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SPECTRAL APPROXIMATIONS FOR WIENER-HOPF OPERATORS

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ABSTRACT. The main purpose of this paper is to compare spectral properties of a Wiener-Hopf operator

$$Kf(s) = \int_0^\infty \kappa(s-t)f(t)\,dt$$

and corresponding finite-section operators

$$K_\beta f(s) = \int_0^\beta \kappa(s-t)f(t)\,dt,$$

where $\kappa \in \mathrm{L}^1(R)$ and f is bounded and continuous. Among other results, we show that any neighborhood of the spectrum of K contains the spectrum of K_β for β sufficiently large. However, the roles of K and K_β cannot be reversed. Examples are given with $\sigma(K)$ a disc and $\sigma(K_\beta) = \{0\}$ for all β. We also compare spectral properties of K_β and corresponding numerical-integral operators $K_{\beta n}$. The spectral properties of K_β and $K_{\beta n}$ match more completely than do the spectral properties of K and K_β.

1. Introduction. Let K be a Wiener-Hopf operator,

$$Kf(s) = \int_0^\infty \kappa(s-t)f(t)\,dt, \quad s \in \mathbf{R}^+,$$

where $\kappa \in \mathrm{L}^1(\mathbf{R})$ and $f \in X^+$, the space of bounded, continuous, complex-valued functions on $\mathbf{R}^+$ with the uniform norm $||f|| = \sup|f(t)|$. Corresponding finite-section operators K_β are given by

$$K_\beta f(s) = \int_0^\beta \kappa(s-t)f(t)\,dt, \quad s \in \mathbf{R}^+, \beta \in \mathbf{R}^+.$$

We shall compare spectral properties of K and K_β as $\beta \to \infty$. This continues a study of integral equations on the half line initiated in **[3]** and carried forward in [**4**] and [**5**]. For related work on Wiener-Hopf operators, see [**2, 6, 8**] and the references in the papers cited.

Let $\mathcal{B}(X^+)$ denote the space of bounded linear operators from X^+ to X^+. Then $K, K_\beta \in \mathcal{B}(X^+)$ and

$$||K|| = ||\kappa||_1 = \int_{-\infty}^{\infty} |\kappa(u)|du,$$

$$||K_\beta|| = \sup_{s\in R^+} \int_{s-\beta}^{s} |\kappa(u)|du \leq ||K||.$$

To avoid trivialities, κ is not the zero function in $L^1(\mathbf{R})$. Then $K \neq O$ and $K_\beta \neq O$ for $\beta > 0$.

Our spectral notation is standard. The resolvent set for K is

$$\rho(K) = \{\lambda \in \mathbf{C} : (\lambda - K)^{-1} \in \mathcal{B}^+(X)\},$$

where $\mathbf{C}$ is the complex plane and $\lambda - K = \lambda I - K$. The spectrum of K is

$$\sigma(K) = \text{complement of } \rho(K),$$

which includes any eigenvalues of K. The resolvent set is open. The spectrum is closed and bounded, hence compact. Moreover,

$$|\lambda| \leq ||K||, \qquad \forall\lambda \in \sigma(K),$$
$$|\lambda| \leq ||K_\beta|| \leq ||K||, \quad \forall\lambda \in \sigma(K_\beta), \ \beta \in \mathbf{R}^+.$$

The Wiener-Hopf operator K is not compact. However, the finite-section operators K_β are compact. This complicates the analysis, but makes it more interesting. Much is known about the spectrum of a Wiener-Hopf operator. Spectral properties of $\sigma(K)$ are summarized in §2 and illustrated there with examples. In particular, $\sigma(K)$ is an infinite connected set in $\mathbf{C}$. On the other hand, since K_β is compact, $\sigma(K_\beta)$ is a discrete set with zero as the only possible point of accumulation. Thus, the relationship between $\sigma(K)$ and $\sigma(K_\beta)$ raises intriguing questions. They are explored in §3. Theorem 3.14 states that

every neighborhood of $\sigma(K)$
contains $\sigma(K_\beta)$ for β sufficiently large.

In the converse direction, one might imagine that $\sigma(K_\beta)$ is asymptotically dense in $\sigma(K)$, i.e., for any $\varepsilon > 0$, the ε-neighborhood of $\sigma(K_\beta)$ contains $\sigma(K)$ for β sufficiently large. But this is not generally true. For two examples in §4, $\sigma(K)$ is a disc and $\sigma(K_\beta) = \{0\}$. The most we can say in this direction (Theorem 3.11) is that

$$\lambda \in \sigma(K) \;\Rightarrow\; \lambda \text{ is an} \textit{asymptotic eigenvalue} \text{ of } K_\beta \text{ as } \beta \to \infty,$$

in the sense that there are *asymptotic eigenfunctions* $x_\beta \in X^+$ which satisfy

$$||x_\beta|| = 1, \quad ||\lambda x_\beta - K_\beta x_\beta|| \to 0 \quad \text{as } \beta \to \infty.$$

The examples mentioned above are simple enough to enable us to demonstrate this property.

Another example in §4 suggests that $\sigma(K_\beta)$ is asymptotically dense in $\sigma(K)$ as $\beta \to \infty$ if κ is real and even. This question will be investigated in a future paper.

In the final section of this paper we compare spectral properties of K_β and discrete approximations $K_{\beta n}$ defined by means of numerical integration. For example, the rectangular quadrature rule gives

$$K_{\beta n} f(s) = \frac{1}{n} \sum_{i=1}^{\beta n} \kappa\left(s - \frac{i}{n}\right) f\left(\frac{i}{n}\right), \quad s \in \mathbf{R}^+, \quad \beta \in \mathbf{R}^+, \quad n \in \mathbf{Z}^+.$$

Fix $\beta \in \mathbf{R}^+$. Under suitable hypotheses on the quadrature rule and the kernel function, $||K_{\beta n} f - K_\beta f|| \to 0$ as $n \to \infty$ for $f \in X^+$ and $\{K_{\beta n} : n \in \mathbf{Z}^+\}$ is collectively compact, i.e., $\{K_{\beta n} f : ||f|| \le 1, n \in \mathbf{Z}^+\}$ is precompact. The theory in **[1]** relates spectral properties of K_β and $K_{\beta n}$ as $n \to \infty$. For each $\beta \in \mathbf{R}^+$,

every neighborhood of $\sigma(K_\beta)$
contains $\sigma(K_{\beta n})$ for n sufficiently large.

Moreover, $\sigma(K_{\beta n})$ is asymptotically dense in $\sigma(K_\beta)$ as $n \to \infty$.

2. The spectrum of K. For later convenience we summarize some known spectral properties of the Wiener-Hopf operator K. Standard references are **[9]** and **[10]**.

