A COMPREHENSIVE APPROACH TO THE APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS OVER THE ARC $(-1, 1)$

DAVID ELLIOTT[1]

ABSTRACT. Both a collocation and a Galerkin method are described for the approximate solution of the complete singular integral equation, with Cauchy principal value integral, defined on the arc $(-1, 1)$. Algorithms are described for all values of the index and an analysis of the discrete equations is given for each case. The approach is based on polynomial approximation over the arc $(-1, 1)$ and depends, in particular, upon the use of the Chebyshev polynomials of the first kind in the range space. The paper concludes with a convergence analysis which gives very satisfactory results. One surprising feature of the method is that except in the determination of the fundamental function, there is no need to evaluate any Cauchy principal value integrals. Furthermore, the method is not restricted to the particular cases of either constant or real coefficients and so is of wide applicability.

1. **Introduction.** In this paper singular integral equations of the form

$$(1.1) \qquad M\phi + K\phi = f,$$

are considered where, if we let

$$(1.2) \qquad T\phi(t) = \frac{1}{\pi}\lambda \int_{-1}^{1} \frac{\phi(\tau)d\tau}{\tau - t}$$

for $t \in (-1, 1)$, the integral being taken as the Cauchy principal value, then

$$(1.3) \qquad M\phi(t) = a(t)\phi(t) + T(b\phi)(t)$$

and

$$(1.4) \qquad K\phi(t) = \int_{-1}^{1} k(t, \tau)\phi(\tau)\,d\tau.$$

Assume that the complex-valued functions a, b, f and k are given and we require ϕ or, in the present context, approximations to ϕ. Then look for ϕ in the space of functions which is denoted by $\mathcal{H}(p_1, p_2)$ where $1 \leq p_1, p_2 \leq \infty$. Such a function is Hölder continuous on the open interval $(-1, 1)$ but is p_1-integrable at the end point -1 and p_2-integrable at the other end point $+1$. The functions a and b are Hölder continuous on $[-1, 1]$ and we shall put conditions on k and f as required below.

There has been much written in the past few decades on approximate solutions to (1.1)-(1.4); for two recent reviews see Golberg [3] and Venturino [10]. Generally M has been defined as

$$(1.5) \qquad\qquad M\phi = a\phi + bT\phi$$

but, as shown below, it is more convenient here to choose M as defined by (1.3). Many approximate methods for solving singular integral equations have exploited the property that when a and b are real, the operator M generates two sets of orthogonal polynomials. These polynomials have been used for both Galerkin and collocation methods to give approximate solutions. The theory is mathematically elegant although it has imposed upon it a further restriction that b be a polynomial. The two sets of orthogonal polynomials turn out to be generalized Jacobi polynomials and their computation, in particular the calculation of their zeros and Christoffel numbers, has been a problem. In special cases, when the three term recurrence relation satisfied by these polynomials is known explicitly, the computation is straightforward. But when the recurrence relation is not known analytically it is much more difficult. In any case it has not been at all clear how to generalize these ideas to the case when a and b are complex-valued. Although the theory of singular integral equations allows a and b to be complex, these computational techniques appear to fail completely. Motivated by these considerations the author has been reviewing the usual approaches to the approximate solution of these equations. This paper is the outcome of these deliberations. One consequence has been to make a minor change, as noted above, to the definition of M. For completeness some of the theory of these equations is restated in §2, together with some other results which shall be needed later. In §3 we consider a Galerkin method based on polynomial approximation. First in §3.1 the dominant equation only is

considered and then, in §3.2, the complete equation. All values of the index κ are allowed, although we must consider separately the two cases of $\kappa \geq 1$ and $\kappa \leq 0$. In §4 a similar analysis for a collocation method is shown. Both the Galerkin and collocation methods are based on the use of Chebyshev polynomials of the first kind. As a consequence, the discretization of the dominant operator gives rise in all cases to matrices whose properties are readily determined. This considerably simplifies the computational problems. Another interesting aspect of these methods is that, other than when evaluating the fundamental function Z, there is no need to evaluate either exactly or approximately any Cauchy principal value integrals. We do, however, need to evaluate improper integrals and these may best be done by using appropriate adaptive quadrature schemes. In §5 we give an error analysis which embraces both methods described in §3 and §4. The upper bounds turn out to be very satisfactory and demonstrate convergence for many functions f and k.

Because of its length we give no numerical results, but hope to repair this omission in a later paper.

2. Theory. Only a summary of results is given. For more information the reader is referred to the classic treatise of Muskhelishvili [7] (but see also Elliott [2]). For $t \in [-1, 1]$ define

$$(2.1) \qquad r^2(t) = a^2(t) + b^2(t)$$

and assume throughout that r does not vanish on $[-1, 1]$. A function θ, which must be continuous on $[-1, 1]$, is defined so that

$$(2.2) \qquad \exp\{\pm i\pi\theta(t)\} = (a(t) \mp ib(t))/r(t),$$

where the upper and lower signs go together. The *fundamental function* Z is defined on $(-1, 1)$ by

$$(2.3) \qquad Z(t) = (1 + t)^{n_1}(1 - t)^{n_2} \exp\{\pi(T\theta)(t)\},$$

where n_1 and n_2 are integers. These are determined by requiring that $Z \in \mathcal{H}(p_1, p_2)$, where $1 \leq p_1, p_2 \leq \infty$, and $1/Z \in \mathcal{H}(q_1, q_2)$ where $1/p_i + 1/q_i = 1, i = 1, 2$. If $\operatorname{Re}\theta(-1)$ is an integer then choose

$$(2.4) \qquad n_1 = \operatorname{Re}\theta(-1)$$

and similarly for n_2 if $\operatorname{Re}\theta(1)$ is an integer. However when neither is an integer look for n_1 and n_2 such that

$$(2.5a) \qquad \{\operatorname{Re}\theta(-1) - 1/p_1\} < n_1 < 1 + \{\operatorname{Re}\theta(-1) - 1/p_1\}$$

and

$$(2.5b) \qquad -\{\operatorname{Re}\theta(1) + 1/p_2\} < n_2 < 1 - \{\operatorname{Re}\theta(1) + 1/p_2\}.$$

