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PROJECTION METHODS FOR SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. Both necessary and sufficient conditions are given under which the direct and indirect methods of finding the approximate solution of singular integral equations, with Cauchy kernel, are the same. The theory is applied to two examples and the paper concludes by considering the Sloan iteration applied to the direct method.

1. Introduction. We consider projection methods for the approximate solution of the singular integral equation

$$a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)\,d\tau}{\tau - t} + \int_{-1}^{1} k(t,\tau)\phi(\tau)\,d\tau = f(t), \tag{1.1}$$

on the arc (-1,1). The first integral is to be interpreted as the Cauchy principal value. The functions a, b, k and f are given and the unknown function ϕ is required or, through the projection methods, approximations to ϕ. Rewrite (1.1) as

$$M\phi + K\phi = f \tag{1.2}$$

where

$$M\phi(t) = a(t)\phi(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)\,d\tau}{\tau - t} \tag{1.3}$$

and

$$K\phi(t) = \int_{-1}^{1} k(t,\tau)\phi(\tau)\,d\tau. \tag{1.4}$$

Suppose that the linear operators M and K each map a normed space X into a normed space Y. The spaces are chosen so that M is bounded and K compact. The function f is an element out of Y and $\phi \in X$.

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Much of the theory of such equations is given by Muskhelishvili [**6**] to which the reader is referred for further details (in particular, Chapter 14). Assume that M has index κ which may be positive, negative or zero. To regularize equation (1.2) define a bounded linear operator $\hat{M}^I : Y \to X$ by

$$(1.5) \qquad \hat{M}^I\psi(t) = \frac{a(t)\psi(t)}{r^2(t)} - \frac{b(t)Z(t)}{r(t)} \cdot \frac{1}{\pi}\int_{-1}^{1} \frac{\psi(\tau)\,d\tau}{r(\tau)Z(\tau)(\tau - t)},$$

for $t \in (-1, 1)$ where Z denotes the fundamental function of M and $r^2 = a^2 + b^2$. This operator has the property that

$$(1.6(a)) \qquad \hat{M}^I M = I \text{ on } X, \text{ when } \kappa \le 0,$$

and

$$(1.6(b)) \qquad M\hat{M}^I = I \text{ on } Y, \text{ when } \kappa \ge 0.$$

$\hat{M}^I$ is the inverse of M only when $\kappa = 0$ (see also Elliott [**1**]). From (1.2) we have, premultiplying by $\hat{M}^I$, that

$$(1.7) \qquad \phi + \hat{M}^I K\phi = \hat{M}^I f + \phi^{(0)}$$

where $\phi^{(0)}$ denotes any solution of the homogeneous dominant equation $M\phi = 0$. It is known that dim ker$(M) = \max(\kappa, 0)$ so that $\phi^{(0)} = 0$ whenever $\kappa \le 0$. We now make an assumption which will be taken to be true throughout the remainder of this paper.

ASSUMPTION A. The operator $(I + \hat{M}^I K)^I : X \to X$ exists.

This means that we are supposing (-1) is not an eigenvalue of the compact operator $\hat{M}^I K$. Given Assumption A (1.7) shows that when $\kappa \le 0$ the solution ϕ, if it exists, will be unique. However, when $\kappa < 0$ the existence of the solution requires that certain consistency conditions be satisfied. These are given by

$$(1.8) \qquad \int_{-1}^{1} \frac{\tau^j}{r(\tau)Z(\tau)}\left\{f(\tau) - K\phi(\tau)\right\} d\tau = 0, \qquad j = 0(1)(-\kappa - 1).$$

In other words we require that $f - K\phi \in$ ran (M) and we shall assume, throughout the remainder of the paper, that

$$(1.9) \qquad f - Kx \in \text{ ran } (M)\, \forall x \in X.$$

Returning to the case when $\kappa > 0$ we see from Assumption A and since $\phi^{(0)} \neq 0$ in this case, that the solution of (1.7) will not be unique. We can make it unique by imposing κ additional conditions on ϕ. Assume these to be specified and that any approximations to ϕ also satisfy the same conditions. For $\kappa > 0$ there is no need for any consistency conditions so that a solution of (1.2) exists for all f and K.

