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ON A GENERALIZED INTEGRAL EQUATION WHICH ORIGINATES FROM A PROBLEM IN DIFFUSION THEORY

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1. Introduction. Let $B_x(s)$ be a reflecting Brownian motion on $(0,\infty)$ with $B_x(0) = x$. Let τ^+ be the first time, s, that the sojourn time of $(1,\infty)$, for B_x up to time s, exceeds the sojourn time of $[0,1]$ up to time s, and define $Y^+ = B_x(\tau^+)$. In [**2**] it was established that, for $0 < x < 1$, the probability density of Y^+ is $\Pi(x,y)$ in the sense that $\mathbf{P}^x(Y^+ \in (1+y, 1+y+dy)) = \Pi(x,y)\,dy$, where Π satisfies

$$\int_0^\infty (\cosh\theta\cos\theta y + \sinh\theta\sin\theta y)\Pi(x,y)\,dy = \cosh\theta x, \qquad \theta > 0. \tag{1}$$

For further discussion of this remarkable result and additional references see [**1**].

In [**2**] the closed form solution to the above problem is obtained by ad hoc methods in the form .

$$\Pi(x,y) = \frac{\cosh\frac{1}{2}\pi y(\sinh\frac{1}{2}\pi y\cos\frac{1}{2}\pi x)^{\frac{1}{2}}}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y + \cos^2\frac{1}{2}\pi x)}. \tag{2}$$

A constructive proof of this result was given in [**3**] by using Laplace transform methods to show that the associated integral equation

$$\int_0^\infty \Big(\sin\Big(\frac{\pi}{4}+\theta\Big)e^{-\theta y} + \sin\Big(\frac{\pi}{4}-\theta\Big)e^{\theta y}\Big)\Pi(x,y)\,dy = \sqrt{2}\cosh\theta x, \tag{3}$$

where x and θ are complex, admits a solution $\Pi(x,y)$ in convolution form, namely,

$$\Pi(x,y) = \sqrt{(2\pi)}\int_0^y \frac{G(x,y-\nu)\,d\nu}{\sqrt{(\sinh\frac{1}{2}\pi\nu)}}, \tag{4}$$

where

(5)

$$G(x,\tau) = G(x,-\tau) = \frac{\sqrt{\pi}}{16}\Big\{\Big[\cosh\frac{1}{2}\pi(x+\tau)\Big]^{-3/2} + \Big[\cosh\frac{1}{2}\pi(x-\tau)\Big]^{-3/2}\Big\}.$$

Furthermore, by setting $a = e^{\pi y/2}$, $b = -e^{-\pi y/2}$, $p = e^{\pi x/2}$ and $q = e^{-\pi x/2}$ in

$$\int \frac{d\nu}{(a\nu + b)^{1/2}(p\nu + q)^{3/2}} = \frac{2}{(aq - bp)}\left(\frac{a\nu + b}{p\nu + q}\right)^{1/2}, \tag{6}$$

it was shown in [**3**] that $\Pi(x, y)$, as given by (4) and (5), may be written as

$$\Pi(x, y) = \frac{\cosh \frac{1}{2}\pi y(\sinh \frac{1}{2}\pi y \cosh \frac{1}{2}\pi x)^{1/2}}{\sqrt{2}(\sinh^2 \frac{1}{2}\pi y + \cosh^2 \frac{1}{2}\pi x)}.$$

Equation (1) is the special case of (3) obtained by replacing θ and x by $i\theta$ and ix respectively. Consequently, (2) was obtained as a solution of (1) by taking ix in place of x.

In this paper we consider, for x and θ complex, the integral equation

$$\int_0^\infty (\sin(\alpha\pi + \theta)e^{-\theta y} + \sin(\alpha\pi - \theta)e^{\theta y})\Pi(x, y)\, dy = \sqrt{2}\cosh \theta x, \tag{7}$$

where α is real. Clearly, $\alpha = 1/4$ gives (3). The procedures used in [**3**] will be modified to obtain solutions $\Pi(x, y) = \Pi_\alpha(x, y)$ of (7). Due to the periodicity of the sine function α can be restricted to a half-open interval of length 2; we choose $-1 \le \alpha < 1$, and if $\alpha = \beta + 1$ with $-1 \le \beta < 0$ then α may be further restricted to $0 \le \alpha < 1$. It would appear that the four cases $0 < \alpha < 1/2, \alpha = 0, \alpha = 1/2, 1/2 < \alpha < 1$ should be dealt with separately. We begin with the case $0 < \alpha < 1/2$.