The spectrum of K is given by

$$\sigma(K) = \sigma_0(K) \cup \sigma^+(K) \cup \sigma^-(K),$$

where $\sigma_0(K), \sigma^+(K)$ and $\sigma^-(K)$ are disjoint sets defined in terms of the Fourier transform of κ:

$$\hat{\kappa}(p) = \int_{-\infty}^{\infty} \kappa(u) e^{ipu} du, \quad p \in \mathbf{R}.$$

First,

$$\sigma_0(K) = \{\hat{\kappa}(p) : p \in \mathbf{R}\} \cup \{0\}.$$

Since $\hat{\kappa}(p)$ is continuous and $\hat{\kappa}(p) \to 0$ as $p \to \pm\infty$, $\sigma_0(K)$ is a continuous curve in $\mathbf{C}$ which we may regard as beginning and ending at 0. For each $\lambda \notin \sigma_0(K)$, the *index* (or winding number) of λ is the integer defined by

$$\operatorname{ind}(\lambda) = -\frac{1}{2\pi} \arg[\lambda - \hat{\kappa}(p)] \Big|_{p=-\infty}^{p=\infty}.$$

The sets $\sigma^+(K)$ and $\sigma^-(K)$ are defined by

$$\sigma^+(K) = \{\lambda \in \mathbf{C} : \operatorname{ind}(\lambda) > 0\},$$
$$\sigma^-(K) = \{\lambda \in \mathbf{C} : \operatorname{ind}(\lambda) < 0\}.$$

Loosely speaking, $\sigma^+(K)$ and $\sigma^-(K)$ consist of the points in $\mathbf{C}$ encircled by $\hat{\kappa}(p)$ a nonzero number of times as p increases from $-\infty$ to ∞.

There is another characterization of the three parts of $\sigma(K)$ that will be useful. Thus $\sigma_0(K), \sigma^+(K)$, and $\sigma^-(K)$ are the sets of eigenvalues (and zero for $\sigma_0(K)$) of the three integral operators

$$\mathcal{K} f(s) = \int_{-\infty}^{\infty} \kappa(s-t) f(t)\, dt, \quad s \in \mathbf{R}, f \in X,$$

$$K f(s) = \int_{0}^{\infty} \kappa(s-t) f(t)\, dt, \quad s \in \mathbf{R}^+, f \in X^+,$$

$$K^- f(x) = \int_{-\infty}^{0} \kappa(s-t) f(t)\, dt, \quad s \in \mathbf{R}^-, f \in X^-,$$

where X, X^+, and X^- are the spaces of bounded continuous functions on $\mathbf{R}, \mathbf{R}^+$ and $\mathbf{R}^-$.

Denote the range and null space of $\lambda - K$ by $(\lambda - K)X^+$ and $\mathcal{N}(\lambda - K)$. The following classical results are due to Krein [**10**]. They will be used in our analysis. The first result, for $\lambda \in \sigma_0(K)$, will play a particularly important role.

LEMMA 2.1.

$$\begin{aligned}
&(\lambda - K)X^+ \neq X^+, & & \forall\lambda \in \sigma_0(K),\\
&(\lambda - K)X^+ = X^+, & \mathcal{N}(\lambda - K) \neq \{0\}, \quad & \forall\lambda \in \sigma^+(K),\\
&(\lambda - K)X^+ \neq X^+, & \mathcal{N}(\lambda - K) = \{0\}, \quad & \forall\lambda \in \sigma^-(K),
\end{aligned}$$

and

$$\begin{aligned}
&\dim \mathcal{N}(\lambda - K) = \operatorname{ind}(\lambda), & & \forall\lambda \in \sigma^+(K),\\
&\operatorname{codim}(\lambda - K)X^+ = -\operatorname{ind}(\lambda), & & \forall\lambda \in \sigma^-(K).
\end{aligned}$$

Thus, $\lambda - K$ is a Fredholm operator with nonzero index equal to $\operatorname{ind}(\lambda)$ if λ is in $\sigma^+(K)$ or $\sigma^-(K)$.

Next we illustrate spectral properties of K with simple examples. The same examples will reappear when we consider $\sigma(K_\beta)$.

EXAMPLE 2.1. Let

$$\kappa(u) = \begin{cases} 0, & u < 0, \\ e^{-u}, & u > 0. \end{cases}$$

Then K is the Volterra operator

$$Kf(s) = \int_0^s e^{t-s} f(t)\, dt = e^{-s} \int_0^s e^t f(t)\, dt.$$

The Fourier transform of κ is

$$\hat{\kappa}(p) = \frac{1}{1 - ip}.$$

A routine calculation yields

$$\sigma_0(K) = \left\{ \lambda \in \mathbf{C} : \left| \lambda - \frac{1}{2} \right| = \frac{1}{2} \right\},$$

the circle with center 1/2 and radius 1/2. Then $\operatorname{ind}(\lambda) = -1$ for λ inside the circle and $\operatorname{ind}(\lambda) = 0$ for λ outside the circle. Therefore,

$$\sigma^+(K) = \emptyset, \quad \sigma^-(K) = \left\{\lambda \in C : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\}.$$

Another characterization of $\sigma^-(K)$ is

$$\sigma^-(K) = \left\{\lambda \in \mathbf{C} : \operatorname{Re}\left(\frac{1}{\lambda} - 1\right) > 0\right\}.$$

By Lemma 2.1, K has no nonzero eigenvalues. This is easy to verify directly. For $\lambda \neq 0$,

$$Kx = \lambda x \Leftrightarrow x'(s) = \left(\frac{1}{\lambda} - 1\right)x(s), \quad x(0) = 0 \Leftrightarrow x \equiv 0.$$

(Instead we could have appealed to the proposition that Volterra operators have no nonzero eigenvalues.) Also, from Lemma 2.1, $(\lambda - K)X^+ \neq X^+$ for $\lambda \in \sigma^-(K)$. To demonstrate this, fix $y \in X^+$ and $\lambda \in \sigma^-(K)$. Then

$$(\lambda - K)x = y \quad \Leftrightarrow \quad x(s) = \frac{1}{\lambda}y(s) + \frac{1}{\lambda^2}\int_0^s e^{(\frac{1}{\lambda}-1)(s-t)}y(t)\,dt.$$

This is valid with x continuous, but not necessarily bounded. Now x is bounded; hence $x \in X^+$ if and only if

$$\int_0^\infty e^{-(\frac{1}{\lambda}-1)t}y(t)\,dt = 0.$$

Thus, $(\lambda - K)X^+$ consists of the functions $y \in X^+$ which satisfy this condition, which shows that $(\lambda - K)X^+ \neq X^+$.

EXAMPLE 2.2. Let

$$\kappa(u) = \begin{cases} e^u, & u < 0, \\ 0, & u > 0. \end{cases}$$

Then

$$Kf(s) = \int_s^\infty e^{s-t}f(t)\,dt = e^s\int_s^\infty e^{-t}f(t)\,dt$$

and

$$\hat{\kappa}(p) = \frac{1}{1+ip}.$$

Once again, $\sigma_0(K)$ is the circle

$$\sigma_0(K) = \left\{\lambda \in \mathbf{C} : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}.$$

But now $\hat{\kappa}(p)$ traverses the circle in the opposite direction. Hence,

$$\sigma^+(K) = \left\{\lambda \in \mathbf{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\}, \ \sigma^-(K) = \emptyset.$$

From Lemma 2.1, if $\lambda \in \sigma^+(K)$ then λ is an eigenvalue of K and $(\lambda - K)X^+ = X^+$. It is easy to verify both of these facts directly. The details are omitted.