Many approximate methods for the solution of (1.2), or the equivalent equation (1.7), are projection methods. These may be summarized as follows. Let $P_n : X \to X_n$ denote a projection operator from X onto an n-dimensional subspace X_n of X. If κ is the index of M then suppose that $Q_m : Y \to Y_m$ is a projection operator from Y onto an m-dimensional subspace Y_m where

$$n - m = \kappa. \tag{1.10}$$

Consider now the so-called direct and indirect methods for the approximate solution of (1.2). First, for the indirect methods, start with the regularized equivalent equation (1.7) and look for approximate solutions $\psi_n \in X_n$ such that

$$\psi_n + P_n \hat{M}^I K \psi_n = P_n \hat{M}^I f + P_n \phi^{(0)}. \tag{1.11}$$

For many projection operators it is known that given Assumption A, the inverse operator $(I + P_n \hat{M}^I K)^I : X \to X$ will exist for n large enough, say for $n > n_0$. If $\kappa > 0$ assume that ψ_n satisfies the same additional κ conditions as imposed upon ϕ.

For the direct methods start with equation (1.2) and look for an approximate solution $\phi_n \in X_n$ such that

$$Q_m M \phi_n + Q_m K \phi_n = Q_m f. \tag{1.12}$$

Again, when $\kappa > 0$ we assume that ϕ_n satisfies the same additional κ conditions as does ϕ.

The main result of this paper is to give both necessary and sufficient conditions on the projection operators P_n and Q_m so that we have $\phi_n = \psi_n$. Sufficient conditions are given in §2 and, in §3, we will establish necessary conditions. In §4 we consider two methods for the approximate solution of (1.2), both based on polynomial approximations to ϕ taken over (-1,1). For a Galerkin method we show that the sufficient conditions for the two approximate solutions to be the same are satisfied, whereas for a collocation method they are not. This paper concludes by considering, in §5, the Sloan iteration of the approximate solutions.

2. Sufficient conditions. Now consider sufficient conditions on the projection operators P_n and Q_m so that ϕ_n and ψ_n should be the same. The results are given in the following theorem.

THEOREM 2.1. *Suppose, for* $n > n_0$, *that*

$$MP_n = Q_m M, \text{ on } X, \tag{2.1}$$

and

$$\ker(M) \subseteq X_n. \tag{2.2}$$

Then $\phi_n = \psi_n$ *where* ψ_n *and* ϕ_n *are given by equations* (1.11) *and* (1.12) *respectively.*

PROOF. We break the proof into two parts depending upon the value of κ and assume first that $\kappa \leq 0$. In this case (2.2) is satisfied trivially. From (1.12) and (2.1) we have

$$MP_n\phi_n + Q_m(K\phi_n - f) = 0. \tag{2.3}$$

Since, referring to (1.9), $K\phi_n - f \in \operatorname{ran}(M)$ then there exists an element g_n in X such that

$$K\phi_n - f = Mg_n. \tag{2.4}$$

Substituting this into (2.3) and using (2.1) again we have

$$M\phi_n + MP_n g_n = 0, \tag{2.5}$$

since $P_n\phi_n = \phi_n$. But, from (2.4),

$$g_n = \hat{M}^I(K\phi_n - f) \tag{2.6}$$

so that we can rewrite (2.5) as

$$M\{\phi_n + P_n\hat{M}^I K\phi_n - P_n\hat{M}^I f\} = 0. \tag{2.7}$$

Since $\ker(M) = \{0\}$ we conclude that

$$\phi_n + P_n\hat{M}^I K\phi_n = P_n\hat{M}^I f. \tag{2.8}$$

Compare this equation with (1.11) and recall we are assuming $n > n_0$ so that $(I + P_n\hat{M}^I K)^I$ exists then we have $\phi_n = \psi_n$, as required.

