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ON AN EXTENSION OF THE TROTTER-KATO THEOREM FOR RESOLVENT FAMILIES OF OPERATORS

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ABSTRACT. We extend the well-known theorem on convergence and approximation of C_0-semigroups due to Trotter and Kato to the context of resolvent families of operators. In particular, this result also extends those due to Goldstein [**4**] for cosine families and it is applied to the case of a class of Volterra equations which were considered by Prüss [**11**].

1. Introduction. Let X be a Banach space with norm $\|\ \|$. Let A be an unbounded closed linear operator in X, with dense domain $D(A)$, and $k \in L^1_{\text{loc}}(\mathbf{R}_+)$.

A strongly continuous family $\{R(t), t \geq 0\}$ of bounded linear operators in X is called a *resolvent family* (for equation (1.2) below) if it commutes with A and satisfies the resolvent equation

$$(1.1)\qquad R(t)x = x + \int_0^t k(t-s)AR(s)x\,ds, \quad t \geq 0, x \in D(A).$$

We remark that a resolvent family is unique if it exists (cf. [**3**]).

The notion of resolvent family is the natural extension of the concepts of a C_0-semigroup for $k(t) \equiv 1$ and of a cosine family for the case $k(t) \equiv t$.

The existence of a resolvent family allows one to solve the Volterra equation

$$(1.2)\quad u(t) = f(t) + \int_0^t k(t-s)Au(s)ds, \quad t \in [0,T] =: J,\ f \in C(J,X).$$

Equation (1.2) has been considered recently by many authors, since it has applications in different fields (see, for example, [**3, 11, 12**]).

The following generation theorem, due to Da Prato and Iannelli [**2**], is the extension for resolvent families of the well-known generation theorems due to Hille-Yosida for C_0-Semigroups and Sova for the case of cosine families.

THEOREM 1.1. [**11**] *Suppose A is a closed linear densely defined operator in the Banach space X and let $k \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ satisfy*

$$\int_0^\infty |k(t)|e^{-wt}dt < \infty.$$

Then there exists a resolvent family $R(t)$ such that

$$||R(t)|| \leq Me^{wt}, \quad \textit{for all } t \geq 0 \tag{1.3}$$

and some constant $M \geq 1$, if and only if

$$\hat{k}(\mu) \neq 0 \quad \textit{and} \quad 1/\hat{k}(\mu) \in \rho(A), \quad \textit{for all } \mu > w, \tag{1.4}$$

and

$$||[(\mu - \mu\hat{k}(\mu)A)^{-1}]^{(n)}|| \leq \frac{Mn!}{(\mu - w)^{n+1}} \quad \textit{for all } \mu > w,$$

$$n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\},$$

where $\hat{k}(\mu)$ denotes the Laplace transform of $k(t)$ and $[\ ]^{(n)}$ denotes the n^{th} derivative.

The main purpose of this paper is to extend to resolvent families the Trotter-Kato theorem on convergence and approximation of C_0-Semigroups and its analog for the case of cosine families due to Goldstein [**4**]. The proof of this theorem is based on ideas in [**1**; Chapter 3, §3], [**4, 5, 6**; Chapter 8, §1 and Chapter 9, §2], [**7**] and [**9**].

As a consequence, we obtain results of the same type as those in Prüss [**11**], but this time on convergence of two special classes of kernels k and operators A (§3).

2. Approximation and convergence of resolvent families. Let $k \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ and suppose that $\hat{k}(\mu)$, the Laplace transform of $k(t)$, exists and is defined for $\mathrm{Re}\,\mu > w$.

Let $\{A_m, m \in \mathbf{N}_0\}$ be a sequence of closed linear operators on the Banach space X, with dense domains $D(A_m)$. Let $\{R_m(t),\ t \geq 0\}$ be resolvent families with respect to each A_m.