Wiener-Hopf operators with real symmetric kernels $\kappa(s-t)$ are of considerable practical interest. Equivalently, $\kappa(u)$ is real and even. Then $\hat{\kappa}(p)$ is real, $\sigma_0(K)$ is a real interval, $\sigma^+(K) = \emptyset, \sigma^-(K) = \emptyset$, and $\sigma(K) = \sigma_0(K)$. The next example is a prototype.

EXAMPLE 2.3. The Picard kernel. Let

$$\kappa(u) = e^{-|u|}.$$

Then

$$Kf(s) = \int_0^\infty e^{-|s-t|} f(t)\,dt = e^{-s}\int_0^s e^t f(t)\,dt + e^s \int_s^\infty e^{-t} f(t)\,dt.$$

The Fourier transform of κ is

$$\hat{\kappa}(p) = \frac{2}{1+p^2}.$$

Now

$$\sigma(K) = \sigma_0(K) = [0,2], \ \sigma^+(K) = \emptyset, \ \sigma^-(K) = \emptyset.$$

Every $\lambda \in (0,2)$ is an eigenvalue of the operator K. In fact, the eigenvalue problem $Kx = \lambda x$ is equivalent to the initial value problem

$$x''(s) + \gamma^2 x(s) = 0, \quad x'(0) = x(0), \quad \gamma = \left(\frac{2}{\lambda} - 1\right)^{\frac{1}{2}},$$

which has the solution

$$x(s) = \gamma \cos \gamma s + \sin \gamma s.$$

3. Spectral comparisons for K and K$_\beta$. To recapitulate, K and K_β are defined for $f \in X^+$ by

$$Kf(s) = \int_0^\infty \kappa(s-t)f(t)\,dt, \quad s \in \mathbf{R}^+, \tag{3.1}$$

$$K_\beta f(s) = \int_0^\beta \kappa(s-t)f(t)\,dt, \quad s \in \mathbf{R}^+, \beta \in \mathbf{R}^+, \tag{3.2}$$

where $\kappa \in \mathrm{L}^1(\mathbf{R})$ and $||\kappa||_1 \neq 0$. In order to relate $\sigma(K)$ and $\sigma(K_\beta)$ we shall need a number of properties of K and K_β. Much of the following analysis is adapted, with extensions and refinements, from [**3**] and [**5**]. For this reason, most of the proofs are merely sketched.

From (3.1) and (3.2),

$$\{Kf : ||f|| \leq 1\} \text{ is bounded and equicontinuous,} \tag{3.3}$$

$$\{K_\beta f : ||f|| \leq 1, \beta \in \mathbf{R}^+\} \text{ is bounded and equicontinuous.} \tag{3.4}$$

The equicontinuity is uniform on $\mathbf{R}^+$. Since bounded, equicontinuous sets in X^+ are not generally precompact, it does not follow (and is not true) that K is compact or that $\{K_\beta : \beta \in \mathbf{R}^+\}$ is collectively compact.

Also, from (3.1) and (3.2),

$$(K - K_\beta)f(s) = \int_\beta^\infty \kappa(s-t)f(t)\,dt = \int_{-\infty}^{s-\beta} \kappa(u)f(s-u)du. \tag{3.5}$$

By an easy argument, $||K_\beta f - Kf|| \not\to 0$ as $\beta \to \infty$ for $f \equiv 1$. Thus, K_β does not converge strongly to K as $\beta \to \infty$.

Let $X_0^+ = \{f \in X^+ : f(t) \to 0 \text{ as } t \to \infty\}$. This is a closed subspace of X^+. The inequality

$$|K_\beta f(s)| \leq ||f|| \int_{s-\beta}^{s} |\kappa(u)| du, \quad f \in X^+,$$

implies that, for each $\beta \in \mathbf{R}^+$,

$$K_\beta : X^+ \to X_0^+, \tag{3.6}$$

$$K_\beta f(s) \to 0 \text{ as } s \to \infty, \text{ uniformly for } ||f|| \leq 1, \quad f \in X^+. \tag{3.7}$$

In view of (3.4) and (3.7), $\{K_\beta f : ||f|| \leq 1\}$ is bounded, equicontinuous, and equiconvergent to zero at infinity. Since such sets are precompact (see [**3, 6**]),

$$K_\beta \text{ is compact } \forall \beta \in \mathbf{R}^+. \tag{3.8}$$

Now consider K and K_β restricted to X_0^+. Then

$$K : X_0^+ \to X_0^+, \tag{3.9}$$

$$K_\beta : X_0^+ \to X_0^+. \tag{3.10}$$

The latter comes from (3.6). The result for K is a consequence of

$$Kf(s) = \left(\int_0^\alpha + \int_\alpha^\infty \right) \kappa(s-t) f(t)\, dt,$$

$$|Kf(s)| \leq ||f|| \int_{s-\alpha}^{s} |\kappa(u)| du + ||f||_{[\alpha,\infty)} ||\kappa||_1,$$

where $||f||_{[\alpha,\infty)} = \sup_{[\alpha,\infty)} |f(t)|$. From (3.5),

$$|(K - K_\beta) f(s)| \leq ||f||_{[\beta,\infty)} ||\kappa||_1,$$

$$||K_\beta f - Kf|| \to 0 \text{ as } \beta \to \infty \quad \forall f \in X_0^+. \tag{3.11}$$

Thus, K_β converges strongly to K on X_0^+, but not on X^+.

In what follows K and K_β are defined on X^+ except for a few results, clearly identified, which pertain specifically to X_0^+.

Strict convergence, introduced in a more general context by Buck [**7**], plays a major role in our analysis (see also [**3**] and [**6**]). Strict convergence on X^+ is defined as follows. Let $x, x_\beta \in X^+$ for β sufficiently large. Then

$$x_\beta \xrightarrow{s} x \quad \text{as } \beta \to \infty \text{ if } \{x_\beta\} \text{ is bounded}$$

and

$$x_\beta(t) \to x(t) \quad \text{as } \beta \to \infty, \text{ uniformly on } [0,\alpha] \;\; \forall \alpha \in \mathbf{R}^+.$$

Let $||f||_{[0,\alpha]} = \max_{[0,\alpha]} |f(t)|$. Then the uniform convergence on $[0,\alpha]$ is equivalent to

$$||x_\beta - x||_{[0,\alpha]} \to 0 \quad \text{as } \beta \to \infty \quad \forall \alpha \in \mathbf{R}^+.$$

Let $\mathbf{R}^*$ denote any unbounded subset of $\mathbf{R}^+$, such as a sequence tending to infinity. Then $x_\beta \xrightarrow{s} x, \beta \in \mathbf{R}^*$, means that β is restricted to $\mathbf{R}^*$ in the foregoing definition of strict convergence. Successive unbounded subsets of $\mathbf{R}^+$ occur. They are denoted by $\mathbf{R}^{**} \subset \mathbf{R}^{***} \subset \mathbf{R}^{**}$, etc.