Suppose now that $\kappa > 0$. From (2.1) and (1.6(b)) we have

$$MP_n\hat{M}^I = Q_m \text{ on } Y. \tag{2.9}$$

From (1.12) and (2.1) we have

$$MP_n\phi_n + MP_n\hat{M}^I(K\phi_n - f) = 0. \tag{2.10}$$

Since $P_n\phi_n = \phi_n$ we can rewrite this as

$$M\{\phi_n + P_n\hat{M}^I(K\phi_n - f)\} = 0 \tag{2.11}$$

so that

$$\phi_n + P_n\hat{M}^I K\phi_n = P_n\hat{M}^I f + \phi^{(0)}, \tag{2.12}$$

for some $\phi^{(0)} \in \ker(M)$. Using (2.2) we see that ϕ_n satisfies the same equation as that for ψ_n (see (1.11)) so, imposing the same additional κ conditions on both ϕ_n and ψ_n, we find $\phi_n = \psi_n$. □

So much for sufficient conditions on P_n and Q_m to assure that the approximate solutions of the direct and indirect equations are the same. The question is then whether these conditions are necessary. This we address in the next section.

3. Necessary conditions. In this section we prove the following theorem.

THEOREM 3.1. *If, from equations* (1.11) *and* (1.12) *we assume, for* $n > n_0$, *that* $\phi_n = \psi_n$, *then*

$$Q_m M P_n = Q_m M, \text{ on } X. \tag{3.1}$$

PROOF. Suppose that $\kappa \leq 0$ then $\phi^{(0)} = 0$. From (1.11) we have

$$\psi_n + P_n \hat{M}^I (K\psi_n - f) = 0. \tag{3.2}$$

From (1.12), with $\phi_n = \psi_n$, we have

$$Q_m M \psi_n + Q_m (K\psi_n - f) = 0. \tag{3.3}$$

Since, referring to (1.9), $f - Kx \in \operatorname{ran}(M)$ for every $x \in X$ then there exists an element $g_n \in X$ such that

$$K\psi_n - f = M g_n. \tag{3.4}$$

Operating on (3.2) with M, using both (3.4) and (1.6(a)) we find

$$M P_n \psi_n + M P_n g_n = 0, \tag{3.5}$$

since $P_n \psi_n = \psi_n$. Then (3.3) and (3.4) together give

$$Q_m M \psi_n + Q_m M g_n = 0, \tag{3.6}$$

so that subtracting this equation from (3.5) we have

$$(M P_n - Q_m M)(\psi_n + g_n) = 0. \tag{3.7}$$

But from (3.4) and (1.6(a))

$$g_n = \hat{M}^I (K\psi_n - f), \tag{3.8}$$

so (3.7) becomes

$$(M P_n - Q_m M)\{(I + \hat{M}^I K)\psi_n - \hat{M}^I f\} = 0. \tag{3.9}$$

Using the observation that if S and T are operators, with T being invertible, then

$$ST^I = I + (S - T)T^I$$

a routine calculation shows that

$$(3.10) \quad (I + \hat{M}^I K)(I + P_n \hat{M}^I K)^I = I + (I - P_n)\hat{M}^I K(I + P_n \hat{M}^I K)^I.$$

Recall from (1.11) we have $\psi_n = (I + P_n \hat{M}^I K)^I P_n \hat{M}^I f$. Since $n > n_0$, we can substitute for ψ_n in (3.9) and use (3.10) so

$$(3.11) \quad (MP_n - Q_m M)(I - P_n)\{\hat{M}^I K(I + P_n \hat{M}^I K)^I P_n - I\}\hat{M}^I f = 0.$$

