $\alpha := (\alpha_0,\ldots,\alpha_{d-1})$, $\beta := (\beta_0,\ldots,\beta_{d-1}) \in \mathbf{N}_0^d$ means that $\beta_i \le \alpha_i$ for $i \in \mathbf{Z}_d$ and $\binom{\alpha}{\beta} = \Pi_{i\in\mathbf{Z}_d}\binom{\alpha_i}{\beta_i}$. Since $\alpha_i \le |\alpha| = k_\iota$ and $\beta_i \le k$, $\binom{\alpha_i}{\beta_i} \le k_\iota^k$, and thus $\binom{\alpha}{\beta} \le k_\iota^{kd}$. For $\tau' \in \mathbf{Z}_{k+1}$, the number of $\beta \in \mathbf{N}_0^d$ with $|\beta| = \tau'$ is $\binom{\tau'+d-1}{d-1}$. Hence,

$$|D^\alpha \bar{h}_{i'j',i\ell}(t)| \le \theta'\Delta k_\iota! k_\iota^{kd}|s-t|^{-(\sigma+k_\iota)} \sum_{\tau'\in\mathbf{Z}_{k+1}}\binom{\tau'+d-1}{d-1}[|s-t|\mu^{(i-1)}]^{\tau'}.$$

Noting that $c_1 r_s\gamma^{m-\iota} \le |s-t| \le r_s$ and $\binom{\tau'+d-1}{d-1} \le (k+d-1)^{d-1}$, there holds

$$|D^\alpha \bar{h}_{i'j',i\ell}(t)| \le c\theta'\Delta k_\iota! k_\iota^{kd}(k+d-1)^{d-1}(c_1 r_s\gamma^{m-\iota})^{-(\sigma+k_\iota-k)}\mu^{k(i-1)}.$$

Similar to Lemma 5.1, we conclude that there exists a positive constant c independent of m and i such that

$$E_{m,\mathbf{k}}(\bar{h}_{i'j',i\ell}) \le c\mu^{k(i-1)}\gamma^{(d-\sigma)m}. \qquad \square$$

Utilizing the properties of the multi-scale basis and collocation functionals and similar arguments in [**13**] we conclude that there exists a positive constant c such that for all $i', i \in \mathbf{Z}_{n+1}$,

$$\|\widetilde{\mathbf{K}}_{i'i} - \widetilde{\widetilde{\mathbf{K}}}_{i'i}\|_\infty \leq c\mu^{(i-1)k}\gamma^{(d-\sigma)m},$$

where $\widetilde{\widetilde{\mathbf{K}}}_{i'i}$ is the block $\widetilde{\mathbf{K}}_{i'i}$ when its entries are computed numerically using the proposed quadrature rule. To ensure that the numerical integration will not ruin the convergence order of the collocation scheme, we have to carefully choose the value of m in the computation of the matrix elements. We choose different integers $m_{i'i}$, $i', i \in \mathbf{Z}_{n+1}$ for the numerical integration of the entries in different blocks $\widetilde{\mathbf{K}}_{i'i}$. With nearly the same arguments in [**13**], we obtain the following theorem.

Theorem 5.3. *Let u be the solution of equation* (4.2)*. For $i', i \in \mathbf{Z}_{n+1}$, let $m_{i'i}$ satisfy*

$$m_{i'i} \geq \frac{-k\log\mu}{(d-\sigma)\log\gamma}(2i + i' - 1). \tag{5.5}$$

Suppose that the kernel $K(s,t)$ as a function of t satisfies (5.1) *and* (5.2)*, and we apply the quadrature rule to the computation of the blocks $\widetilde{\mathbf{K}}_{i'i}$ with $m := m_{i'i}$ and $\mathbf{k}$ defined by* (5.3)*. Let $\tilde{\tilde{u}}_n$ denote the corresponding approximate solution. If $u \in W^{k,\infty}(\Omega)$, then there exists a positive constant c and a positive integer N such that, for all $n > N$,*

$$\|u - \tilde{\tilde{u}}_n\|_\infty \leq c(f(n))^{k/d}(\log f(n))^\tau \|u\|_{k,\infty}, \tag{5.6}$$

where $\tau = 1$ if $b' > k/(2k - \sigma')$ and $\tau = 2$ if $b' = k/(2k - \sigma')$.