The *index* $\kappa(p_1, p_2)$ of M in the space $\mathcal{H}(p_1, p_2)$ is defined by

$$(2.6) \qquad\qquad \kappa(p_1, p_2) = -(n_1 + n_2).$$

(Note that in the above analysis Muskhelishvili generally chooses $p_1 = p_2 = 1$). Consider now the deleted complex plane; that is the complex plane $\mathbf{C}$ with the interval $[-1, 1]$ deleted. The *canonical function* $X(z)$ for $z \in \mathbf{C} \setminus [-1, 1]$ is defined by

$$(2.7) \qquad X(z) = (1 + z)^{n_1}(1 - z)^{n_2}\exp\left\{\int_{-1}^1 \frac{\theta(\tau)\,d\tau}{(\tau - z)}\right\}.$$

From the Sokhotski-Plemelj formulae we find that for $t \in (-1, 1)$

$$(2.8) \qquad\qquad X^{\pm}(t) = \exp\{\pm i\pi\theta(t)\}Z(t).$$

Now consider the solution of the *dominant equation* $M\phi = f$. It can be shown that for $1 \le p_1, p_2 < \infty$, M maps the space $\mathcal{H}(p_1, p_2)$ into itself, so that we first assume that f is a given element out of $\mathcal{H}(p_1, p_2)$ for $1 \le p_1, p_2 < \infty$. The operator $\hat{M}^I$ is defined by

$$(2.9) \qquad\qquad \hat{M}^I f = (Z/r)\{(af/rZ) - T(fb/rZ)\}.$$

For $\kappa \ge 1$, the general solution of $M\phi = f$ is given by

$$(2.10) \qquad\qquad \phi = \hat{M}^I f + (Z/r)\mathbf{P}_{\kappa-1},$$

where $\mathbf{P}_{\kappa-1}$ denotes an arbitrary polynomial of degree $\le (\kappa - 1)$. On the other hand, when $\kappa \le 0$ the solution of $M\phi = f$, if it exists, is unique and is given by

$$(2.11) \qquad\qquad \phi = \hat{M}^I f.$$

Existence is guaranteed if f satisfies the $-\kappa$ *consistency conditions*

$$(2.12) \qquad \int_{-1}^{1} \frac{\tau^{i-1} b(\tau) f(\tau)}{Z(\tau) r(\tau)} \, d\tau = 0, \quad i = 1(1)(-\kappa).$$

We should note that only when $\kappa = 0$ is $\hat{M}^I$ the inverse of M and in this case the solution of $M\phi = f$ exists and is unique for all $f \in \mathcal{H}(p_1, p_2)$.

Now we introduce the principal part of a function defined in the complex plane $\mathbf{C}$. Suppose that the Laurent expansion of a function f about the point at infinity is given by

$$(2.13) \qquad f(z) = \sum_{j=-\infty}^{N} f_j z^j, \quad f_N \neq 0.$$

Then the *prinicpal part* of f at any point $z_0 \in \mathbf{C}$ is denoted and defined by

$$(2.14) \qquad \text{p.p.} \ (f; z_0) = \sum_{j=0}^{N} f_j z_0^j.$$

If $N \leq -1$ then p.p. $(f) \equiv 0$ and if $N \geq 0$ then p.p. (f) is a polynomial of degree N. By the theory of residues it can be shown that

$$(2.15) \qquad \text{p.p.} \ (f; z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z) \, dz}{z - z_0},$$

where all the singularities of f in the finite plane and the point z_0 are contained within the circle C_R, center at the origin and of radius R.

From (2.6) and (2.7) it can be shown that $X(z)$ behaves like $z^{-\kappa}$ at infinity; trivially, $X^{-1}(z)$ behaves like z^{κ}. It is straightforward to show that if P is any polynomial then, for $t \in (-1, 1)$,

$$(2.16) \qquad \frac{a(t) Z(t) P(t)}{r(t)} + T\left(\frac{bZP}{r}\right)(t) = \text{p.p.} \ (PX; t)$$

and

$$(2.17) \qquad \frac{a(t) P(t)}{r(t) Z(t)} - T\left(\frac{bP}{rZ}\right)(t) = \text{p.p.} \ (PX^{-1}; t).$$

These two equations are of fundamental importance when we consider approximate methods of solution of (1.1).

Returning to equations (2.9) and (2.10), it is convenient to define a new dependent variable ψ where

$$(2.18) \qquad\qquad \psi = r\phi/Z \quad \text{or} \quad \phi = Z\psi/r.$$

In many cases ψ will be a continuous function on $[-1,1]$. For this reason it is better to consider approximations to ψ rather than ϕ which has integrable singularities at the end points. To be more specific, Z has been chosen so that it is in $\mathcal{H}(p_1, p_2)$. Let operators A and $\hat{A}^I$ be defined by

$$(2.19) \qquad A\psi = M(Z\psi/r), \quad \hat{A}^I\psi = (r/Z)\hat{M}^I\psi.$$

It can be shown (see Appendix) that $\hat{A}^I$ maps Hölder continuous functions of order ν on $[-1,1]$ into Hölder continuous functions of order $\nu - 1/q$ on $[-1,1]$ where $q = \min(q_1, q_2)$. Furthermore, it turns out that $\hat{A}^I$ is a bounded linear operator on the Hölder space $H_\nu[-1,1]$ into $C[-1,1]$. In terms of the operators A and $\hat{A}^I$ we can rewrite (2.16) and (2.17) respectively as

$$(2.20) \qquad AP = \text{p.p.}\,(PX), \quad \hat{A}^I P = \text{p.p.}\,(PX^{-1}),$$

for all polynomials P. Thus both A and $\hat{A}^I$ map polynomials into polynomials and this will be exploited in our algorithms.