Strict convergence on the space X of bounded continuous functions on the real line $\mathbf{R}$ is defined in a similar manner:

$$x_\beta \xrightarrow{s} x \quad \text{as } \beta \to \infty \text{ with } x, x_\beta \in X \text{ if } \{x_\beta\} \text{ is bounded}$$

and

$$x_\beta(t) \to x(t) \quad \text{as } \beta \to \infty, \text{ uniformly on } [-\alpha,\alpha] \;\; \forall \alpha \in \mathbf{R}^+.$$

The operators K and K_β on X^+ have the strict convergence properties with $f, f_\beta \in X^+$:

$$K_\beta f \xrightarrow{s} K f, \tag{3.12}$$

$$f_\beta \xrightarrow{s} f \;\Rightarrow\; K f_\beta \xrightarrow{s} K f, \tag{3.13}$$

$$f_\beta \xrightarrow{s} f \;\Rightarrow\; K_\beta f_\beta \xrightarrow{s} Kf, \tag{3.14}$$

These follow in turn from the inequalities

$$||K_\beta f - Kf||_{[0,\alpha]} \leq ||f|| \int_{-\infty}^{\alpha-\beta} |\kappa(u)|\, du,$$

$$||Kf_\beta - Kf||_{[0,\gamma]} \leq ||f_\beta - f||_{[0,\alpha]} ||\kappa||_1 + (||f_\beta|| + ||f||) \int_{-\infty}^{\gamma-\alpha} |\kappa(u)|\, du,$$

and, for $\beta \geq \alpha$,

$$||K_\beta f_\beta - Kf||_{[0,\gamma]} \leq ||f_\beta - f||_{[0,\alpha]} ||\kappa||_1 + (||f_\beta|| + ||f||) \int_{-\infty}^{\gamma-\alpha} |\kappa(u)| du.$$

There is an analogue of the Arzela-Ascoli theorem for strict convergence:

LEMMA 3.1. *Let* $\{x_\beta \in X^+ : \beta \in \mathbf{R}^*\}$ *be bounded and equicontinuous. Then there exists* $\mathbf{R}^{**} \subset \mathbf{R}^*$ *and* $x \in X^+$ *such that* $x_\beta \xrightarrow{s} x$ *with* $\beta \in \mathbf{R}^{**}$.

The proof is based on applications of the classical Arzela-Ascoli theorem to successively larger intervals, followed by a diagonal argument (see [**3, 6**]).

A variant of Lemma 3.1, proved in the same way, holds for the space X of bounded continuous functions on $\mathbf{R}$.

LEMMA 3.2. *Assume* $\{f_\beta : \beta \in \mathbf{R}^*\}$ *is bounded. Then there exist* $\mathbf{R}^{**} \subset \mathbf{R}^*$ *and* $g, h \in X^+$ *such that*

$$Kf_\beta \xrightarrow{s} g, \quad K_\beta f_\beta \xrightarrow{s} h, \quad \beta \in \mathbf{R}^{**}.$$

This follows from (3.3), (3.4) and Lemma 3.1. We shall make frequent use of Lemma 3.2.

Next we compare solutions of equations

$$(\lambda - K)x = y, \quad (\lambda - K_\beta)x_\beta = y_\beta, \quad y_\beta \xrightarrow{s} y,$$

with λ fixed and $\lambda \neq 0$. Since K_β is compact, the Fredholm alternative gives

$$\lambda - K_\beta \text{ one-to-one } \Leftrightarrow (\lambda - K_\beta)X^+ = X^+,$$

in which case $(\lambda - K_\beta)^{-1} \in \mathcal{B}(X^+)$. For $\lambda \neq 0$,

$$(\lambda - K_\beta)x_\beta = y_\beta \Leftrightarrow x_\beta = \frac{1}{\lambda}(K_\beta x_\beta + y_\beta).$$

Lemma 3.2 and (3.14) yield the following three lemmas, which are slight extensions of results in [**3**] for the case with $\lambda = 1$. Throughout, $\lambda \neq 0$ and $\mathbf{R}^*$ is an arbitrary unbounded subset of $\mathbf{R}^+$.

LEMMA 3.3. *Assume* $\{x_\beta : \beta \in \mathbf{R}^*\}$ *bounded and*

$$(\lambda - K_\beta)x_\beta = y_\beta, \quad y_\beta \xrightarrow{s} y, \quad \beta \in \mathbf{R}^*.$$

Then there exist $\mathbf{R}^{**} \subset \mathbf{R}^*$ *and* $x \in X^+$ *such that*

$$x_\beta \xrightarrow{s} x, \quad \beta \in \mathbf{R}^{**}, \quad (\lambda - K)x = y.$$

If $\lambda - K$ *is one-to-one, then* x *is unique and*

$$x_\beta \xrightarrow{s} x, \quad \beta \in \mathbf{R}^*.$$

LEMMA 3.4. *Assume* $\lambda \in \rho(K_\beta)$ *and there exists* $(\lambda - K_\beta)^{-1}$ *bounded uniformly for* $\beta \in \mathbf{R}^*$. *Then* $(\lambda - K)X^+ = X^+$.

LEMMA 3.5. *Assume* $\lambda \in \rho(K_\beta)$ *and there exists* $(\lambda - K_\beta)^{-1}$ *bounded uniformly for* $\beta \in \mathbf{R}^*$. *Assume also that* $\lambda - K$ *is one-to-one. Then* $\lambda \in \rho(K)$. *Let*

$$(\lambda - K)x = y, \quad (\lambda - K_\beta)x_\beta = y_\beta, \quad y_\beta \xrightarrow{s} y, \quad \beta \in \mathbf{R}^*.$$

Then $x_\beta \xrightarrow{s} x, \beta \in \mathbf{R}^*$.

The next lemma is adapted from [**3**, Theorem 9.1]. See also [**9, 10**]. Again $\lambda \neq 0$.

LEMMA 3.6. *Assume* $(\lambda - K)X^+ = X^+$ *and* $Kx = \lambda x$ *with* $x \in X^+$. *Then* $x \in X_0^+$.