Now we turn to analyzing the computational complexity for generating the matrix $\widetilde{\widetilde{\mathbf{K}}}_n$ in terms of the number of functional evaluations. For $i', i \in \mathbf{Z}_{n+1}$, we denote by $\mathcal{M}_{i'i}$ the total number of functional evaluations for computing the entries of $\widetilde{\widetilde{\mathbf{K}}}_{i'i}$. Then the total number of functional evaluations used for computing all the entries of $\widetilde{\widetilde{\mathbf{K}}}_n$ is given by

$$\mathcal{M}_n := \sum_{i' \in \mathbf{Z}_{n+1}} \sum_{i \in \mathbf{Z}_{n+1}} \mathcal{M}_{i'i}.$$

The following theorem gives an estimate of $\mathcal{M}_n$.

Theorem 5.4. *Suppose that the number of points used in the quadrature rule is of order k_ι^d, i.e., $\nu(\iota, \tau) \leq c k_\iota^d$ for some positive constant c, $m_{i'i}$, $i', i \in \mathbf{Z}_{n+1}$ are chosen to be the smallest integer satisfying* (5.5)*, and the truncation parameters $\varepsilon_{i'i}^n$ are chosen as* (4.6) *with $b' \leq 1$, $b \leq 1$. Then, there exists a positive constant c such that for $n \in \mathbf{N}$*

$$\mathcal{M}_n \leq c f(n)(\log f(n))^{d+2}. \tag{5.7}$$

Proof. For $i', i \in \mathbf{Z}_{n+1}$, we let $\mathcal{M}_{i'i,j'}$ denote the number of functional evaluations used in computing the j'th row of the block $\tilde{\tilde{\mathbf{K}}}_{i'i}$. Recalling that there are no more than $\mu^{d(i'+1)}$ rows in this block, we obtain

$$\mathcal{M}_{i'i} \leq \mu^{d(i'+1)} \max_{j'} \mathcal{M}_{i'i,j'}.$$

For a function h, we use $\mathcal{M}(h)$ to denote the number of functional evaluations in the numerical integration of h. It follows from the definition of $\bar{h}_{i'j',i\ell}$ that

$$\begin{aligned} \mathcal{M}_{i'i,j'} &\leq c \sum_{j \in \mathbf{Z}_{i'j',i}} \mathcal{M}(K(s,\cdot)w_{i,j}) = c \sum_{\ell \in \mathbf{Z}_r} \sum_{j \in \mathbf{Z}_{i'j',i}^{\ell}} \mathcal{M}(K(s,\cdot)w_{i,j}) \\ &= c \sum_{\ell \in \mathbf{Z}_r} \mathcal{M}(\bar{h}_{i'j',i\ell}), \end{aligned} \tag{5.8}$$

where the constant c is the upper bound of the number of the points involved in the functionals $\ell_{i',j'}$, which is independent of n, i' and i. By our assumptions on the quadrature rule, there holds

$$\mathcal{M}(\bar{h}_{i'j',i\ell}) \leq c \sum_{\iota=1}^{l} k_\iota^d \left[c_3 r_s (\gamma^{d(m_{i'i}-\iota-1)} - \gamma^{d(m_{i'i}-\iota)}) \mu^{d(i-1)} \right]$$

for some $l \leq m$. We remark that if the truncation strategy is not applied, then $l = m$. Since $k_\iota < \varepsilon\iota + 1$, we obtain that

$$\mathcal{M}_{i'i,j'} \leq c \left[c_3 r_s \left(\frac{1}{\gamma} - 1 \right)^d \mu^{d(i-1)} \sum_{\iota=1}^{l} (\varepsilon\iota + 1)^d \gamma^{d(m_{i'i}-\iota)} \right].$$

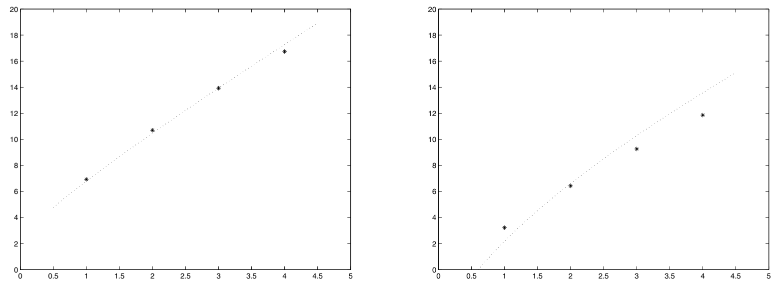

FIGURE 5. The numbers of nonzero elements (left) and computational costs (right) of the coefficient matrices.