This will be true for every $f \in \operatorname{ran}(M)$. If we write $f = Mg$ where $g \in X$ then we have

$$(3.12) \quad (MP_n - Q_m M)(I - P_n)\{\hat{M}^I K(I + P_n \hat{M}^I K)^I P_n - I\}g = 0,$$

for all $g \in X$. It is not difficult to show that the only $g \in X$ for which

$$(3.13) \qquad \{\hat{M}^I K(I + P_n \hat{M}^I K)^I P_n - I\}g = 0$$

is $g = 0$. Operating on (3.13) with P_n and adding an appropriate term to each side gives

$$(I + P_n \hat{M}^I K)(I + P_n \hat{M}^I K)^I P_n g = P_n g + (I + P_n \hat{M}^I K)^I P_n g$$

from which it follows that

$$(3.14) \qquad P_n g = 0.$$

Substituting this condition on g into (3.13) gives $g = 0$. Thus the operator given in the brackets $\{\ \}$ in (3.13) is one-to-one from X into itself. That it is also onto X follows from the Fredholm alternative on recalling that $\hat{M}^I$ and P_n are bounded and K is compact so that the operator in $\{\ \}$ is a second kind Fredholm integral operator. As g varies in X, the term $\{\ \}g$ in (3.13) takes all values in X and we have

$$(3.15) \qquad (MP_n - Q_m M)(I - P_n) = 0 \text{ on } X.$$

The necessary condition as given by (3.1) now follows.

So much for the case when $k \leq 0$. Suppose now that $\kappa > 0$. From (1.11), since $\psi_n = P_n\psi_n$, we have

$$(3.16) \qquad MP_n\psi_n + MP_n(\hat{M}^I K\psi_n - \hat{M}^I f) = MP_n\phi^{(0)}.$$

From (1.12) and recalling (1.6(b)) we have, since $\phi_n = \psi_n$, that

$$(3.17) \qquad Q_m M\psi_n + Q_m M(\hat{M}^I K\psi_n - \hat{M}^I f) = 0.$$

Subtracting (3.17) from (3.16) gives

$$(3.18) \qquad (MP_n - Q_m M)\{\psi_n + \hat{M}^I K\psi_n - \hat{M}^I f\} = MP_n\phi^{(0)}.$$

Let us consider the term in brackets $\{\ \}$. Assuming that for $n > n_0$, $(I + P_n\hat{M}^I K)^I$ exists, then from (1.11) we have

$$\{\ \} = (I + \hat{M}^I K)(I + P_n\hat{M}^I K)^I(P_n\hat{M}^I f + P_n\phi^{(0)}) - \hat{M}^I f.$$

Recalling (3.10) we find, after some algebra, that

$$(3.19) \quad \{\ \} = (I - P_n)\{\hat{M}^I K(I + P_n\hat{M}^I K)^I P_n - I\}(\hat{M}^I f + \phi^{(0)}) + \phi^{(0)}.$$

Since $M\phi^{(0)} = 0$, by definition of $\phi^{(0)}$, then

$$(3.20) \qquad (MP_n - Q_m M)\phi^{(0)} = MP_n\phi^{(0)}.$$

Substituting (3.19) and (3.20) into (3.18) the result is

$$(3.21) \qquad \begin{aligned} &(MP_n - Q_m M)(I - P_n)\{\hat{M}^I K(I + P_n\hat{M}^I K)^I P_n - I\} \\ &\quad \cdot (\hat{M}^I f + \phi^{(0)}) = 0. \end{aligned}$$

It follows from (1.6(b)) that every element $g \in X$ can be written in the form $\hat{M}^I f + \phi^{(0)}$ by choosing $f = Mg$ and $\phi^{(0)} = (I - \hat{M}^I M)g$. Consequently arguing as above we have that

$$(MP_n - Q_m M)(I - P_n) = 0 \quad \text{on } X,$$

from which (3.1) follows at once, concluding the proof. □

It is of interest to compare the sufficient condition (2.1) with the necessary condition (3.1). Obviously $MP_n = Q_m M$ immediately implies $Q_m M P_n = Q_m M$, since $Q_m^2 = Q_m$. The converse is, of course, not true.