2. The case $0 < \alpha < 1/2$. Let x and θ be complex with $-1 < \operatorname{Im} x < 1$ and $-\alpha\pi < \operatorname{re}\theta < \alpha\pi$, $0 < \alpha < 1/2$; the conditions on x, θ and α are sufficient to guarantee that the various integrals exist and for the gamma and beta functions to be defined. Observe that if $0 < \alpha < 1/2$ then the analogues of (4) and (5) are, respectively,

$$\Pi(x, y) = K(\alpha) \int_0^y \frac{G(x, y - \nu)\, d\nu}{(\sinh \frac{1}{2}\pi\nu)^{2\alpha}} \tag{8}$$

and

(9)

$$G(x, \tau) = G(x, -\tau) = L(\alpha)\left\{\left[\cosh \frac{1}{2}\pi(x+\tau)\right]^{2\alpha-2} + \left[\cosh \frac{1}{2}\pi(x-\tau)\right]^{2\alpha-2}\right\},$$

where $K(\alpha) = \pi 2^{1-2\alpha}/\Gamma(1-2\alpha)$ and $L(\alpha) = 2^{2\alpha}\Gamma(2-2\alpha)/8\sqrt{2}$. The coefficients $K(\alpha)$ and $L(\alpha)$ are chosen in this way to yield (4) and (5) when $\alpha = 1/4$. Furthermore, by setting $a = e^{\pi y/2}$, $b = -e^{-\pi y/2}$, $p = e^{\pi x/2}$ and $q = e^{-\pi x/2}$ in

$$(10)\quad \int \frac{d\nu}{(a\nu+b)^{2\alpha}(p\nu+q)^{2-2\alpha}} = \frac{1}{(1-2\alpha)(aq-bp)}\left(\frac{a\nu+b}{p\nu+q}\right)^{1-2\alpha}$$

(which is (6) when $\alpha = 1/4$), one can evaluate the convolution (8) to get

$$(11)\qquad \Pi(x,y) = \frac{\cosh\frac{1}{2}\pi y(\sinh\frac{1}{2}\pi y)^{1-2\alpha}(\cosh\frac{1}{2}\pi x)^{2\alpha}}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y + \cosh^2\frac{1}{2}\pi x)}.$$

If $\Pi(x,y)$ is given by (8) and (9) then it will be convenient to set $\tilde{\Pi}(x,\theta) = \mathcal{L}[\Pi(x,y)](\theta)$, and if $G(x,\tau) = G_\alpha(x,\tau)$ is given by (9) then let $\tilde{G}(x,\theta) = \mathcal{L}[G(x,\tau)](\theta)$ and proceed to establish the identity

$$(12)\quad \tilde{G}(x,\theta)+\tilde{G}(x,-\theta) = (\pi\sqrt{2})^{-1}\Gamma\Big(1-\alpha+\frac{\theta}{\pi}\Big)\Gamma\Big(1-\alpha-\frac{\theta}{\pi}\Big)\cosh\theta x.$$

We begin by showing that

$$(13)\qquad \tilde{G}(x,\theta)+\tilde{G}(x,-\theta) = 2L(\alpha)\int_{-\infty}^{\infty}\frac{\cosh\theta\tau\, d\tau}{[\cosh\frac{1}{2}\pi(x+\tau)]^{2-2\alpha}}.$$