According to the truncation strategy,

$$\gamma^{m_{i'i}-\iota} \leq \varepsilon^n_{i'i} + d_i + d_{i'}, \quad \text{for all} \quad \iota \leq l,$$

where d_i is the upper bound of the diameter of the support of the basis functions at level i. By our choice of $\varepsilon^n_{i'i}$ and $m_{i'i}$, there is a positive constant c such that

$$\mathcal{M}_{i'i,j'} \leq c\left[n^d\mu^{d(i-1)}\left(\mu^{-n+b'(n-i')+b(n-i)} + \mu^{-i+1} + \mu^{-i'+1}\right)^d\right].$$

Therefore,

$$\mathcal{M}_n \leq c\sum_{i'\in\mathbf{Z}_{n+1}}\sum_{i\in\mathbf{Z}_{n+1}}\left[n^d\mu^{d(i+i')}\left(\mu^{-n+b'(n-i')+b(n-i)} + \mu^{-i+1}+\mu^{-i'+1}\right)^d\right].$$

A simple computation leads to $\mathcal{M}_n \leq cn^{d+2}\mu^{dn}$, proving the theorem. $\square$

6. A numerical experiment. In this section we present a numerical experiment of solving equation (4.2) on a three-dimensional simplex by the fast collocation method. Consider the equation

$$u(s) - \int_\Omega K(s,t)u(t)\,dt = f(s), \tag{6.1}$$

where $\Omega := \{t = (x, y, z) : 0 \le x \le y \le z \le 1\}$ and $K(s, t) := 1/|s - t|$, $s, t \in \Omega$. The righthand side function f is chosen to be $f(s) = |t|^2 - \int_\Omega |s|^2/|s - t|\, dt$ so that the solution of (6.1) has the form $u(s) = x^2 + y^2 + z^2$.

We use the three-dimensional linear basis and the corresponding collocation functionals to discretize equation (6.1). The compressed coefficient matrix of the resulting linear system is obtained by using the choice (4.6) of the truncation parameters with $a = 0.125$, $b = 1$, $b' = 0.9$, $\rho = 1.01$. The resulting linear system with a sparse matrix is solved by the multilevel augmentation method developed in [**12**] with the initial level 2. We denote by $\tilde{u}_{2,n-2}$ the approximate solution obtained from the multilevel augmentation method at level n with the initial level 2.

The numerical experiment is done in a Pentium 4 personal computer with 3GHz CPU and 512M memory.

We report in Table 3 the error of the approximate solution $\tilde{u}_{2,n-2}$, convergence order (conv. order), the number of nonzero entries of the matrix $\widetilde{\mathbf{K}}_n$, compression rate (comp. rate), computational time (CT) measured in seconds for computing the entries of the matrix $\widetilde{\mathbf{K}}_n$, and computational time (ST) for solving the linear system. The convergence order α is computed according to the formula

$$\alpha := \log \frac{\|u - \tilde{u}_{2,n-3}\|_\infty}{\|u - \tilde{u}_{2,n-2}\|_\infty} \Big/ \log \frac{f(n)}{f(n-1)}.$$

Note that the theoretical convergence order is $\alpha = 2/3$ up to a logarithm factor, because $k = 2$ and $d = 3$. We observe that the computed convergence order is approximately equal to 2/3.