From the definitions of A and $\hat{A}^I$ it can be shown that

$$(2.21) \qquad\qquad \begin{cases} \hat{A}^I A = I & \text{when } \kappa \leq 0, \\ A\hat{A}^I = I & \text{when } \kappa \geq 0, \end{cases}$$

so that $\hat{A}^I$ is a left inverse of A when $\kappa \leq 0$ and a right inverse when $\kappa \geq 0$. Only when $\kappa = 0$ is $\hat{A}^I$ the inverse of A. When $\kappa \geq 1$, the null space of A is of dimension κ and

$$(2.22) \qquad\qquad \ker(A) = \text{span}\,\{1, t, \ldots, t^{\kappa-1}\}.$$

Finally we make some comments on the polynomials which will be used throughout the algorithms to be discussed. We approximate

to both f and ψ by polynomials so that we are concerned with global approximations on $[-1, 1]$. For the range space of A we shall approximate by means of the Chebyshev polynomials of the first kind. These are denoted and defined by

$$(2.23) \qquad T_n(x) = \cos n\theta, \quad x = \cos \theta.$$

A description of their properties is given in the book by Rivlin [9??], to which the reader will be frequently referred. In the domain of A we introduce polynomials u_n, as follows. For $n \geq \max(0, \kappa)$ define

$$(2.24) \qquad u_n(t) = (\hat{A}^I T_{n-\kappa})(t) = \text{p.p.}\,(X^{-1} T_{n-\kappa}; t),$$

on using (2.20). From the first equation of (2.24) it follows that

$$(2.25) \qquad (A u_n)(t) = T_{n-\kappa}(t) + \sum_{\ell=1}^{-\kappa} \sigma_{\ell,n} T_{\ell-1}(t),$$

where the constants $\sigma_{\ell,n}$ are to be determined. We observe the convention that an empty sum is zero. When $\kappa \geq 0$ the sum is empty, when $\kappa \leq -1$ we have essentially an element out of the null space of the operator $\hat{A}^I$. It is readily seen from (2.20) that

$$(2.26) \qquad \ker(\hat{A}^I) = \text{span}\,\{1, t, \ldots, t^{-\kappa-1}\}.$$

For a given value of n, to determine the $(-\kappa)$ constants $\sigma_{l,n}$ we have that since, by construction, the solution u_n of (2.25) exists, then the right hand side must satisfy the $(-\kappa)$ consistency conditions (2.12). Thus we find

$$(2.27) \qquad \sum_{\ell=1}^{-\kappa} \left(\int_{-1}^{1} \frac{\tau^{k-1} b T_{\ell-1}}{rZ}\, d\tau \right) \sigma_{\ell,n} = -\int_{-1}^{1} \frac{\tau^{k-1} b T_{n-\kappa}}{rZ}\, d\tau,$$

for $k = 1(1)(-\kappa)$. This gives $(-\kappa)$ equations for $(-\kappa)$ unknowns $\sigma_{\ell,n}$ which, in principle, may be solved.

From time to time we need to evaluate finite sums of the form

$$(2.28) \qquad s_n(t) = \sum_{k=\lambda}^{n} c_{k-1} u_{k-1}(t), \quad \lambda = \max(1, 1+\kappa),$$

for given values of $t \in [-1, 1]$, the coefficients c_{k-1} being assumed known. A convenient method for doing this is based on a minor adaptation of Clenshaw's algorithm [1??]. Now for $n \geq 1$

$$(2.29) \qquad T_{n+1}(t) - 2tT_n(t) + T_{n-1}(t) = 0$$

with $T_0(t) = 1$ and $T_1(t) = 1$. For $n \geq \lambda$ we have, from (2.24) and (2.15) that
(2.30)

$$u_{n+1}(t) - 2tu_n(t) + u_{n-1}(t) = 2\text{p.p.} \left((z-t)X^{-1}(z)T_{n-\kappa}(z); t \right)$$
$$= \frac{1}{\pi i} \int_{C_R} X^{-1}(z)T_{n-\kappa}(z)dz,$$

where $R > 1$, the right hand side being independent of t. On deforming the contour C_R to either side of the interval $[-1, 1]$ and recalling from (2.2) and (2.8) that

$$(2.31) \qquad (X^{-1})^+ - (X^{-1})^- = 2ib/rZ,$$

we find that for $n \geq \lambda$ the polynomials u_n satisfy the inhomogeneous recurrence relation

$$(2.32) \qquad u_{n+1}(t) - 2tu_n(t) + u_{n-1}(t) = \alpha_n$$

where

$$(2.33) \qquad \alpha_n = -\frac{2}{\pi} \int_{-1}^{1} \frac{b(\tau)T_{n-\kappa}(\tau)}{r(\tau)Z(\tau)} \, d\tau.$$

Assume that the coefficients α_n can be evaluated either analytically or, more generally, by quadrature to any desired accuracy. In order to evaluate $s_n(t)$ for a given value of $t \in [-1, 1]$ construct the sequence $\{b_k\}$ for $k = (n+2)(-1)\lambda$ such that

$$(2.34) \qquad \begin{cases} b_{n+2} = b_{n+1} = 0 & \text{and} \\ b_k - 2tb_{k+1} + b_{k+2} = c_{k-1}. \end{cases}$$

Then it is readily shown that

$$s_n(t) = b_\lambda u_{\lambda-1}(t) + b_{\lambda+1}(u_\lambda(t) - 2t u_{\lambda-1}(t))$$

(2.35)
$$+ \sum_{k=\lambda+2}^{n} b_k \alpha_{k-2}.$$

Note that in (2.24) the integer $(n - \kappa)$ was introduced and this has appeared subsequently. It will appear increasingly when we discretize (1.1) in various ways. We shall henceforth write

(2.36)
$$m = n - \kappa,$$

and always assume that m and n are non-negative integers.