THEOREM 2.1. *Suppose that the following condition on the kernel* $k(t)$ *holds:*

$$\lim_{\mu \to \infty} |\hat{k}(\mu)| = 0, \tag{2.1}$$

and assume that the resolvent families $R_m(t)$ *exist and satisfy the "condition of stability,"*

$$||R_m(t)|| \leq Me^{wt} \text{ for all } t \geq 0, \quad m \in \mathbf{N}_0. \tag{2.2}$$

Then the following statements are equivalent:

(i) $\lim_{m\to\infty}(\mu - \mu\hat{k}(\mu)A_m)^{-1}x = (\mu - \mu\hat{k}(\mu)A_0)^{-1}x$ *for all* $\mu > w,\ x \in X$.

(ii) $\lim_{m\to\infty} R_m(t)x = R_0(t)x$ *for all* $x \in X,\ t \geq 0$. *Moreover, the convergence is uniform in* t *on every compact subset of* $\mathbf{R}_+$.

PROOF. (ii)⇒(i). This is an immediate consequence of (2.2) by using the dominated convergence theorem (for the Bochner integral) and the identity

$$(\mu - \mu\hat{k}(\mu)A_m)^{-1}x = \int_0^\infty e^{-\mu t} R_m(t)x\, dt, \quad \operatorname{Re}\mu > w,\ x \in X. \tag{2.3}$$

(Cf. Da Prato and Iannelli [**2**; formula 27, p. 213].)

(i)⇒(ii). Let

$$\mathcal{K} = \{x = (x_n)_0^\infty \subseteq X / \lim_{n\to\infty} x_n = x_0\}.$$

It is easy to see that $\mathcal{K}$ is a Banach space under the norm $|x| := \operatorname{Sup}_n ||x_n||$.

Define the operator $\mathcal{A}$ on $\mathcal{K}$ by

$$\mathcal{D}(\mathcal{A}) = \{x \in \mathcal{K} / x_n \in D(A_n) \text{ for all } n \in \mathbf{N}_0\}$$

and

$$\mathcal{A}x = y = (y_n)_0^\infty \in \mathcal{K} \Leftrightarrow x \in \mathcal{D}(\mathcal{A}) \text{ and } A_n x_n = y_n \text{ for all } n \in \mathbf{N}_0.$$

It is clear that $\mathcal{A}$ is a closed linear operator in $\mathcal{K}$. Now, let $x = (x_n)_0^\infty \in \mathcal{K}$ and $\mu > w$ be fixed. Then

$$\begin{aligned} &\|(\mu - \mu\hat{k}(\mu)A_m)^{-1}x_m - (\mu - \mu\hat{k}(\mu)A_0)^{-1}x_0\| \\ (2.4) \qquad &\leq \|(\mu - \mu\hat{k}(\mu)A_m)^{-1}\| \; \|x_m - x_0\| \\ &+ \|(\mu - \mu\hat{k}(\mu)A_m)^{-1}x_0 - (\mu - \mu\hat{k}(\mu)A_0)^{-1}x_0\|. \end{aligned}$$

On the other hand, from (2.2) and Theorem 1.1,

$$(2.5) \qquad \|[(\mu - \mu\hat{k}(\mu)A_m)^{-1}]^{(n)}\| \leq \frac{Mn!}{(\mu - w)^{n+1}} \quad \text{for all } n, m \in \mathbf{N}_0.$$

Therefore, we show by using (ii) and (2.5) in (2.4) that the operator $\mu - \mu\hat{k}(\mu)\mathcal{A}$ has an inverse in $\mathcal{K}$ defined by

$$(2.6) \qquad (\mu - \mu\hat{k}(\mu)\mathcal{A})^{-1}x = ((\mu - \mu\hat{k}(\mu)A_m)^{-1}x_m)_0^\infty,$$

$$\text{for all } \mu > w, \; x \in \mathcal{K}.$$

Now, in order to apply Theorem 1.1 to the operator $\mathcal{A}$ defined in $\mathcal{K}$, we require to prove that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{K}$. This can be done by the argument given in Goldstein [4] (we omit details) and using the fact that (2.1) implies

(2.7)

$$(I - \hat{k}(\mu)A_0)^{-1}x_0 - x_0 \text{ converges to } 0 \text{ as } \mu \to \infty, \text{ for all } x_0 \in X.$$