4. Two approximate methods. The two methods considered here are the Galerkin method [**4**] and the method of classical collocation [**2**]. In each case assume in (1.1) that the coefficients a and b are real and as a consequence that $r > 0$ and that the fundamental Z is real. Instead of solving directly for ϕ a new dependent variable ψ is introduced where

$$\phi = Z\psi/r. \tag{4.1}$$

Thus, for example, if a, b, k and f are Hölder continuous it turns out that ψ is Hölder continuous on [-1,1], unlike ϕ which may be unbounded at the end points ± 1. Consequently, a new operator A is defined such that

$$A\psi = M(Z\psi/r). \tag{4.2}$$

But A will possess similar properties to M with regard to index, etc. so that the results of the preceding sections can be applied to A. For both the methods we are considering the weight functions w_1 and w_2 are defined by

$$w_1 = Z/r, \quad w_2 = 1/(Zr) \tag{4.3}$$

which are integrable and induce on (-1,1) sets of orthonormal polynomials $\{t_n\}$ and $\{u_n\}$ respectively; that is

$$\begin{aligned} &\int_{-1}^{1} w_1(\tau) t_j(\tau) t_k(\tau)\, d\tau = \delta_{j,k} \\ \text{and } &\int_{-1}^{1} w_2(\tau) u_j(\tau) u_k(\tau)\, d\tau = \delta_{j,k} \end{aligned} \tag{4.4}$$

for $j, k = 0, 1, 2, 3, \ldots$. The relationships between these two sets of polynomials and the operators A and $\hat{A}^I$ are given in [**2**] and [**4**].

In the Galerkin-Petrov method let X and Y be the weighted Hilbert spaces H_1 and H_2 respectively where the inner product on $H_i, i = 1, 2$, is denoted and defined by

$$< \psi_1, \psi_2 >_i = \int_{-1}^{1} w_i(\tau) \psi_1(\tau) \psi_2(\tau)\, d\tau \tag{4.5}$$

for all $\psi_1,\ \psi_2 \in H_i$. The projection operators P_n and Q_m on H_1 and H_2 respectively, are defined by

$$P_n\psi = \sum_{j=0}^{n-1} < t_j,\ \psi >_1 t_j, \tag{4.6}$$

and

$$Q_m\psi = \sum_{j=0}^{m-1} < u_j, \psi >_2 u_j. \tag{4.7}$$

It can be shown that when b is a polynomial then for n large enough

$$AP_n = Q_m A \text{ and } P_n\{\ker(A)\} = \ker(A), \tag{4.8}$$

see [**4**, §2]. Thus the sufficient conditions of Theorem 2.1 are satisfied and the direct and indirect methods give the same approximate solution. This was already observed in [**4**] and generalized a result given by Ioakimidis [**5**] for the case when the coefficients a and b are constants.

For the classical collocation method, see [**2**]. In order to define the projection operators we need to introduce the zeros of the polynomials t_n and u_m respectively. Suppose that

$$t_n(\tau_{j,n}) = 0,\ j = 1(1)n \text{ and } u_m(t_{i,m}) = 0,\ i = 1(1)m. \tag{4.9}$$

The projection operators P_n and Q_m are now essentially the Lagrange interpolation operators defined on t_n and u_m respectively. We have

$$(P_n x)(t) = \sum_{j=1}^{n} \frac{t_n(t)x(\tau_{j,n})}{t_n'(\tau_{j,n})(t - \tau_{j,n})} \tag{4.10}$$

and

$$(Q_m y)(t) = \sum_{i=1}^{m} \frac{u_m(t)y(t_{i,m})}{u_m'(t_{i,m})(t - t_{i,m})}. \tag{4.11}$$

Although for n large enough we have $P_n\{\ker(A)\} = \ker(A)$ we do not have in this case that $AP_nx = Q_mAx$ for every $x \in X$. This relation turns out to be true when $x \in \mathbf{P}_{n-1}$, the space of all polynomials of degree $\leq (n-1)$, but not otherwise. To see this, choose $x(t) = t_n(t)$, then $P_n t_n(t) = n t_n$ and, from [**3**], we have $AP_n t_n = (-1)^\kappa n u_m$. On the other hand, since $At_n = (-1)^\kappa u_m$, then $Q_m A t_n = (-1)^\kappa Q_m u_m = (-1)^\kappa m u_m$. Since, in general, $n \neq m$ the statement follows. Thus for classical collocation the direct and indirect methods will give different approximate solutions.