Clearly,

$$\tilde{G}(x,\theta)+\tilde{G}(x,-\theta) = 2\int_0^\infty \cosh\theta\tau G(x,\tau)\,d\tau,$$

and substituting for $G(x,\tau)$ from (9) readily produces (13). To complete the proof of (12) we show that the right-hand sides of (12) and (13) are equal. Beginning with the well-known beta function formula

$$B(m,n) = \int_0^\infty \frac{v^{m-1}\,dv}{(1+v)^{m+n}}\quad (\operatorname{Re} m > 0, \operatorname{Re} n > 0),$$

set $e^{2u} = v$ to get, with $m = 1-\alpha+s$ and $n = 1-\alpha-s$,

$$\int_{-\infty}^{\infty}\frac{e^{2su}\,du}{(\cosh u)^{2-2\alpha}} = 2^{1-2\alpha}B(1-\alpha+s, 1-\alpha-s).$$

From this one can deduce that

$$\int_{-\infty}^{\infty} \frac{e^{2st}\,dt}{[\cosh(z+t)]^{2-2\alpha}} = 2^{1-2\alpha} e^{-2sz} B(1-\alpha+s, 1-\alpha-s),$$

and replacing s by $-s$ gives, on adding the two formulae,

$$\int_{-\infty}^{\infty} \frac{\cosh 2st\,dt}{[\cosh(z+t)]^{2-2\alpha}} = 2^{1-2\alpha} \cosh 2sz B(1-\alpha+s, 1-\alpha-s).$$

Replacing z by $\pi x/2, t = \pi\tau/2, s = \theta/\pi$, we need only some well-known properties of the beta and gamma functions together with $L(\alpha) = 2^{2\alpha}\Gamma(2-2\alpha)/8\sqrt{2}$ to complete the proof of (12).

With $\tilde{\Pi}(x,\theta) = \mathcal{L}[\Pi(x,y)](\theta)$ our integral equation (7) takes the form

$$\sin(\alpha\pi+\theta)\tilde{\Pi}(x,\theta) + \sin(\alpha\pi-\theta)\tilde{\Pi}(x,-\theta) = \sqrt{2}\cosh\theta x. \tag{14}$$

We proceed to show that this Wiener-Hopf identity is satisfied by

$$\tilde{\Pi}(x,\theta) = \frac{2}{\Gamma(1-2\alpha)} \tilde{G}(x,\theta) B\Big(\alpha + \frac{\theta}{\pi}, 1-2\alpha\Big). \tag{15}$$

Substituting into the left-hand side of (14) for $\tilde{\Pi}(x,\theta)$ and $\tilde{\Pi}(x,-\theta)$, from (15), gives

$$\begin{aligned}\frac{2}{\Gamma(1-2\alpha)}\Big(&\sin(\alpha\pi+\theta)\tilde{G}(x,\theta)B\Big(\alpha+\frac{\theta}{\pi},1-2\alpha\Big)\\ &+\sin(\alpha\pi-\theta)\tilde{G}(x,-\theta)B\Big(\alpha-\frac{\theta}{\pi},1-2\alpha\Big)\Big)\\ &= 2\bigg(\frac{\sin(\alpha\pi+\theta)\Gamma(\alpha+\frac{\theta}{\pi})\tilde{G}(x,\theta)}{\Gamma(1-\alpha+\frac{\theta}{\pi})} + \frac{\sin(\alpha\pi-\theta)\Gamma(\alpha-\frac{\theta}{\pi})\tilde{G}(x,-\theta)}{\Gamma(1-\alpha-\frac{\theta}{\pi})}\bigg).\end{aligned}$$

With $z = \alpha + \theta/\pi$ and $z = \alpha - \theta/\pi$ in $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, the left-hand side of (14) may be reduced to

$$2\pi\frac{(\tilde{G}(x,\theta)+\tilde{G}(x,-\theta))}{\Gamma(1-\alpha+\frac{\theta}{\pi})\Gamma(1-\alpha-\frac{\theta}{\pi})}.$$

Using (12) this becomes $\sqrt{2}\cosh\theta x$, and we have shown that (14) is satisfied by $\tilde{\Pi}(x,\theta)$ as given in (15). To establish (15), apply the Laplace transform to the convolution (8) to get

$$\tilde{\Pi}(x,\theta) = K(\alpha)\tilde{G}(x,\theta)\mathcal{L}\Big[\Big(\sinh\frac{1}{2}\pi\nu\Big)^{-2\alpha}\Big](\theta),$$

and it only remains to prove that

$$\mathcal{L}\Big[\Big(\sinh\frac{1}{2}\pi\nu\Big)^{-2\alpha}\Big](\theta) = 2^{2\alpha}\pi^{-1}B\Big(\alpha+\frac{\theta}{\pi},1-2\alpha\Big). \tag{16}$$