In fact, from the identity

$$(I - \hat{k}(\mu)A_0)^{-1}z - z = \hat{k}(\mu)(I - \hat{k}(\mu)A_0)^{-1}A_0 z, \quad \text{for all } z \in D(A_0),$$

and, by using (2.5), we obtain

$$(2.8) \quad \|(I - \hat{k}(\mu)A_0)^{-1}z - z\| \leq \frac{|\hat{k}(\mu)|M\mu}{(\mu - w)}\|A_0 z\|, \text{ for all } z \in D(A_0).$$

Now, (2.1) shows that on the left side (2.8) converges to 0 as $\mu \to \infty$ for all $z \in D(A_0)$. Finally, by using the fact that A_0 is densely defined, we obtain (2.7).

Therefore, there exists a resolvent family, $\mathcal{R}(t)$, defined in $\mathcal{K}$ and such that (1.3) holds.

Now it follows, from (2.3) and the uniqueness theorem for Laplace transforms, that

$$\mathcal{R}(t)x = (R_m(t)x_m)_0^\infty \quad \text{for all } t \geq 0,\ x \in \mathcal{K}. \tag{2.9}$$

Let $x \in X$ be fixed and choose $x_m \to x$. Then

$$||R_m(t)x - R_0(t)x|| \leq ||R_m(t)||\ ||x - x_m|| + ||R_m(t)x_m - R_0(t)x||. \tag{2.10}$$

Applying (2.2) and (2.9) in (2.10), obtains (ii).

Finally, in order to see that the convergence is uniform in t on every compact subset of $\mathbf{R}_+$, it is sufficient to observe that, for all $x_0 \in X$ and $0 \leq s, t \leq r$, with r fixed,

$$\begin{aligned} &||R_m(s)x_0 - R_0(s)x_0|| \\ &\quad \leq ||R_m(s)x_0 - R_m(t)x_0|| + ||R_m(t)x_0 - R_0(t)x_0|| \\ &\qquad + ||R_0(t)x_0 - R_0(s)x_0|| \\ &\quad \leq 2|\mathcal{R}(s)x - \mathcal{R}(t)x| + ||R_m(t)x_0 - R_0(t)x_0||, \end{aligned} \tag{2.11}$$

where $x = (x_n)_0^\infty \in \mathcal{K}$ is defined as $x_n = x_0$ for all $n \in \mathbf{N}_0$.

Now, using (ii) and the fact that $\mathcal{R}(t)$ is strongly continuous in $\mathcal{K}$ (see also Goldstein [**5**]) we obtain from (2.11) the assertion of the theorem. □

REMARK 2.2. The equivalences (i) and (ii) for resolvent families can be obtained by making use of a general result on convergence (cf. Lizama [**9**, Theorem 1.2]). However, it is more suitable to use a direct proof in order to obtain the condition (2.1).

REMARK 2.3. A variant of Theorem 2.1 is valid for operators acting on different spaces as follows: Let X_0 be a Banach space, and let

$\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces that approximate X_0 in the sense that, for each n, there exists a bounded linear operator P_n that maps X_0 into X_n and $\lim_{n\to\infty} ||P_n x||_{X_n} = ||x||_{X_0}$ for all $x \in X_0$. These conditions imply that $\sup_n ||P_n|| < +\infty$.

We will say that $\lim_{n\to\infty} x_n = x, x_n \in X_n, x \in X_0$ if and only if $\lim_{n\to\infty} ||P_n x - x_n||_{X_n} = 0$ (see also Kurtz [**8**]). Then, if A_n is a closed linear operator densely defined on X_n, $R_n(t)$ is a resolvent family (for A_n) acting on X_n, and, moreover, $(\mu - \mu\hat{k}(\mu)A_n)^{-1}x_n$ converges to $(\mu - \mu\hat{k}(\mu)A_0)^{-1}x_0$ as $x_n \to x_0$, then $R_n(t)x_n$ converges to $R_0(t)x_0$ as $x_n \to x_0$. The proof is very similar to that of Theorem 2.1. We take

$$\mathcal{K} = \{x = (x_n)_0^{\infty}/x_n \in X_n \text{ and } \lim_{n\to\infty} x_n = x_0\}$$

and define $|x| := \mathrm{Sup}_n ||x_n||_{X_n}$.