The compression rate is defined as the ratio of the number of the nonzero entries of the compressed matrix $\widetilde{\mathbf{K}}_n$ and that of the full matrix $\mathbf{K}_n$, i.e., $\mathcal{N}(\widetilde{\mathbf{K}}_n)/f(n)^2$. For $n = 4$, the compression rate 0.069176 tells us that the truncation strategy saves our time by ignoring the calculation of more than 93% entries of $\mathbf{K}_n$. The theoretical estimate for the nonzero entries of matrix $\widetilde{\mathbf{K}}_n$ reveals that

$$\mathcal{N}(\widetilde{\mathbf{K}}_n) = \mathcal{O}(f(n) \log f(n)).$$

In order to confirm this estimate, we plot in Figure 5 (left) four points $(n, \log \mathcal{N}(\widetilde{\mathbf{K}}_n))$ for $n = 1, 2, 3, 4$, marked with the mark '$*$'. Since

$f(n) = 4 \cdot 8^n$, the points should match the graph of the function $y = \log(4\rho \cdot 8^x \log(4 \cdot 8^x))$ for some positive constant ρ if our theoretical estimate matches the computed result. We let $\rho = 1$ and plot the graph in dotted curve. It shows that the data points match the curve very well.

Recall that the total number of functional evaluations in generating the matrix $\widetilde{\mathbf{K}}_n$ is given by

$$\mathcal{M}_n = \mathcal{O}(f(n)(\log f(n))^5).$$

The value $\mathcal{M}_n$ is asymptotically proportional to the total time spent for generating the matrix $\widetilde{\mathbf{K}}_n$, which are shown in the "CT" column. We utilize the data in column "CT" to plot four points (n, CT_n) with '$*$' marks in Figure 5 (right) and compare them with the dotted curve of the graph of the function $y = \log(4 \cdot 8^x) + 5\log(\log(4 \cdot 8^x)) - 7.5$.

The last column "ST" records the time for solving the resulting linear system via the multilevel augmentation method. We observe the high efficiency of the algorithm by noticing that it spends only 0.156 seconds to solve a linear system of dimension 16384.

REFERENCES

1. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, Cambridge, UK, 1997.

2. K.E. Atkinson and G. Chandler, *The collocation method for solving the radiosity equation for unoccluded surfaces*, J. Integral Equations Appl. **10** (1998), 253–290.

3. K.E. Atkinson and D. Chien, *A fast matrix-vector multiplication method for solving the radiosity equation*, Adv. Comput. Math. **12** (2000), 151–174.

4. K.E. Atkinson, I. Graham and I. Sloan, *Piecewise continuous collocation for integral equations*, SIAM J. Numer. Anal. **20** (1983), 172–186.

5. G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms* I, Comm. Pure Appl. Math. **44** (1991), 141–183.

6. Z. Chen, C.A. Micchelli and Y. Xu, *The Petrov-Galerkin methods for second kind integral equations* II: *Multiwavelet scheme*, Adv. Comp. Math. **7** (1997), 199–233.

7. ———, *A construction of interpolating wavelets on invariant sets*, Math. Comp. **68** (1999), 1560–1587.

8. ———, *Hermite interpolating wavelets*, in *Lecture notes in scientific computation*, Luoluo Li, Zhongying Chen and Yongdong Zhang, eds., International Culture Publishing Inc., Beijing, 2000.

9. ———, *A multilevel method for solving operator equations*, J. Math. Anal. Appl. **262** (2001), 688–699.

10. ———, *Discrete wavelet Petrov-Galerkin methods*, Adv. Comp. Math. **16** (2002), 1–28.

11. ———, *Fast collocation methods for second kind integral equations*, SIAM J. Numer. Anal. **40** (2002), 344–375.

12. Z. Chen, B. Wu and Y. Xu, *Multilevel augmentation methods for solving operator equations*, Numerical Mathematics: A Journal of Chinese Universities (English Series) **14** (2005), 31–55.

13. ———, *Error control strategies for numerical integrations in fast collocation methods*, Northeast Math. J. **21** (2005), 233–252.

14. ———, *Fast numerical collocation solutions of integral equations*, Comm. Pure Appl. Anal., to appear.

15. W. Dahmen, S. Proessdorf and R. Schneider, *Wavelet approximation methods for pseudodifferential equations* II: *Matrix compression and fast solutions*, Adv. Comput. Math. **1** (1993), 259–335.