5. Sloan iteration. For Galerkin methods applied to Fredholm integral equations of the second kind, Sloan [**7**] has proposed an iteration which gives an improvement on the approximate solution first obtained. This can be described as follows. With a solution ψ_n of (1.11) we return to (1.7) and consider an improvement ψ_n^* such that

$$\psi_n^* + \hat{M}^I K \psi_n = \hat{M}^I f + \phi^{(0)}. \tag{5.1}$$

From this equation we have

$$P_n \psi_n^* = P_n\{\hat{M}^I f + \phi^{(0)} - \hat{M}^I K \psi_n\} = \psi_n, \tag{5.2}$$

using (1.11). It follows that the component of ψ_n^* in X_n is given by ψ_n but, in general, $\psi_n^* \notin X_n$. From (5.2) we see on substituting into (5.1) that ψ_n^* satisfies the equation

$$\psi_n^* + \hat{M}^I K P_n \psi_n^* = \hat{M}^I f + \phi^{(0)}. \tag{5.3}$$

Let us see how to apply this technique to the direct method of solving (1.2) once an approximation ϕ_n satisfying (1.12) has been found. Proceeding as in the preceding paragraph suggests that we solve for ϕ_n^* where

$$M\phi_n^* + K\phi_n = f. \tag{5.4}$$

Thus irrespective of the value of κ, provided $f - K\phi_n \in \operatorname{ran}(M)$ we have,

$$\phi_n^* = \hat{M}^I (f - K\phi_n) + \phi^{(0)}, \tag{5.5}$$

and

$$P_n \phi_n^* = P_n \hat{M}^I (f - K\phi_n) + P_n \phi^{(0)}. \tag{5.6}$$

If we assume that our projection operators P_n and Q_m satisfy the sufficiency condition of Theorem 2.1 then $\phi_n = \psi_n$ and from (1.11) we obtain

$$P_n\phi_n^* = \psi_n = \phi_n. \tag{5.7}$$

This gives rise to the following theorem.

THEOREM 5.1. *Under the sufficiency conditions of Theorem* 2.1 *the Sloan iteration* ϕ_n^*, *as defined by equation* (5.4), *satisfies the equation*

$$M\phi_n^* + KP_n\phi_n^* = f. \tag{5.8}$$

This is a good generalization of the comparable result for Fredholm integral equations of the second kind.

REFERENCES

1. D. Elliott, *A convergence theorem for singular integral equations*, J. Austral. Math. Soc. Ser. B **22** (1981), 539-552.

2. ———, *The classical collocation method for singular integral equations*, SIAM J. Numer. Anal. **19** (1982), 816-832.

3. ———, *Orthogonal polynomials associated with singular integral equations having a Cauchy kernel*, SIAM J. Math. Anal. **13** (1982), 1041-1052.

4. ———, *A Galerkin-Petrov method for singular integral equations*, J. Austral. Math. Soc. Ser. B **25** (1983), 261-275.

5. N.I. Ioakimidis, *On the weighted Galerkin method of numerical solution of Cauchy type singular integral equations*, SIAM J. Numer. Anal. **18** (1981), 1120-1127.

6. N.I. Muskhelishvili, *Singular Integral Equations*, P. Noordhoff, Groningen, 1953.

7. I.H. Sloan, *Improvement by iteration for compact operator equations*, Math. Comp. **30** (1976), 758-764.

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