Following Davies [**1**] and Kato [**6**], we can obtain our main result on convergence of resolvent families:

THEOREM 2.4. *Assume that the conditions* (2.1) *and* (2.2) *in Theorem* 2.1 *hold. Suppose, moreover, that*

$$\lim_{m\to\infty} (\mu - \mu\hat{k}(\mu)A_m)^{-1}x =: L(\mu)x \tag{2.12}$$

exists for all $\mu > w$, $x \in X$, *and that the condition*

$$\lim_{\mu\to\infty} \mu L(\mu)x = x \tag{2.13}$$

holds for all $x \in X$. *Then there exists a closed linear and densely defined operator* A *in* X *and a resolvent family* $R(t)$ *such that*

(i) $1/\hat{k}(\mu) \in \rho(A)$ *and* $L(\mu)x = (\mu - \mu\hat{k}(\mu)A)^{-1}x$ *for all* $\mu > w$, $x \in X$.

(ii) $\lim_{m\to\infty} R_m(t)x = R(t)x$ *for all* $t \geq 0$, $x \in X$. *Moreover, the convergence is uniform in* t *on every compact subset of* $\mathbf{R}_+$.

PROOF. Let $\mu, \lambda > w$ be fixed. We have the identity

$$\begin{aligned}(1/\hat{k}(\mu) - A_m)^{-1} &- (1/\hat{k}(\lambda) - A_m)^{-1} \\ &= (1/\hat{k}(\lambda) - 1/\hat{k}(\mu))(1/\hat{k}(\lambda) - A_m)^{-1}(1/\hat{k}(\mu) - A_m)^{-1}.\end{aligned} \tag{2.14}$$

Therefore, making $m \to \infty$ in (2.14) and using (2.12) obtains

$$\mu\hat{k}(\mu)L(\mu) - \lambda\hat{k}(\lambda)L(\lambda) = (\hat{k}(\mu) - \hat{k}(\lambda))\lambda\mu L(\lambda)L(\mu). \tag{2.15}$$

This shows that $L(\mu)$ commutes with $L(\lambda)$ and, moreover, that $\operatorname{Ker} L(\mu) = \operatorname{Ker} L(\lambda)$ and $\operatorname{Rang} L(\mu) = \operatorname{Rang} L(\lambda)$, for all $\mu, \lambda > w$.

Define $D := \operatorname{Rang} L(\mu)$ and $N := \operatorname{Ker} L(\mu)$. Then, it follows from (2.13) that, for all $x \in X$,

$$x = \lim_{n\to\infty} y_n, \quad \text{where} \quad y_n = nL(n)x \in \operatorname{Rang} L(n) = D$$

and, if $L(\mu)x = 0$ for all $\mu > w$, that

$$x = \lim_{\mu\to\infty} \mu L(\mu)x = 0.$$

This shows that D is dense in X and $N = \{0\}$.

Define

$$Ax = (1/\mu\hat{k}(\mu))(\mu - L(\mu)^{-1})x \quad \text{for all } x \in D,\ \mu > w. \tag{2.16}$$

Then it follows that A is a closed linear and densely defined operator in X. Moreover, from (2.16), it is clear that (i) holds.

Now define

$$H_m(\mu) = (\mu - \mu\hat{k}(\mu)A_m)^{-1}, \quad a(\mu) = 1/\hat{k}(\mu), \quad b(\mu) = a(\mu)/\mu. \tag{2.17}$$

It follows by induction that

(2.18)

$$H_m(\mu)^{(n)}x = \sum_{k=0}^{n}\sum_{j=1}^{k}\binom{n}{j}(-1)^j b(\mu)^{(n-k)}c_j(\mu)(\mu\hat{k}(\mu))^{k+1}H_m(\mu)^k x,$$

where

$$\begin{aligned} c_j(\mu) =&(a(\mu)^j)^{(k)} - ja(\mu)(a(\mu)^{j-1})^{(k)}/1! + \cdots \\ &+ (-1)^{j-1}ja(\mu)^{j-1}(a(\mu))^{(k)}. \end{aligned}$$