16. W. Fang and M. Lu, *A fast collocation method for an inverse boundary value problem*, Inter. J. Numerical Methods Engineering **59** (2004), 1563–1585.

17. W. Fang, F. Ma and Y. Xu, *Multilevel iteration methods for solving integral equations of the second kind*, J. Integral Equations Appl. **14** (2002), 355–376.

18. W.F. Ford, Y. Xu and Y. Zhao, *Derivative correction for quadrature formulas*, Adv. Comput. Math. **6** (1996), 139–157.

19. R. Horn and C. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

20. G.C. Hsiao, and A. Rathsfeld, *Wavelet collocation methods for a first kind boundary integral equation in acoustic scattering*, Adv. Comput. Math. **17** (2002), 281–308.

21. J.E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

22. H. Kaneko, R. Noren and Y. Xu, *Regularity of the solution of Hammerstein equations with weakly singular kernel*, Integral Equations Operator Theory **13** (1990), 660–670.

23. H. Kaneko and Y. Xu, *Gauss-type quadratures for weakly singular integrals and their application to Fredholm integral equations of the second kind*, Math. Comp. **62** (1994), 739–753.

24. R. Kress, *Linear integral equations*, Springer-Verlag, Berlin, 1989.

25. C.A. Micchelli, T. Sauer and Y. Xu, *A construction of refinable sets for interpolating wavelets*, Results Math. **34** (1998), 359–372.

26. ———, *Subdivision schemes for iterated function systems*, Proc. Amer. Math. Soc. **129** (2001), 1861–1872.

27. C.A. Micchelli and Y. Xu, *Using the matrix refinement equation for the construction of wavelets on invariant sets*, Appl. Comput. Harmonic Anal. **1** (1994), 391–401.

28. ———, *Reconstruction and decomposition algorithms for biorthogonal wavelets*, Multidimensional Sys. Signal Processing **8** (1997), 31–69.

29. C.A. Micchelli, Y. Xu and Y. Zhao, *Wavelet Galerkin methods for second-kind integral equations*, J. Comput. Appl. Math. **86** (1997), 251–270.

30. T. von Petersdorff and C. Schwab, *Wavelet approximation of first kind integral equations in a polygon*, Numer. Math. **74** (1996), 479–516.

31. T. von Petersdorff, C. Schwab and R. Schneider, *Multiwavelets for second-kind integral equations*, SIAM J. Numer. Anal. **34** (1997), 2212–2227.

32. J. Pitkäranta, *Estimates for the derivatives of solutions to weakly singular Fredholm integral equations*, SIAM J. Math. Anal. **11** (1980), 952–968.

33. A. Rathsfeld, *A wavelet algorithm for the solution of a singular integral equation over a smooth two-dimensional manifold*, J. Integral Equations Appl. **10** (1998), 445–501.

34. C. Schneider, *Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind*, Integral Equations Operator Theory **2** (1979), 62–68.

35. R. Schneider, *Multiskalen- und wavelet-Matrixkompression: Analyiss-sasierte Methoden zur effizienten Lösung großer vollbesetzter Gleichungs-systeme'*, Habilitationsschrift, Technische Hochschule Darmstadt, 1995.

36. C. Schwab, *Variable order composite quadrature of singular and nearly singular integrals*, Computing **53** (1994), 173–194.

37. G. Vainikko, *Multidimensional weakly singular integral equations*, Lecture Notes Math. **1549**, Springer-Verlag, Berlin, 1993.

38. Y. Wang and Y. Xu, *A fast wavelet collocation method for integral equations on polygons*, J. Integral Equations Appl. **17** (2005), 277–330.

39. Y. Xu, H.-L. Chen and Q. Zou, *Limit values of derivatives of the Cauchy integrals and computation of the logarithmic potential*, Computing **73** (2004), 295–237.

Department of Scientific Computing and Computer Applications, Sun Yat-sen University, Guangzhou 510275, China
Email address: lnsczy@mail.sysu.edu.cn

Department of Scientific Computing and Computer Applications, Sun Yat-sen University, Guangzhou 510275, China
Email address: wubin@mail.sysu.edu.cn

Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150
Email address: yxu06@syr.edu