PROOF. Suppose that $x \notin X_0^+$. Then there exists $\mathbf{R}^*$ and c such that

$$|x(\beta)| \geq c > 0 \quad \forall \beta \in \mathbf{R}^*.$$

Consider the translates $x(t+\beta)$ for $t \geq -\beta$ and $\beta \in \mathbf{R}^*$. Since $x = Kx/\lambda$, (3.3) and a variant of Lemma 3.1 yield $\mathbf{R}^{**} \subset \mathbf{R}^*$ and $h \in X$ such that

$$x(t+\beta) \xrightarrow{s} h(t) \quad \text{on } R, \quad \beta \in \mathbf{R}^{**}.$$

Then $|h(0)| \geq c > 0$, so that $h \neq 0$. From $Kx = \lambda x$,

$$\int_{-\beta}^{\infty} \kappa(s-t)x(t+\beta)\,dt = \lambda x(s+\beta), \quad s \geq -\beta.$$

Let $\beta \to \infty$ through $\mathbf{R}^{**}$ to obtain

$$\int_{-\infty}^{\infty} \kappa(s-t)h(t)\,dt = \lambda h(s), \quad s \in \mathbf{R}.$$

Thus, $\mathcal{K}h = \lambda h$, which implies that $\lambda \in \sigma_0(K)$. By Lemma 2.1, $(\lambda - K)X^+ \neq X^+$. This contradicts the hypothesis that $(\lambda - K)X^+ = X^+$. Therefore, $x \in X_0^+$. □

The following lemma pertains exclusively to X_0^+. Recall that $K, K_\beta : X_0^+ \to X_0^+$ and $||K_\beta f - Kf|| \to 0$ as $\beta \to \infty$ for $f \in X_0^+$.

LEMMA 3.7. *Restrict* K *and* K_β *to* X_0^+. *Assume* $(\lambda - K_\beta)^{-1}$ *exists and is bounded uniformly for* $\beta \in \mathbf{R}^*$. *Then* $(\lambda - K)^{-1}$ *exists and is bounded.* (Nothing is inferred about $(\lambda - K)X_0^+$.)

PROOF. This is a standard argument. Recall that

$$||(\lambda - K_\beta)^{-1}|| \leq b \quad \Leftrightarrow \quad ||(\lambda - K_\beta)f|| \geq \frac{1}{b} \quad \text{for } ||f|| = 1.$$

Let $\beta \to \infty$ with $\beta \in \mathbf{R}^*$ to show that $(\lambda - K)^{-1}$ exists and is bounded. □

THEOREM 3.8. *Assume there exists* $\mathbf{R}^*$ *such that* $\lambda \in \rho(K_\beta)$ *and* $(\lambda - K_\beta)^{-1}$ *is bounded uniformly for* $\beta \in \mathbf{R}^*$. *Then* $\lambda \in \rho(K)$.

PROOF. By Lemma 3.4, $(\lambda - K)X^+ = X^+$. Let $Kx = \lambda x$ with $x \in X^+$. By Lemma 3.6, $x \in X_0^+$. By Lemma 3.7, $x = 0$. Thus, $\lambda - K$ is one-to-one on X^+, $(\lambda - K)^{-1} \in \mathcal{B}(X^+)$, and $\lambda \in \rho(K)$. □

Since the operators K_β are compact, Theorem 3.8 has an equivalent form:

THEOREM 3.9. *Assume there exist* $\mathbf{R}^*$ *and* r *such that*

$$||(\lambda - K_\beta)f|| \geq r > 0 \quad \text{for } ||f|| = 1, \quad \beta \in \mathbf{R}^*.$$

Then $\lambda \in \rho(K)$.

The next theorem characterizes the spectrum of K in terms of properties of the operators K_β. It is adapted from [**3**, Theorem 10.1].

THEOREM 3.10. *Assume there exist* $\mathbf{R}^*$ *and* $x_\beta \in X^+$ *for* $\beta \in \mathbf{R}^*$ *such that*

$$||x_\beta|| = 1, \quad ||\lambda x_\beta - K_\beta x_\beta|| \to 0, \quad \beta \in \mathbf{R}^*.$$

Then $\lambda \in \sigma(K)$.

PROOF. By Lemma 3.3 there exist $\mathbf{R}^{**} \subset \mathbf{R}^*$ and $x \in X^+$ such that

$$x_\beta \xrightarrow{s} x, \quad \beta \in \mathbf{R}^{**}, \quad (\lambda - K)x = 0.$$

If $x \neq 0$ then $\lambda \in \sigma(K)$ and we are done. Assume $x = 0$. Then

$$x_\beta \xrightarrow{s} 0, \quad \beta \in \mathbf{R}^{**}.$$

Since $||x_\beta|| = 1$, there exist $t_\beta \in \mathbf{R}^+$ such that

$$|x_\beta(t_\beta)| \geq \frac{1}{2}, \quad t_\beta \to \infty \ \text{ as } \beta \to \infty, \ \beta \in \mathbf{R}^{**}.$$

Consider the translates $x_\beta(t+t_\beta)$ for $t \geq -t_\beta$. Since $||\lambda x_\beta - K_\beta x_\beta|| \to 0$, (3.4) and a variant of Lemma 3.1 yield $\mathbf{R}^{***} \subset \mathbf{R}^{**}$ and $h \in X$ such that

$$x_\beta(t + t_\beta) \xrightarrow{s} h(t) \ \text{ on } \mathbf{R}, \quad \beta \in \mathbf{R}^{***}.$$

Then $|h(0)| \geq 1/2$, so that $h \neq 0$. Now

$$K_\beta x_\beta(s + t_\beta) = \int_{-t_\beta}^{\beta - t_\beta} \kappa(s - t) x_\beta(t + t_\beta)\, dt, \quad s \geq -t_\beta.$$

There exist $\alpha \in [-\infty, \infty]$ and $\mathbf{R}^{****} \subset \mathbf{R}^{***}$ such that

$$\beta - t_\beta \to \alpha \ \text{ as } \beta \to \infty, \quad \beta \in \mathbf{R}^{****}.$$

It follows that

$$K_\beta x_\beta(s + t_\beta) \to \int_{-\infty}^{\alpha} \kappa(s - t) h(t)\, dt \ \text{ as } \beta \to \infty, \quad \beta \in \mathbf{R}^{****}.$$

Since $||\lambda x_\beta - K_\beta x_\beta|| \to 0$,

$$\int_{-\infty}^{\alpha} \kappa(s - t) h(t)\, dt = \lambda h(s), \quad s \in R.$$

Since $h \neq 0, \alpha \neq -\infty$. If $\alpha = +\infty$ then $\mathcal{K}h = \lambda h$, so that $\lambda \in \sigma_0(K)$ and $\lambda \in \sigma(K)$. If $-\infty < \alpha < \infty$, let $g(s) = h(s + \alpha)$. Then $K^- g = \lambda g$ so that $\lambda \in \sigma^-(K)$ and $\lambda \in \sigma(K)$. Thus, in all cases, $\lambda \in \sigma(K)$. □

We combine Theorems 3.9 and 3.10 to obtain the first of our principal results.