Observe that $\lim_{m\to\infty} H_m(\mu)^k x = L(\mu)^k x$ for all $k \in \mathbf{N}$, $x \in X$. Therefore, we obtain from (2.16) and (2.18) that

$$(2.19)\quad \lim_{m\to\infty} H_m(\mu)^{(n)}x = L(\mu)^{(n)}x \text{ for all } \mu > w,\ x \in X,\ n \in \mathbf{N}_0.$$

Hence, by making use of (2.5), from (2.19),

$$||((\mu - \mu\hat{k}(\mu)A)^{-1})^{(n)}|| \leq \frac{Mn!}{(\mu - w)^{n+1}},\quad n \in \mathbf{N}_0,\ \mu > w.$$

Therefore, we can apply Theorem 1.1 and obtain a resolvent family $R(t)$, such that (1.3) holds. Finally, applying Theorem 2.1 yields that (ii) holds, and the proof is complete. □

REMARK 2.5. The condition (2.12) holds, for example, when $\lim_{\mu\to\infty}(I - \hat{k}(\mu)A_m)^{-1} = I$ uniformly in m (see Kato [**5**; Chapter 9, Theorem 2.17]) or when the range of $L(\mu)$ is dense in X (see Davies [**1**, Theorem 2.6]).

3. Applications to the study of hyperbolic Volterra equations. In [**11**] J. Prüss studies aspects on existence, positivity, regularity and compactness, as well as integrability of the resolvent for (1.2) for two special classes of kernels k and operators A. These classes are:

(I) A is the generator of a C_0-semigroup $T(t)$ in X; $k(t) > 0$ is nonincreasing and $\log k(t)$ is convex;

(II) A is the generator of a cosine family $C(t)$ in X; $k(t) = k_0 + k_\infty t + \int_0^t k_1(s)ds$, where $k_0, k_\infty \geq 0$, $k_1(t) > 0$ is nonincreasing, $\log k_1(t)$ convex, and $\lim_{t\to\infty} k_1(t) = 0$.

For equivalent conditions, see also Prüss [**11**, p. 326-327 and p. 336-337].

Let X be a Banach space and $\{A_n\}_0^\infty$ a sequence of linear closed operators in X with dense domains $D(A_n)$.

We consider the equations

$$(3.1)\qquad u_n(t) = f_n(t) + \int_0^t k(t-s)A_n u_n(s)ds,\ n \in \mathbf{N}_0,\ t \geq 0,$$

where $f_n \in W^{1,1}_{\text{loc}}([0,T],X)$.

THEOREM 3.1. *Suppose that one of the following conditions holds:*

(i) *Each* A_n *generates a* C_0*-semigroup* $T_n(t)$ *in* X, $k(t)$ *satisfies condition* (I) *and* $\|T_n(t)\| \leq M_1 e^{w_1 t}$ *for all* $t \geq 0$, $n \in \mathbf{N}_0$.

(ii) *Each* A_n *generates a strongly continuous cosine family* $C_n(t)$ *in* $X, k(t)$ *satisfies condition* (II), *and* $\|C_n(t)\| \leq M_2 e^{w_2 t}$ *for all* $t \geq 0$, $n \in \mathbf{N}_0$.

Assume, moreover, that (3.2) $\lim_{m\to\infty}(\mu - A_m)^{-1}x = (\mu - A_0)^{-1}x$ *for every* $x \in X$ *and all* μ *sufficiently large. Then*

(a) *The equations* (3.1) *admit resolvent families* $R_n(t)$ *in* $\mathbf{R}_+$, *for all* $n \in \mathbf{N}_0$, *and moreover,*

$$\|R_n(t)\| \leq M_3 e^{w_3 t} \text{ for all } t \geq 0,\ n \in \mathbf{N}_0. \tag{3.3}$$

(b) $\lim_{n\to\infty} R_n(t)x = R_0(t)x$ *for all* $t \geq 0$, $x \in X$. *Moreover, the convergence is uniform in* t *on every compact subset of* $\mathbf{R}_+$.