3. A Galerkin method.

3.1 The dominant equation. Consider now an approximate method for solving the dominant equation for ψ, that is

(3.1.1)
$$A\psi = f.$$

The two cases of $\kappa \geq 1$ and $\kappa \leq 0$ will be considered separately. First suppose that $\kappa \geq 1$. Recalling (2.22) and also, from (2.24), that the polynomials u_n are defined only for $n \geq \kappa$, we shall look for an approximate solution of (3.1.1) of the form

(3.1.2)
$$\psi_n(t) = \sum_{j=1}^{\kappa} a_j T_{j-1}(t) + \sum_{j=\kappa+1}^{n} a_j u_{j-1}(t).$$

If the residual r_n is defined by

(3.1.3)
$$r_n = A\psi_n - f$$

then on substituting (3.1.2) into (3.1.3) we find that

(3.1.4)
$$r_n = \sum_{j=\kappa+1}^{n} a_j T_{j-1-\kappa} - f.$$

For the Galerkin method, in order to determine the m coefficients $\{a_{\kappa+1}, \ldots, a_n\}$ impose the m conditions that

$$(3.1.5) \qquad \int_{-1}^{1} (1-\tau^2)^{-1/2} r_n(\tau) T_{i-1}(\tau) \, d\tau = 0, \qquad i = 1(1)m.$$

If we define, for $i \in \mathbf{N}$,

$$(3.1.6) \qquad f_{i-1} = (1/h_{i-1}) \int_{-1}^{1} (1-\tau^2)^{-1/2} f(\tau) T_{i-1}(\tau) \, d\tau$$

where

$$(3.1.7) \qquad h_{i-1} = 2/(1 + \delta_{i-1,0})\pi$$

then on substituting (3.1.4) into (3.1.5) we find that

$$(3.1.8) \qquad a_i = f_{i-\kappa-1}, \qquad i = (\kappa+1)(1)n.$$

Thus from (3.1.2) and (3.1.8) we have

$$(3.1.9) \qquad \psi_n(t) = \sum_{j=\kappa+1}^{n} f_{j-\kappa-1} u_{j-1}(t) + \sum_{j=1}^{\kappa} a_j T_{j-1}(t)$$

where the coefficients $a_1, a_2, \ldots, a_\kappa$ remain to be determined. But we know that when $\kappa \geq 1, \psi$ is not determined uniquely unless a further κ conditions are imposed upon it. Suppose that these same conditions are imposed upon ψ_n in order to determine the coefficients $a_j, j = 1(1)\kappa$.

So much for the case when $\kappa \geq 1$. Assume that $\kappa \leq 0$. To find an approximate solution of (3.1.1) now write

$$(3.1.10) \qquad \psi_n(t) = \sum_{j=1}^{n} a_j u_{j-1}(t).$$

Defining the residual as before by (3.1.3) then on substituting (3.1.10) into (3.1.3) and recalling (2.25) we find

$$(3.1.11) \qquad r_n(t) = \sum_{j=1}^{n} a_j \left\{ T_{j-1-\kappa}(t) + \sum_{\ell=1}^{-\kappa} \sigma_{\ell,j-1} T_{\ell-1}(t) \right\} - f(t).$$

Again apply the m conditions as given by (3.1.5). For $i = (-\kappa+1)(1)m$ we obtain

(3.1.12)
$$\sum_{j=1}^{n} a_j \int_{-1}^{1} (1 - \tau^2)^{-1/2} T_{i-1}(\tau) T_{j-1-\kappa}(\tau) \, d\tau$$
$$= \int_{-1}^{1} (1 - \tau^2)^{-1/2} T_{i-1}(\tau) f(\tau) \, d\tau,$$

from which is obtained

(3.1.13) $a_i = f_{i-\kappa-1}$ for $i = 1(1)n$.

Thus we obtain all the unknown coefficients in (3.1.10) and hence ψ_n. What of the remaining equations? For $i = 1(1)(-\kappa)$ we obtain from (3.1.5) and (3.1.11) that

(3.1.14)
$$\sum_{j=1}^{n} a_j \sum_{\ell=1}^{-\kappa} \sigma_{\ell,j-1} \int_{-1}^{1} (1 - \tau^2)^{-1/2} T_{i-1}(\tau) T_{\ell-1}(\tau) \, d\tau$$
$$= \int_{-1}^{1} (1 - \tau^2)^{-1/2} T_{i-1}(\tau) f(\tau) \, d\tau,$$

from which the orthogonality of the Chebyshev polynomials gives

(3.1.15) $\displaystyle\sum_{j=1}^{n} \sigma_{i,j-1} a_j = f_{i-1}$, $i = 1(1)(-\kappa)$.

Combining this equation with (3.1.13) the equations can be written in partitioned matrix form as

(3.1.16)
$$\left[\frac{\sum_{-\kappa}}{\mathbf{I}_n} \right] \mathbf{x}_n = \mathbf{y}_m,$$

where

(3.1.17)
$$\mathbf{x}_n^T = (a_1, a_2, \ldots, a_n)$$

and

(3.1.18)
$$\mathbf{y}_m^T = (f_0, f_1, \ldots, f_{m-1}).$$

The matrix $\boldsymbol{\Sigma}_{-\kappa}$ is a $(-\kappa) \times n$ matrix such that

$$(3.1.19) \qquad \boldsymbol{\Sigma}_{-\kappa} = (\sigma_{i,j-1}), \quad i = 1(1)(-\kappa), \quad j = 1(1)n.$$

From the theory of linear equations (3.1.16) possesses a solution if $\mathbf{y}_m$ is orthogonal to the null space of the matrix.

$$(3.1.20) \qquad\qquad \left[\boldsymbol{\Sigma}_{-\kappa}^T | \mathbf{I}_n\right],$$

this being an $n \times m$ matrix. Suppose vectors $\mathbf{u}_i, i = 1(1)(-\kappa)$ are defined by

$$(3.1.21) \qquad \mathbf{u}_i^T = (0, \ldots, 1, \ldots 0, -\sigma_{i,0}, -\sigma_{i,1}, \ldots, -\sigma_{i,n-1}),$$

where the 1 appears in the ith place. It is readily verified that the $(-\kappa)$ vectors (3.1.21) span the null space of the matrix (3.1.20). Consequently (3.1.16) possesses a solution if

$$(3.1.22) \qquad \mathbf{u}_i^T \mathbf{y}_m = 0 \qquad \text{for} \qquad i = 1(1)(-\kappa).$$

In other words we require

$$(3.1.23) \qquad f_{i-1} = \sum_{j=-\kappa+1}^{m} f_{j-1}\sigma_{i,j-1+\kappa} \qquad \text{for } i = 1(1)(-\kappa).$$