THEOREM 3.11. *$\lambda \in \sigma(K)$ if and only if λ is an approximate eigenvalue of K_β as $\beta \to \infty$, i.e., $\exists x_\beta \in X^+$ $\forall \beta \in \mathbf{R}^+$ such that*

$$||x_\beta|| = 1, \quad ||\lambda x_\beta - K_\beta x_\beta|| \to 0 \ \text{ as } \beta \to \infty,$$

PROOF. The forward implication is the contrapositive of Theorem 3.9. The reverse implication is Theorem 3.10 with $\mathbf{R}^* = \mathbf{R}^+$. □

By Theorems 3.10 and 3.11, if there exist $\mathbf{R}^*$ and $x_\beta \in X^+$ for $\beta \in \mathbf{R}^*$ with

$$||x_\beta|| = 1, \quad ||\lambda x_\beta - K_\beta x_\beta|| \to 0 \text{ as } \beta \to \infty, \quad \beta \in \mathbf{R}^*,$$

then there exist $x_\beta \in X^+$ for $\beta \in \mathbf{R}^+$ such that

$$||x_\beta|| = 1, \quad ||\lambda x_\beta - K_\beta x_\beta|| \to 0 \text{ as } \beta \to \infty, \quad \beta \in \mathbf{R}^+.$$

The next theorem relates points in $\rho(K)$ and $\rho(K_\beta)$.

THEOREM 3.12. $\lambda \in \rho(K) \Leftrightarrow$ *there exist* $r(\lambda)$ *and* $\gamma(\lambda)$ *in* $\mathbf{R}^+$ *such that*

$$\lambda \in \rho(K_\beta), \quad ||(\lambda - K_\beta)^{-1}|| \le r(\lambda) \quad \forall \beta \ge \gamma(\lambda).$$

In this case, the solutions of

$$(\lambda - K)x = y, \quad (\lambda - K_\beta)x_\beta = y_\beta, \quad y_\beta \xrightarrow{s} y, \quad \beta \ge \gamma(\lambda),$$

satisfy $x_\beta \xrightarrow{s} x$ *as* $\beta \to \infty$.

PROOF. The implication from $\lambda \in \rho(K)$ is the contrapositive of Theorem 3.10. The reverse implication is Theorem 3.9 with $\mathbf{R}^* = [\gamma(\lambda), \infty)$. Finally, Lemma 3.5 gives $x_\beta \xrightarrow{s} x$ as $\beta \to \infty$. □

In order to indicate the significance of the uniform boundedness of the operators $(\lambda - K_\beta)^{-1}$ in Theorem 3.12, we recall a few facts from elementary spectral theory. As mentioned before, $\sigma(K)$ and $\sigma(K_\beta)$ are compact and

$$\begin{aligned} |\lambda| &\le ||K|| && \forall \lambda \in \sigma(K), \\ |\lambda| &\le ||K_\beta|| \le ||K|| && \forall \lambda \in \sigma(K_\beta), \beta \in \mathbf{R}^+, \end{aligned}$$

$$|\lambda| > ||K|| \Rightarrow \lambda \in \rho(K), \quad ||(\lambda - K)^{-1}|| \le \frac{1}{|\lambda| - ||K||},$$

$$|\lambda| > ||K|| \Rightarrow \lambda \in \rho(K_\beta), \quad ||(\lambda - K_\beta)^{-1}|| \le \frac{1}{|\lambda| - ||K||}, \quad \forall \beta \in \mathbf{R}^+.$$

So, $(\lambda - K)^{-1}$ and $(\lambda - K_\beta)^{-1}$ are bounded uniformly for $|\lambda| > 2||K||$ and $\beta \in \mathbf{R}^+$.

The resolvent set $\rho(K)$ is open and

$$\lambda \in \rho(K), \quad |\mu - \lambda| < \frac{1}{||(\lambda - K)^{-1}||}$$

implies

$$\mu \in \rho(K), \quad ||(\mu - K)^{-1}|| \le \frac{||(\lambda - K)^{-1}||}{1 - |\mu - \lambda|\,||(\lambda - K)^{-1}||}.$$

Consequently

$$\lambda \in \rho(K), \quad |\mu - \lambda| < \frac{1}{2||(\lambda - K)^{-1}||}$$

implies

$$\mu \in \rho(K), \ ||(\mu - K)^{-1}|| < 2||(\lambda - K)^{-1}||.$$

This result and a standard compactness argument yield

$$(\mu - K)^{-1} \text{ bounded uniformly for } \mu \in \Lambda, \quad \forall \text{ closed sets } \Lambda \subset \rho(K).$$

(By the preceding remarks, it suffices to consider Λ closed and bounded, hence compact.) A similar compactness argument will give an analogous result for $(\mu - K_\beta)^{-1}$ which is uniform for β sufficiently large. The following lemma will facilitate the proof. Replace K by K_β above to obtain

LEMMA 3.13. *As in Theorem* 3.12 *assume*

$$\lambda \in \rho(K_\beta), \quad ||(\lambda - K_\beta)^{-1}|| \le r(\lambda), \quad \forall \beta \ge \gamma(\lambda).$$

(a) *If* $|\mu - \lambda| < 1/r(\lambda)$ *then* $\mu \in \rho(K_\beta)$ $\forall \beta \ge \gamma(\lambda)$.

(b) *If* $|\mu - \lambda| < 1/2r(\lambda)$ *then* $\mu \in \rho(K_\beta)$ *and* $||(\mu - K_\beta)^{-1}|| < 2r(\lambda)$ $\forall \beta \ge \gamma(\lambda)$.

Now we can augment Theorem 3.12. If $\lambda \in \rho(K)$, then $(\mu - K)^{-1}$ and $(\mu - K_\beta)^{-1}$ exist and are bounded uniformly for all μ in a neighborhood of λ and for all β sufficiently large. This is a local result in that it pertains to a fixed $\lambda \in \rho(K)$. The next theorem is a corresponding global result.

THEOREM 3.14. *Let Λ be any closed subset of $\rho(K)$. Then there exists $\gamma \in \mathbf{R}^+$ such that $\Lambda \subset \rho(K_\beta)$ for $\beta \geq \gamma$ and $(\mu - K_\beta)^{-1}$ is bounded uniformly for $\mu \in \Lambda$ and $\beta \geq \gamma$.*

PROOF. In view of the preceding remarks, we may assume that Λ is compact. Let $\lambda \in \Lambda$. Then $\lambda \in \rho(K)$ and, by Theorem 3.12,

$$\lambda \in \rho(K_\beta), \quad ||(\lambda - K_\beta)^{-1}|| \leq r(\lambda), \quad \forall \beta \geq \gamma(\lambda).$$

Define open sets $S(\lambda)$ for $\lambda \in \Lambda$ by

$$S(\lambda) = \left\{\mu \in \mathbf{C} : |\mu - \lambda| < \frac{1}{2r(\lambda)}\right\}.$$

Clearly, $\lambda \in S(\lambda)$. By Lemma 3.13(b),

$$\mu \in \rho(K_\beta), \quad ||(\mu - K_\beta)^{-1}|| < 2r(\lambda) \quad \forall \mu \in S(\lambda), \quad \forall \beta \geq \gamma(\lambda).$$

Since $\{S(\lambda) : \lambda \in \Lambda\}$ is an open cover for Λ, there is a finite subcover:

$$\Lambda \subset \cup_{i=1}^{n} S(\lambda_i).$$

Let $\gamma = \max \gamma(\lambda_i)$ and $r = 2 \max r(\lambda_i)$ for $i = 1, \ldots, n$. Then

$$\mu \in \Lambda \Rightarrow \mu \in S(\lambda_i) \text{ for some } i \Rightarrow$$

$$||(\mu - K_\beta)^{-1}|| < 2r(\lambda_i) \leq r \text{ for } \beta \geq \gamma \geq \gamma(\lambda_i),$$

which completes the proof. □

Take complements in Theorem 3.14 to obtain a global comparison of the spectra of K and K_β:

THEOREM 3.15. *Let $\sigma(K) \subset \Omega$ with Ω open. Then $\sigma(K_\beta) \subset \Omega$ for all β sufficiently large.*

For $\varepsilon > 0$ let $\Omega_\varepsilon[\sigma(K)]$ denote the (open) ε-neighborhood of $\sigma(K)$. Since $\sigma(K)$ is compact, Theorem 3.15 is expressed equivalently by

$$\forall \varepsilon > 0 \ \exists \beta_\varepsilon \text{ such that } \sigma(K_\beta) \subset \Omega_\varepsilon[\sigma(K)] \ \ \forall \beta \geq \beta_\varepsilon.$$

The question arises whether the reciprocal property, with K and K_β interchanged, holds. We shall say that

$$\sigma(K_\beta) \text{ is } \textit{asymptotically dense} \text{ in } \sigma(K) \text{ as } \beta \to \infty$$

if

$$\forall \varepsilon > 0 \ \exists \beta_\varepsilon \text{ such that } \sigma(K) \subset \Omega_\varepsilon[\sigma(K_\beta)] \ \ \forall \beta \geq \beta_\varepsilon.$$

If this is true, along with Theorem 3.15, then $\sigma(K_\beta) \to \sigma(K)$ as $\beta \to \infty$, in the sense of the Hausdorff semi-metric for the distance between two sets. However, as we shall see in §4, it is not generally true that $\sigma(K_\beta)$ is asymptotically dense in $\sigma(K)$ as $\beta \to \infty$.

4. Examples. We consider again the examples of §2 in order to illustrate our principal results. In the first two examples, with non-symmetric kernels, $\sigma(K_\beta)$ is not asymptotically dense in $\sigma(K)$ as $\beta \to \infty$. The third example, for the Picard kernel, suggests that $\sigma(K_\beta)$ is asymptotically dense in $\sigma(K)$ as $\beta \to \infty$ if κ is real and even. This topic will be pursued in a subsequent paper.

EXAMPLE 4.1. As in Example 2.1 let

$$\kappa(u) = \begin{cases} 0, & u < 0, \\ e^{-u}, & u > 0. \end{cases}$$

Recall that

$$\sigma_0(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}, \quad \sigma^+(K) = \emptyset,$$

$$\sigma^-(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\} = \left\{\lambda : \operatorname{Re}\left(\frac{1}{\lambda} - 1\right) > 0\right\},$$

$$\sigma(K) = \sigma_0(K) \cup \sigma^-(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}.$$

Now consider K_β. Since K_β is compact, $\sigma(K_\beta)$ can consist only of eigenvalues and 0. For $\lambda \neq 0, K_\beta x = \lambda x$ if and only if

$$\int_0^s e^{t-s} x(t)\,dt = \lambda x(s), \quad 0 \le s \le \beta,$$

$$x(s) = \frac{1}{\lambda}\int_0^\beta e^{t-s} x(t)\,dt, \quad \beta \le s < \infty.$$

Consider the first of these equations. By the arguments in Example 2.1, $x(t) = 0$ for $0 \le t \le \beta$. Then the second equation implies that $x \equiv 0$. Thus, K_β has no nonzero eigenvalues and $\sigma(K_\beta) = \{0\}$, whereas $\sigma(K)$ is the disc with center 1/2 and radius 1/2. This is consistent with Theorem 3.15. We also see that $\sigma(K_\beta)$ is not asymptotically dense in $\sigma(K)$ as $\beta \to \infty$.

By Theorem 3.11, each $\lambda \in \sigma(K)$ is an asymptotic eigenvalue of K_β as $\beta \to \infty$, i.e., there exist $x_\beta \in X^+$ such that

$$||x_\beta|| = 1, \quad ||\lambda x_\beta - K_\beta x_\beta|| \to 0 \text{ as } \beta \to \infty.$$

We exhibit such asymptotic eigenfunctions x_β. First let $\lambda \in \sigma^-(K)$. Define

$$x_\beta(s) = \begin{cases} e^{(\frac{1}{\lambda}-1)(s-\beta)}, & 0 \le s \le \beta, \\ e^{\beta - s}, & \beta \le s < \infty. \end{cases}$$

Since $x_\beta(\beta) = 1$ and $\mathrm{Re}(1/\lambda - 1) > 0$ for $\lambda \in \sigma^-(K)$, $||x_\beta|| = 1$. Direct calculations yield

$$\lambda x_\beta(s) - K_\beta x_\beta(s) = \lambda e^{-(\frac{1}{\lambda}-1)\beta - s}$$

$$||\lambda x_\beta - K_\beta x_\beta|| = |\lambda| e^{-\mathrm{Re}(\frac{1}{\lambda}-1)\beta} \to 0 \text{ as } \beta \to \infty.$$

Thus, each $\lambda \in \sigma^-(K)$ is an asymptotic eigenvalue of K_β as $\beta \to \infty$. Now let $\lambda \in \sigma_0(K)$. Choose $\lambda_n \in \sigma^-(K)$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Then asymptotic eigenfunctions x_β of K_β corresponding to λ can be chosen from among the asymptotic eigenfunctions of K_β corresponding to $\lambda_n, n = 1, 2, \ldots$.

EXAMPLE 4.2. As in Example 2.2, let

$$\kappa(u) = \begin{cases} e^u, & u < 0 \\ 0, & u > 0. \end{cases}$$

In this case,

$$\sigma_0(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}, \quad \sigma^-(K) = \emptyset,$$

$$\sigma^+(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\} = \left\{\lambda : \mathrm{Re}\left(\frac{1}{\lambda} - 1\right) > 0\right\},$$

$$\sigma(K) = \sigma_0(K) \cup \sigma^+(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}.$$

Consider K_β. For $\lambda \neq 0$, $K_\beta x = \lambda x_i$ if and only if

$$\int_s^\beta e^{s-t} x(t)\, dt = \lambda x(s), \quad 0 \leq s \leq \beta,$$

$$x(s) = 0, \quad \beta \leq s < \infty.$$

In the first equation let $y(t) = x(\beta - t)$ to obtain

$$\int_0^s e^{t-s} y(t)\, dt = \lambda y(s), \quad 0 \leq s \leq \beta.$$

As in Example 4.1, $y \equiv 0$. Hence, $x \equiv 0$ and $\sigma(K_\beta) = \{0\}$, whereas $\sigma(K)$ is a disc. So $\sigma(K_\beta)$ is not asymptotically dense in $\sigma(K)$ as $\beta \to \infty$.