To obtain (16), simply set $e^{-\pi\nu} = u$ in the Laplace integral to get

$$\frac{2^{2\alpha}}{\pi}\int_0^1 u^{\theta/\pi-1+\alpha}(1-u)^{-2\alpha}\,du,$$

which is the required beta function integral. It is now clear that the convolution (8) satisfies our integral equation (7).

3. Evaluation of the convolution. To evaluate the convolution (8), set $y-\nu=-\tau$ to get

$$\Pi(x,y) = -K(\alpha)\int_0^{-y}\frac{G(x,\tau)\,d\tau}{[\sinh\frac{1}{2}\pi(y+\tau)]^{2\alpha}}, \tag{17}$$

and, if we substitute in for $G(x,\tau)$ from (9), there are two integrals to determine. Let $X=\pi x/2, Y=\pi y/2$ and consider

$$I(X,Y) = \int_0^{-Y}\frac{dt}{[\sinh(Y+t)]^{2\alpha}[\cosh(X+t)]^{2-2\alpha}};$$

with $e^{2t}=\nu$ this becomes

$$\frac{1}{2}I(X,Y) = \int_1^{e^{-2Y}}\frac{d\nu}{(e^Y\nu-e^{-Y})^{2\alpha}(e^X\nu+e^{-X})^{2-2\alpha}},$$

and (10) with $a=e^Y$, $b=-e^{-Y}$, $p=e^X$ and $q=e^{-X}$ gives

$$\frac{1}{2}I(X,Y) = \frac{-1}{(1-2\alpha)(e^{Y-X}+e^{X-Y})}\left(\frac{e^Y-e^{-Y}}{e^X+e^{-X}}\right)^{1-2\alpha}.$$

Replacing X by $-X$ and adding gives

$$\frac{1}{2}(I(X,Y)+I(-X,Y)) = \frac{-\cosh Y\cosh X}{(1-2\alpha)(\sinh^2 Y+\cosh^2 X)}\left(\frac{\sinh Y}{\cosh X}\right)^{1-2\alpha},$$

and setting $\tau = 2t/\pi$ in our integral (17) gives, on using $K(\alpha)L(\alpha) = \pi(1-2\alpha)/4\sqrt{2}$,

$$\Pi(x,y) = -\frac{(1-2\alpha)}{2\sqrt{2}}\left\{I\left(\frac{1}{2}\pi x, \frac{1}{2}\pi y\right) + I\left(-\frac{1}{2}\pi x, \frac{1}{2}\pi y\right)\right\},$$

which is (11).

4. The case $\alpha = 0$. In this case (7) reduces to

$$-\sin\theta\int_0^\infty 2\sinh\theta y\Pi(x,y)\,dy = \sqrt{2}\cosh\theta x,$$

and, with θ replaced by $i\theta$, this becomes

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sin\theta y\Pi(x,y)\,dy = \frac{\cos\theta x}{\sqrt{\pi}\sinh\theta}.$$

From a study of Fourier sine transforms,

$$\Pi(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta y\cos\theta x\,d\theta}{\sqrt{\pi}\sinh\theta},$$

which may be written as

$$\Pi(x,y) = \frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{(\sin\theta(x+y)-\sin\theta(x-y))d\theta}{\sinh\theta}.$$

It is well-known that if $|\mathrm{Im}\,a| < 1$ then

$$\int_0^\infty \frac{\sin a\theta d\theta}{\sinh\theta} = \frac{\pi}{2}\tanh\frac{a\pi}{2},$$

and consequently

$$\begin{aligned}\Pi(x,y) &= \left\{\tanh\frac{1}{2}\pi(x+y) - \tanh\frac{1}{2}\pi(x-y)\right\}/2\sqrt{2}\\ &= \frac{\sinh\frac{1}{2}\pi y\cosh\frac{1}{2}\pi y}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y+\cosh^2\frac{1}{2}\pi x)},\end{aligned}$$

which is (11) when $\alpha = 0$.