It is of interest to derive the consistency conditions for the original equation (3.1.1). Suppose

$$(3.1.24) \qquad\qquad f(t) = \sum_{j=1}^{\infty} f_{j-1}T_{j-1}(t),$$

the series converging absolutely and uniformly for all $t \in [-1, 1]$ then, from (2.12), we must have

$$(3.1.25) \qquad\qquad \sum_{j=1}^{\infty} f_{j-1}\left(\int_{-1}^{1} \frac{\tau^{i-1}b(\tau)T_{j-1}(\tau)}{r(\tau)Z(\tau)} \, d\tau\right) = 0,$$

for $i = 1(1)(-\kappa)$, if the dominant equation $A\psi = f$ is to possess a solution. On splitting the sum into two sums with $j = 1(1)(-\kappa)$ and $j = (-\kappa + 1)(1)\infty$ and using (2.27) we find

(3.1.26)

$$\sum_{j=1}^{-\kappa} \left(\int_{-1}^{1} \frac{\tau^{i-1} b(\tau) T_{j-1}(\tau)}{r(\tau) Z(\tau)} \, d\tau \right) \left\{ f_{j-1} - \sum_{\ell=-\kappa+1}^{\infty} f_{\ell-1} \sigma_{j,\ell-1+\kappa} \right\} = 0.$$

In order that this be satisfied we shall impose the $(-\kappa)$ conditions

$$(3.1.27) \qquad f_{j-1} = \sum_{\ell=-\kappa+1}^{\infty} f_{\ell-1} \sigma_{j,\ell-1+\kappa}, \qquad j = 1(1)(-\kappa).$$

On comparing this equation with (3.1.23) we see that the latter is a truncated version of (3.1.27) where essentially we put $f_{m+\ell} = 0$ for $\ell \in \mathbf{N}$.

This completes the discussion of this particular Galerkin method for the dominant equation. This method is now extended to the complete equation.

3.2 The complete equation. In terms of ψ the complete equation will be written as

$$(3.2.1) \qquad\qquad A\psi + \mathcal{K}\psi = f$$

where

$$(3.2.2) \qquad \mathcal{K}\psi(t) = \int_{-1}^{1} (Z(\tau)/r(\tau)) k(t, \tau) \psi(\tau) \, d\tau.$$

To find approximate solutions to (3.2.1), proceed as before by considering separately the two cases of $\kappa \geq 1$ and $\kappa \leq 0$.

Suppose first that $\kappa \geq 1$. Choose an approximate solution ψ_n as given by (3.1.2). The residual r_n is now defined as

$$(3.2.3) \qquad\qquad r_n = A\psi_n + \mathcal{K}\psi_n - f.$$

On substituting (3.1.2) into (3.2.3) and using (2.25), noting that the sum is now empty, we obtain

$$r_n(t) = \sum_{j=\kappa+1}^{n} a_j T_{j-1-\kappa}(t) + \sum_{j=1}^{\kappa} a_j \int_{-1}^{1} (Z(\tau)/r(\tau))k(t,\tau)T_{j-1}(\tau)\,d\tau$$

$$(3.2.4) \quad + \sum_{j=\kappa+1}^{n} a_j \int_{-1}^{1} (Z(\tau)/r(\tau))k(t,\tau)u_{j-1}(\tau)\,d\tau - f(t).$$

Now impose the m conditions, as before, that

$$(3.2.5) \qquad \int_{-1}^{1} (1-t^2)^{-1/2} r_n(t)T_{i-1}(t)\,dt = 0, \qquad i = 1(1)m.$$

Writing

$$(3.2.6) \qquad c_{j-1}(\tau) = (1/h_{j-1}) \int_{-1}^{1} (1-t^2)^{-1/2} k(t,\tau)T_{j-1}(t)\,dt$$

then (3.2.4) and (3.2.5) together give a system of m linear algebraic equations in n unknowns which can be written as

$$(3.2.7) \qquad\qquad (\mathbf{A}_n^G + \mathbf{K}_n^G)\mathbf{x}_n = \mathbf{y}_m.$$

The vectors $\mathbf{x}_n$ and $\mathbf{y}_m$ are as defined by (3.1.17) and (3.1.18) respectively. The matrix $\mathbf{A}_n^G$ is an $m \times n$ matrix which can be written in partitioned form as

$$(3.2.8) \qquad\qquad \mathbf{A}_n^G = (\mathbf{O}_\kappa | \mathbf{I}_m),$$

where $\mathbf{O}_\kappa$ denotes the null matrix of order $m \times \kappa$ and $\mathbf{I}_m$ is the identity matrix of order m. The matrix $\mathbf{K}_n^G$ can also be written in partitioned form as

$$(3.2.9) \qquad\qquad \mathbf{K}_n^G = \left(\mathbf{K}_{n,1}^G | \mathbf{K}_{n,2}^G \right)$$

where, if $\mathbf{K}_{n,1}^G = \left(k_{i,j}^{(1)} \right)$ then

$$(3.2.10) \qquad\qquad k_{i,j}^{(1)} = \int_{-1}^{1} (Z(\tau)/r(\tau))c_{i-1}(\tau)T_{j-1}(\tau)\,d\tau,$$

for $i = 1(1)m$ and $j = 1(1)\kappa$. Again, if $\mathbf{K}^G_{n,2} = \left(k^{(2)}_{i,j} \right)$ then

$$(3.2.11) \qquad k^{(2)}_{i,j} = \int_{-1}^{1} (Z(\tau)/r(\tau))c_{i-1}(\tau)u_{j-1}(\tau)\, d\tau,$$

for $i = 1(1)m$ and $j = (\kappa + 1)(1)n$.

Before proceeding to discuss the solution of (3.2.7) consider briefly the integrals appearing in (3.2.10) and (3.2.11). If it is assumed that the functions c_{i-1} are well behaved then the integrals will in general be improper integrals due to the singularities of the fundamental function Z at the end points of the interval of integration. In the case of the integrals for the elements $k^{(2)}_{i,j}$, the polynomials u_{j-1}, for a given value of τ, may be evaluated from the recurrence relation (2.32) which itself requires the values of the improper integrals α_n as defined by (2.33). It is suggested that adaptive quadrature rules may be used to evaluate all these integrals to any prescribed accuracy. The evaluation of the integrals $k^{(1)}_{i,j}$ may be done similarly.