Let $\lambda \in \sigma^+(K)$. From Example 2.2, λ is an eigenvalue of K with the eigenfunction

$$x(s) = e^{-(\frac{1}{\lambda} - 1)s}.$$

Since $x(0) = 1$ and $\mathrm{Re}(1/\lambda - 1) > 0, ||x|| = 1$. A simple calculation yields

$$\lambda x(s) - K_\beta x(s) = \begin{cases} \lambda e^{-\frac{1}{\lambda}\beta + s}, & 0 \leq s \leq \beta, \\ \lambda e^{-(\frac{1}{\lambda} - 1)s}, & \beta \leq s < \infty. \end{cases}$$

It follows that

$$||\lambda x - K_\beta x|| = |\lambda| e^{-\mathrm{Re}(\frac{1}{\lambda} - 1)\beta} \to 0 \text{ as } \beta \to \infty.$$

Therefore, λ is an asymptotic eigenvalue of K_β as $\beta \to \infty$.

EXAMPLE 4.3. The Picard kernel. As in Example 2.3 let

$$\kappa(u) = e^{-|u|}.$$

In this case,

$$\sigma(K) = \sigma_0(K) = [0, 2].$$

Let $\lambda \in (0, 2)$. Then $K_\beta x = \lambda x$ if and only if

$$\int_0^\beta e^{-|s-t|} x(t)\, dt = \lambda x(s), \quad 0 \le s \le \beta,$$

$$x(s) = \frac{1}{\lambda} \int_0^\beta e^{-s+t} x(t)\, dt, \quad \beta \le s < \infty.$$

The first equation is equivalent to the two-point boundary value problem

$$x''(s) + \gamma^2 x(s) = 0, \quad 0 \le s \le \beta, \quad \gamma = \left(\frac{2}{\lambda} - 1\right)^{\frac{1}{2}},$$

$$x'(0) = x(0), \quad x'(\beta) = -x(\beta).$$

By elementary arguments, this problem has the nontrivial solution

$$x(s) = \gamma \cos \gamma s + \sin \gamma s$$

if and only if γ is a positive root of the transcendental equation

$$\tan \beta\gamma = \frac{2\gamma}{\gamma^2 - 1}.$$

By graphical or other means, there is at least one solution γ in almost every interval of length π/β. The corresponding numbers

$$\lambda = \frac{2}{\gamma^2 + 1}$$

are eigenvalues of K_β. Therefore, $\sigma(K_\beta)$ is asymptotically dense in $\sigma(K) = [0, 2]$ as $\beta \to \infty$.

By Theorem 3.11, every $\lambda \in (0, 2)$ is an asymptotic eigenvalue of K_β as $\beta \to \infty$. It is easy to verify this. Fix $\lambda \in (0, 2)$. From the preceding results, there exist $\lambda_\beta \in \sigma(K_\beta)$ for $\beta \in \mathbf{R}^+$ such that $\lambda_\beta \to \lambda$ as

$\beta \to \infty$. Let x_β be a corresponding normalized eigenfunction of K_β. Then

$$\lambda x_\beta - K_\beta x_\beta = (\lambda - \lambda_\beta) x_\beta,$$

$$||\lambda x_\beta - K_\beta x_\beta|| = |\lambda - \lambda_\beta| \to 0 \text{ as } \beta \to \infty.$$

Thus, λ is an asymptotic eigenvalue of K_β as $\beta \to \infty$.

5. Spectral comparisons for $\mathbf{K}_\beta$ and $\mathbf{K}_{\beta n}$. Approximations $K_{\beta n}$ for K_β will be defined by means of numerical integration. The procedure is merely sketched. For further details, see [**5**].

A quadrature rule, such as a standard repeated or composite rule, is defined formally on $[0, \infty)$ and then restricted to finite intervals $[0, \beta]$. We assume that

$$\text{(5.1)} \quad \sum_0^\beta {}^* \omega_{ni} f(t_{ni}) \to \int_0^\beta f(t)\, dt \quad \text{as } n \to \infty \quad \forall f \in C[0, \beta], \beta \in \mathbf{R}^+,$$

where $\omega_{ni} > 0, 0 \le t_{n1} < t_{n2} < \cdots$ and the sum on i is over the terms with $0 \le t_{ni} \le \beta$. The star in (5.1) means that if $t_{ni} = \beta$ for some t_{ni} and β then ω_{ni} may have to be multiplied by some factor in order to recover the correct repeated or composite rule on $[0, \beta]$. The factor is $1/2$ for the trapezoidal rule.

The operators K_β and $K_{\beta n}$ are defined on X^+ by

$$\text{(5.2)} \qquad K_\beta f(s) = \int_0^\beta \kappa(s - t) f(t)\, dt, \quad \beta \in \mathbf{R}^+,$$

$$\text{(5.3)} \qquad K_{\beta n} f(s) = \sum_0^\beta {}^* \omega_{ni} \kappa(s - t_{ni}) f(t_{ni}), \quad \beta \in \mathbf{R}^+, n \in \mathbf{Z}^+.$$

Restrictions must be imposed on the kernel function κ in order to facilitate numerical integration. Assume that

$$\text{(5.4)} \qquad \kappa \in \mathrm{L}^1(\mathbf{R}),$$

(5.5) κ is bounded and uniformly continuous on $\mathbf{R}$.

It follows from (5.4) and (5.5) that

$$\kappa(u) \to 0 \quad \text{as } u \to \pm\infty. \tag{5.6}$$

The following kernel functions satisfy (5.4)–(5.6).

EXAMPLE 5.1. (Picard kernel). $\kappa(u) = e^{-|u|}$.

EXAMPLE 5.2. $\kappa(u) = 1/(1+u^2)$.

EXAMPLE 5.3. $\kappa(u) = \sin u/(1+u^2)$.

Under the foregoing hypotheses on the quadrature formula and the kernel function, it is shown in [**5**] that the operators K_β and $K_{\beta n}$ satisfy

THEOREM 5.1. *For each* $\beta \in \mathbf{R}^+$,

(5.7) K_β *is compact,*

(5.8) $\{K_{\beta n} : n \in \mathbf{Z}^+\}$ *is collectively compact,*

(5.9) $\|K_{\beta n} f - K_\beta f\| \to 0$ *as* $n \to \infty$ $\forall f \in X^+$.

The spectral approximation theory in [**1**], particularly Theorems 4.8 and 4.16, applies to the operators K_β and $K_{\beta n}$. We obtain

THEOREM 5.2. *Fix* $\beta \in \mathbf{R}^+$. *Then*

(a) *for each open set* $\Omega \supset \sigma(K_\beta)$ *there exists* $n(\beta, \Omega)$ *such that* $\Omega \supset \sigma(K_{\beta n})$ *for* $n \geq n(\beta, \Omega)$;

(b) $\sigma(K_{\beta n})$ *is asymptotically dense in* $\sigma(K_\beta)$ *as* $n \to \infty$.

Under more restrictive conditions on the quadrature formula and the kernel function κ, which are given in [**5**], it is possible to derive stronger relationships between $\sigma(K_\beta)$ and $\sigma(K_{\beta n})$ that are uniform in β.

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