5. The case $\alpha = 1/2$. Here our integral equation (7) reduces to

$$\cos\theta \int_0^\infty 2\cosh\theta y \Pi(x,y)\,dy = \sqrt{2}\cosh\theta x,$$

and, with θ replaced by $i\theta$, this becomes

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \cos\theta y \Pi(x,y)\,dy = \frac{\cos\theta x}{\sqrt{\pi}\cosh\theta}.$$

Fourier cosine transform theory then gives

$$\Pi(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos\theta y \cos\theta x\,d\theta}{\sqrt{\pi}\cosh\theta},$$

which may be written as

$$\Pi(x,y) = \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{(\cos\theta(x+y) + \cos\theta(x-y))\,d\theta}{\cosh\theta}.$$

It is known that if $|\mathrm{Im}\, a| < 1$ then

$$\int_0^\infty \frac{\cos a\theta\,d\theta}{\cosh\theta} = \frac{\pi}{2}\operatorname{sech}\frac{a\pi}{2},$$

and hence

$$\begin{aligned}\Pi(x,y) &= \left\{\operatorname{sech}\frac{1}{2}\pi(x+y) + \operatorname{sech}\frac{1}{2}\pi(x-y)\right\}/2\sqrt{2}\\ &= \frac{\cosh\frac{1}{2}\pi x \cosh\frac{1}{2}\pi y}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y + \cosh^2\frac{1}{2}\pi x)},\end{aligned}$$

which is (11) when $\alpha = 1/2$.

6. The case $1/2 < \alpha < 1$. We shall now demonstrate that the solution $\Phi(x,y) = \Phi_\alpha(x,y)$ of the equation

$$(18)\quad \int_0^\infty (\sin(\alpha\pi+\theta)e^{-\theta y} + \sin(\alpha\pi-\theta)e^{\theta y})\Phi_\alpha(x,y)\,dy = \sqrt{2}\cosh\theta x,$$

when $1/2 < \alpha < 1$, is simply

$$\Phi_\alpha(x,y) = -\Pi_{1-\alpha}(x,y),$$

where $\Pi(x,y) = \Pi_\alpha(x,y)$ is given in (11). To see this set $\alpha = 1-\beta$ with $0 < \beta < 1/2$; (18) then takes the form

$$\int_0^\infty (\sin(\beta\pi - \theta)e^{-\theta y} + \sin(\beta\pi + \theta)e^{\theta y})\Phi_\alpha(x,y)\,dy = \sqrt{2}\cosh\theta x,$$

which may be written as

$$\sin(\beta\pi - \theta)\tilde{\Phi}_\alpha(x,\theta) + \sin(\beta\pi + \theta)\tilde{\Phi}_\alpha(x,-\theta) = \sqrt{2}\cosh\theta x \tag{19}$$

with the usual notation. At this point we extend the definition of $\Pi(x,y)$ to $\mathbf{C}\times\mathbf{R}$ by insisting that

$$\Pi(x,y) = \Pi(x,-y) = \Pi(-x,y) = \Pi(-x,-y),$$

which may be achieved by replacing $\sinh\pi y/2$ by $\sinh\pi|y|/2$ in the numerator of (11). A comparison of (19) and (14) shows that $\tilde{\Phi}_\alpha(x,\theta) = \tilde{\Pi}_\beta(x,-\theta)$ solves (19). To complete the analysis we need a result which relates $\tilde{f}(\theta) = \mathcal{L}[f(t)](\theta)$ and $\tilde{f}(-\theta)$, where $-a < \operatorname{Re}\theta < a$. I conjecture that if f is continuous on $\mathbf{R}$ and $f(t) \sim e^{-a|t|}$ as $|t| \to \infty$ and if $\tilde{f}(\theta) = \mathcal{L}[f(t)](\theta)$ is defined and analytic in the strip $-a < \operatorname{Re}\theta < a$, then, by using analytic continuation,