Returning now to (3.2.7) partition the vector $\mathbf{x}_n$ so that

$$(3.2.12) \qquad \mathbf{x}_n^T = \left(\mathbf{x}_n^{(1)T} | \mathbf{x}_n^{(2)T} \right) = (a_1, \dots, a_\kappa | a_{\kappa+1}, \dots, a_n).$$

Then rewrite (3.2.7) as

$$(3.2.13) \qquad (\mathbf{O}_\kappa | \mathbf{I}_m) \begin{pmatrix} \mathbf{x}_n^{(1)} \\ \mathbf{x}_n^{(2)} \end{pmatrix} + \left((\mathbf{K}^G_{n,1} | \mathbf{K}^G_{n,2}) \right) \begin{pmatrix} \mathbf{x}_n^{(1)} \\ \mathbf{x}_n^{(2)} \end{pmatrix} = \mathbf{y}_m$$

so that on multiplication we obtain

$$(3.2.14) \qquad \left(\mathbf{I}_m + \mathbf{K}^G_{n,2} \right) \mathbf{x}_n^{(2)} = \mathbf{y}_m - \mathbf{K}^G_{n,1} \mathbf{x}_n^{(1)}.$$

Now as has been seen in the previous section, the elements of $\mathbf{x}_n^{(1)}$ are arbitrary and can only be determined when κ additional conditions are given. As is shown later (see §5) for n (or m) large enough $(\mathbf{I}_m + \mathbf{K}^G_{n,2})^I$ exists and then (3.2.14) can be solved.

Suppose now that $\kappa \leq 0$. With ψ_n as defined by (3.1.10) and r_n as defined by (3.2.3), if we again impose the m conditions given by (3.2.5), and recalling (2.25), a system of m linear algebraic equations is obtained in n unknowns which is written as

$$(3.2.15) \qquad \left(\frac{\boldsymbol{\Sigma}_{-\kappa}}{\mathbf{I}_n}\right)\mathbf{x}_n + \left(\frac{\mathbf{K}^G_{n,1}}{\mathbf{K}^G_{n,2}}\right)\mathbf{x}_n = \mathbf{y}_m.$$

The vectors $\mathbf{x}_n$ and $\mathbf{y}_m$ are as defined by (3.1.17) and (3.1.18) respectively. The matrix $\boldsymbol{\Sigma}_{-\kappa}$ is defined by (3.1.19) and $\mathbf{I}_n$ is the identity matrix of order n. If we write

$$(3.2.16) \qquad \mathbf{K}^G_n = \left(\frac{\mathbf{K}^G_{n,1}}{\mathbf{K}^G_{n,2}}\right) = (k_{i,j})$$

then we have

$$(3.2.17) \qquad k_{i,j} = \int_{-1}^{1} (Z(\tau)/r(\tau))c_{i-1}(\tau)u_{j-1}(\tau)\,d\tau$$

for $i = 1(1)m, j = 1(1)n$ where c_{i-1} is defined in (3.2.6). To solve (3.2.15) observe that the last n equations give

$$(3.2.18) \qquad (\mathbf{I}_n + \mathbf{K}^G_{n,2})\mathbf{x}_n = \mathbf{y}^{(2)}_m$$

where

$$(3.2.19) \qquad \mathbf{y}^{(2)T}_m = (f_{-\kappa}, f_{-\kappa+1}, \ldots, f_{m-1}).$$

For n large enough $(\mathbf{I}_n + \mathbf{K}^G_{n,2})^I$ exists; see §5. However, there still remain $-\kappa$ conditions to be satisfied arising from the first $(-\kappa)$ equations of (3.2.15). That is, we must have

$$(3.2.20) \qquad \boldsymbol{\Sigma}_{-\kappa}\mathbf{x}_n + \mathbf{K}^G_{n,1}\mathbf{x}_n = \mathbf{y}^{(1)}_m,$$

say where

$$(3.2.21) \qquad \mathbf{y}^{(1)T}_m = (f_0, f_1, \ldots, f_{-\kappa-1}).$$

If we take the second term on the left of (3.2.20) to the right hand side and assume, for the moment, that it is known, then we can apply

the same arguments as were used for the dominant equation in §3.1. Consequently define, for $i = 1(1)m$,

$$(3.2.22) \qquad g_{i-1} = f_{i-1} - \sum_{j=1}^{n} k_{i,j} a_j$$

then from (3.1.23) the following $(-\kappa)$ conditions must be satisfied

$$(3.2.23) \qquad g_{i-1} = \sum_{j=-\kappa+1}^{m} g_{j-1} \sigma_{i,j-1+\kappa}.$$

These may be verified *a posteriori*, once $\mathbf{x}_n$ has been determined from (3.2.18). Note as before, that these consistency conditions are a discretised version of those for the complete equation which, in terms of the dependent variable ψ, are given by

$$(3.2.24) \qquad \int_{-1}^{1} \frac{\tau^{i-1} b(\tau)}{r(\tau) Z(\tau)} \{f(\tau) - (\mathcal{K}\psi)(\tau)\} \, d\tau = 0.$$

for $i = 1(1)(-\kappa)$.

This completes the discussion of the Galerkin method. As has been seen the structure of the matrix arising from the operator A is particularly simple in this case. However, this simplicity is obtained at a cost of evaluating many integrals and, in particular, the double integrals which arise in the coefficients $k_{i,j}$ of the matrix $\mathbf{K}_n^G$ (see (3.2.6) and (3.2.17). Inevitably these integrals will have to be evaluated approximately by some quadrature rule. This could be avoided by replacing (3.2.5) by m different conditions. In particular one might consider a discrete Galerkin method where (3.2.5) is replaced by

$$(3.2.25) \qquad \sum_{j=1}^{m} r_n(\xi_{j,m}) T_{i-1}(\xi_{j,m}) = 0,$$

for $i = 1(1)m$ where $\xi_{j,m}$ are the m zeros of T_m. We shall not pursue this any further in this paper but will turn our attentions now to a collocation method.