PROOF. Under our hypothesis, the first assertion in (a) follows directly from Prüss [**11**], Theorem 5 and Theorem 6 respectively, and a review of the proofs in these theorems shows that (3.3) holds.

On the other hand, from (I) (or (II)), it is shown that the condition (2.1) in Theorem 2.1 holds. In particular, it follows from this condition and (3.2) that

$$\begin{aligned}\lim_{m\to\infty}(\mu - \mu\hat{k}(\mu)A_m)^{-1}x &= (1/\mu\hat{k}(\mu)) \lim_{m\to\infty}(1/\hat{k}(\mu) - A_m)^{-1}x \\ &= (\mu - \mu\hat{k}(\mu)A_0)^{-1}x\end{aligned}$$

for all $x \in X$ and μ sufficiently large. Consequently, by using Theorem 2.1, we obtain that the assertion in (b) holds. □

EXAMPLE 3.2. Suppose that the kernel $k(t)$ satisfies condition (II) and let B be a linear bounded operator defined in X.

We define the operators

$$A_m = A + (1/m)B, \quad m \in \mathbf{N}, \tag{3.4}$$

where A generates a strongly continuous cosine function $C(t)$ in X which satisfies

$$(3.5)\qquad ||C(t)|| \leq Me^{wt}, \quad t \geq 0.$$

It follows from Nagy [**10**] that, for every $m \in \mathbf{N}$, the operators in (3.4) generate strongly continuous cosine functions $C_m(t)$ in X which, in turn, satisfy

$$(3.6)\qquad ||C_m(t)|| \leq Me^{(w+(1/m)||B||)t} \leq Me^{(w+||B||)t}, \quad t \geq 0.$$

On the other hand, by using (3.6), the identity

$$(\mu - A_m)^{-1} - (\mu - A)^{-1} = (1/m)B(\mu - A)^{-1}(\mu - A_m)^{-1},$$

for μ sufficiently large, and the generation theorem for cosine families (i.e., Theorem 1.1 with $k(t) \equiv t$), we show that

$$(\mu - A_m)^{-1}x \text{ converges to } (\mu - A)^{-1}x \text{ as } m \to \infty$$

for all $x \in X$ and μ sufficiently large.

Therefore, Theorem 3.1 shows that, for every $n \in \mathbf{N}$, the perturbed equations

(3.7)

$$u_n(t) = f_n(t) + \int_0^t k(t-s)Au_n(s)ds + (1/n)\int_0^t k(t-s)Bu_n(s)\,ds,$$

where $f_n \in W^{1,1}_{\text{loc}}([0,T], X)$, admit resolvent families $R_n(t)$ in $\mathbf{R}_+$ such that $||R_n(t)|| \leq Me^{wt}$ and

$$(3.8)\qquad \lim_{n\to\infty} R_n(t)x = R(t)x, \quad t \geq 0, \ x \in X,$$

where the convergence is uniform in t on every compact subset of $\mathbf{R}_+$ and $R(t)$ is a resolvent family for

$$(3.9)\quad u(t) = f(t) + \int_0^t k(t-s)Au(s)\,ds, \quad t \geq 0, f \in W^{1,1}_{\text{loc}}([0,T], X).$$

In particular, it is well-known that the solutions of (3.7) are represented by the variation of parameters formula

$$u_n(t) = R_n(t)f_n(0) + \int_0^t R_n(t-s)f_n'(s)\,ds, \quad t \geq 0, \tag{3.10}$$

$$u(t) = R(t)f(0) + \int_0^t R(t-s)f'(s)\,ds, \quad t \geq 0, \tag{3.11}$$

respectively.

Therefore if, for example, $(f_n)_{n=1}^{\infty}$ is a sequence of continuously differentiable functions in $\mathbf{R}_+$ such that the sequence $(f_n(0))$ in X converges and, moreover, the derivatives f_n' converge uniformly to a function g, then

$$\lim_{n\to\infty} u_n(t) = u(t), \quad t \geq 0,$$

with uniform convergence in t on every compact subset of $\mathbf{R}_+$. □

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