$$\tilde{f}(-\theta) = \mathcal{L}[-f(-t)](\theta).$$

In a private communication, R.R. London has proved a special case of this result. As a consequence, our solution $\Phi_\alpha(x,y)$ of (19) in the case $1/2 < \alpha < 1$ is

$$\begin{aligned}\Phi_\alpha(x,y) &= -\Pi_\beta(x,-y) = -\Pi_{1-\alpha}(x,-y)\\ &= -\frac{\cosh\frac{1}{2}\pi y(\sinh\frac{1}{2}\pi|y|)^{2\alpha-1}(\cosh\frac{1}{2}\pi x)^{2-2\alpha}}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y + \cosh^2\frac{1}{2}\pi x)},\end{aligned}$$

which completes the last of the four cases.

7. A probabilistic corollary. From the analysis of §2 with x replaced by ix and $0 < x < 1$ we have that

$$\Pi(x,y) = \Pi_\alpha(x,y) = \frac{\cosh\frac{1}{2}\pi y(\sinh\frac{1}{2}\pi y)^{1-2\alpha}(\cos\frac{1}{2}\pi x)^{2\alpha}}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y + \cos^2\frac{1}{2}\pi x)}$$

solves the integral equation

$$(20)\quad \int_0^\infty (\sin(\alpha\pi+\theta)e^{-\theta y} + \sin(\alpha\pi-\theta)e^{\theta y})\Pi(x,y)\,dy = \sqrt{2}\cosh(i\theta x)$$

when $|\operatorname{Re}\theta| < \alpha\pi$ and $0 < \alpha < 1/2$. For $|\operatorname{Re}\theta| < \alpha\pi$ the integral on the left-hand side of (20) is uniformly and absolutely convergent and represents a regular function of θ in $|\operatorname{Re}\theta| < \alpha\pi$. However, the right-hand side of (20) is an entire function of θ, and in $|\theta| < \alpha\pi$ we can expand each side in a power series about $\theta = 0$ and equate corresponding powers of θ. Clearly, the left-hand side of (20) is

$$\frac{1}{i}\sum_{n=0}^\infty \frac{\theta^{2n}}{(2n)!}\int_0^\infty (e^{i\alpha\pi}(i-y)^{2n} - e^{-i\alpha\pi}(-i-y)^{2n})\Pi(x,y)\,dy,$$

and, equating coefficients of θ^{2n}, $n = 0,1,2,\ldots$, in (20), yields

$$\sqrt{2}(ix)^{2n} = 2\,\operatorname{Im}\int_0^\infty e^{i\alpha\pi}(i-y)^{2n}\Pi(x,y)\,dy.$$

Hence

$$\sqrt{2}(-1)^n x^{2n} = 2\,\operatorname{Im}\{e^{i\alpha\pi}\mathbf{E}(i - Y^+ + 1)^{2n}\},$$

or, for $n = 0,1,2,3,\ldots$,

$$(-1)^n x^{2n} = \mathbf{E}\Big\{\sqrt{2}\sin\Big(\alpha\pi - 2n\tan^{-1}\Big(\frac{1}{Y^+-1}\Big)\Big)\cdot((Y^+-1)^2+1)^n\Big\}.$$

REFERENCES

1. R.R. London, H.P. McKean, L.C.G. Rogers and David Williams, *A martingale approach to some Wiener-Hopf problems*, I, Séminaire de Probabilités XVI, Lecture Notes in Math. **920** Springer, Heidelberg, 1982, 41-67.

2. ———, ———, ——— and ———, *A martingale approach to some Wiener-Hopf problems*, II, Séminaire de Probabilités XVI, Lecture Notes in Math. **920** Springer, Heidelberg, 1982, 68-90.

3. M.T. McGregor, *A solution of the integral equation*

$$\int_0^\infty \left(\sin\left(\frac{\pi}{4}+\theta\right)e^{-\theta y} + \sin\left(\frac{\pi}{4}-\theta\right)e^{\theta y}\right)\Pi(x,y)\,dy = \sqrt{2}\cosh\theta x$$

in convolution form and a problem in diffusion theory, Stochastic Mechanics and Stochastic Processes, Lecture Notes in Math. **1325** Springer, Heidelberg, 1988, 162-166.

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