4. A collocation method. The same pattern shall be followed as in the previous section for the Galerkin method. First we shall consider

the dominant equation and then, in §4.2, the complete equation. In each case we distinguish between two cases but now we choose them to be $\kappa \geq 0$ and $\kappa \leq -1$. The initial developments for this method are the same as for the Galerkin method and we shall be referring frequently to equations from the preceding section.

4.1. The dominant equation. Suppose first that $\kappa \geq 0$. With ψ_n and r_n as defined by(3.1.2) and (3.1.3), respectively, we obtain (3.1.4). In order to determine the m coefficients $\{a_{\kappa+1}, \ldots, a_n\}$ now impose the m conditions

$$(4.1.1) \qquad r_n(\xi_{i,m}) = 0, \quad i = 1(1)m,$$

where the points $\xi_{i,m}$ are the m zeros of T_m, that is

$$(4.1.2) \qquad \xi_{i,m} = \cos[(2i - 1)\pi/(2m)], \quad i = 1(1)m.$$

Consequently a system of m equations in m unknowns is obtained and written as

$$(4.1.3) \qquad \mathbf{A}_m \mathbf{x}_m = \mathbf{y}_m$$

where

$$(4.1.4) \qquad \mathbf{x}_m^T = (a_{\kappa+1}, \ldots, a_n)$$

and

$$(4.1.5) \qquad \mathbf{y}_m^T = (f(\xi_{1,m}), \ldots, f(\xi_{m,m})).$$

If we write $\mathbf{A}_m = (a_{i,j})$ then

$$(4.1.6) \qquad a_{i,j} = T_{j-1}(\xi_{i,m}) \quad \text{for} \ \ i, j = 1(1)m.$$

From the orthogonality property of the Chebyshev polynomials with respect to summation over the zeros of T_m i.e.

$$(4.1.7) \qquad \sum_{i=1}^{m} T_{j-1}(\xi_{i,m}) T_{k-1}(\xi_{i,m}) = (m/2)(1 + \delta_{1,j})\delta_{j,k}$$

for $j, k = 1(1)m$ (see [9, Ex. 1.5.26]) we can write down explicitly the inverse of $\mathbf{A}_m$. If $\mathbf{A}_m^I = (a_{i,j}^I)$ then

(4.1.8) $\qquad a_{i,j}^I = (2/m)T_{i-1}(\xi_{j,m})/(1 + \delta_{i,1}) \quad$ for $\;i, j = 1(1)m$.

Consequently from (4.1.3) $\mathbf{x}_m = \mathbf{A}_m^I \mathbf{y}_m$ from which is obtained

(4.1.9) $\qquad a_i = (2/m(1 + \delta_{i,1})) \displaystyle\sum_{j=1}^{m} T_{i-\kappa-1}(\xi_{j,m}) f(\xi_{j,m})$

for $i = (\kappa + 1)(1)n$. The remaining κ coefficients $\{a_1, a_2, \ldots, a_\kappa\}$ must be determined from κ additional conditions on ψ_n which are assumed to be the same as those given for ψ.

When $\kappa \leq -1$ it is assumed that ψ_n is given by (3.1.10) so that r_n is given by (3.1.11). On imposing the m conditions (4.1.1) we obtain a system of linear algebraic equations which is written as

(4.1.10) $\qquad\qquad (\mathbf{A}_{m,-\kappa} + \mathbf{B}_m \boldsymbol{\Sigma}_{-\kappa}) \mathbf{x}_n = \mathbf{y}_m,$

where

(4.1.11) $\qquad\qquad\qquad \mathbf{x}_n^T = (a_1, a_2, \ldots, a_n)$

and $\mathbf{y}_m$ is defined in (4.1.5). If $\mathbf{A}_{m,-\kappa} = (a_{i,j})$ then this is an $m \times n$ matrix where

(4.1.12) $\qquad a_{i,j} = T_{j-1-\kappa}(\xi_{i,m}), \quad i = 1(1)m, \; j = 1(1)n.$

Again, $\mathbf{B}_m = (b_{i,j})$ is an $m \times (-\kappa)$ matrix where

(4.1.13) $\qquad b_{i,j} = T_{j-1}(\xi_{i,m}), \quad i = 1(1)m, \; j = 1(1)(-\kappa).$

Finally, the matrix $\boldsymbol{\Sigma}_{-\kappa}$ is a $(-\kappa) \times n$ matrix which has previously been defined in (3.1.19).

In order to solve (4.1.10) we first write it as

(4.1.14) $\qquad\qquad \mathbf{A}_{m,-\kappa} \mathbf{x}_n = \mathbf{y}_m - \mathbf{B}_m \boldsymbol{\Sigma}_{-\kappa} \mathbf{x}_n$

and assume initially that the right hand side is known. In order that $\mathbf{x}_n$ exists, the right hand side must be orthogonal to the null space of

$\mathbf{A}_{m,-\kappa}^T$. Again, from the orthogonality of the Chebyshev polynomials with respect to summation (see (4.1.7)), we can verify that there are $(-\kappa)$ linearly independent vectors $\mathbf{v}_j$ such that $\mathbf{A}_{m,-\kappa}^T \mathbf{v}_j = \mathbf{0}$ given by

$$(4.1.15) \qquad \mathbf{v}_j^T = (T_{j-1}(\xi_{1,m}), \dots, T_{j-1}(\xi_{m,m}))$$

for $j = 1(1)(-\kappa)$. Again from (4.1.7) it is shown that

$$(4.1.16) \qquad (2/m)\mathbf{A}_{m,-\kappa}^T \mathbf{A}_{m,-\kappa} = \mathbf{I}_n$$

and

$$(4.1.17) \qquad \mathbf{A}_{m,-\kappa}^T \mathbf{B}_m = \mathbf{0}.$$

Consequently from (4.1.14)

$$(4.1.18) \qquad \mathbf{x}_n = (2/m)\mathbf{A}_{m,-\kappa}^T \mathbf{y}_m$$

provided that the $(-\kappa)$ consistency conditions on $\mathbf{y}_m$ given by

$$(4.1.19) \qquad \mathbf{v}_j^T(\mathbf{y}_m - (2/m)\mathbf{B}_m \boldsymbol{\Sigma}_{-\kappa}\mathbf{A}_{m,-\kappa}^T \mathbf{y}_m) = 0$$

for $j = 1(1)(-\kappa)$ are satisfied. Returning to (4.1.18) we see that provided $\mathbf{y}_m$ satisfies the $(-\kappa)$ conditions given by (4.1.19) then

$$(4.1.20) \qquad a_i = (2/m)\sum_{j=1}^{m} T_{i-1-\kappa}(\xi_{j,m})f(\xi_{j,m})$$

for $i = 1(1)n$. This completes the discussion of the dominant equation so we now turn our attention to the collocation method as applied to the complete equation.

4.2. The complete equation. Consider separately the two cases of $\kappa \geq 0$ and $\kappa \leq -1$. When $\kappa \geq 0$ assume that ψ_n is given by (3.1.2) then with r_n as defined by (3.2.3) we find on imposing the m conditions (4.1.1) that a system of m equations in n unknowns is obtained which we write as

$$(4.2.1) \qquad (\mathbf{A}_m^C + \mathbf{K}_m^C)\mathbf{x}_n = \mathbf{y}_m$$

where now

$$(4.2.2) \qquad \mathbf{x}_n^T = (a_1, a_2, \ldots, a_n)$$

and $\mathbf{y}_m$ is as defined in (4.1.5). The $m \times n$ matrix $\mathbf{A}_m^C$ can be written in partitioned form as

$$(4.2.3) \qquad \mathbf{A}_m^C = (\mathbf{O}_\kappa | \mathbf{A}_m)$$

where $\mathbf{O}_\kappa$ denotes the $m \times \kappa$ null matrix and the $m \times m$ matrix $\mathbf{A}_m$ is defined in (4.1.6). The $m \times n$ matrix $\mathbf{K}_m^C$ is written in partitioned form as

$$(4.2.4) \qquad \mathbf{K}_m^C = (\mathbf{K}_{m,1}^C | \mathbf{K}_{m,2}^C).$$

The matrix $\mathbf{K}_{m,1}^C$ is an $m \times \kappa$ matrix where, if $\mathbf{K}_{m,1}^C = (k_{i,j}^{(1)})$, then

$$(4.2.5) \qquad k_{i,j}^{(1)} = \int_{-1}^{1} (Z(\tau)/r(\tau)) k(\xi_{i,m}, \tau) T_{j-1}(\tau) \, d\tau$$

for $i = 1(1)m$ and $j = 1(1)\kappa$. The matrix $\mathbf{K}_{m,2}^C$ is a square matrix of order m where, if $\mathbf{K}_{m,2}^C = (k_{i,j}^{(2)})$, then

$$(4.2.6) \qquad k_{i,j}^{(2)} = \int_{-1}^{1} (Z(\tau)/r(\tau)) k(\xi_{i,m}, \tau) u_{j-1}(\tau) \, d\tau$$

for $i = 1(1)m$ and $j = (\kappa+1)(1)n$. On partitioning the vector $\mathbf{x}_n$ as in (3.2.12) we find on multiplying out the partitioned matrices that

$$(4.2.7) \qquad (\mathbf{A}_m + \mathbf{K}_{m,2}^C)\mathbf{x}_n^{(2)} = \mathbf{y}_m - \mathbf{K}_{m,1}^C \mathbf{x}_n^{(1)}.$$

Again note that the elements $a_1, a_2, \ldots, a_\kappa$ of $\mathbf{x}_n^{(1)}$ are arbitrary and can only be determined when additional κ conditions are imposed upon ψ_n. Since there is an explicit representation for $\mathbf{A}_m$, see (4.1.8), we have from (4.2.7) that

$$(4.2.8) \qquad (\mathbf{I}_m + \mathbf{A}_m^I \mathbf{K}_{m,2}^C)\mathbf{x}_n^{(2)} = \mathbf{A}_m^I \mathbf{y}_m - \mathbf{A}_m^I \mathbf{K}_{m,1}^C \mathbf{x}_n^{(1)}.$$

These equations may be solved for $\mathbf{x}_n^{(2)}$ in terms of the unknown elements of $\mathbf{x}_n^{(1)}$.

Finally, suppose that $\kappa \leq -1$. With ψ_n as given by (3.1.10) and r_n as defined by (3.2.3) then, on imposing the m conditions (4.1.1) and using (2.25) we obtain a system of m linear algebraic equations in n unknowns given by

$$(4.2.9) \qquad (\mathbf{A}_{m,-\kappa} + \mathbf{B}_m \boldsymbol{\Sigma}_{-\kappa} + \mathbf{K}_m^C)\mathbf{x}_n = \mathbf{y}_m.$$

The vectors $\mathbf{x}_n$ and $\mathbf{y}_m$ are as defined in (4.1.11) and (4.1.5), respectively. The matrices $\mathbf{A}_{m,-\kappa}, \mathbf{B}_m$ and $\boldsymbol{\Sigma}_{-\kappa}$ have been defined in (4.1.12), (4.1.13) and (3.1.19), respectively. The matrix $\mathbf{K}_m^C = (k_{i,j})$ is an $m \times n$ matrix where

$$(4.2.10) \qquad k_{i,j} = \int_{-1}^{1} (Z(\tau)/r(\tau))k(\xi_{i,m}, \tau)u_{j-1}(\tau)\, d\tau$$

for $i = 1(1)m$ and $j = 1(1)n$. To solve (4.2.9) proceed as has been done for the dominant equation. Pre-multiplying (4.2.9) by $(2/m)\mathbf{A}_{m,-\kappa}^T$ we obtain

$$(4.2.11) \qquad \left(\mathbf{I}_n + (2/m)\mathbf{A}_{m,-\kappa}^T \mathbf{K}_m^C\right)\mathbf{x}_n = (2/m)\mathbf{A}_{m,-\kappa}^T \mathbf{y}_m$$

which gives us n equations for n unknowns. The solution $\mathbf{x}_n$ of (4.2.11) exists provided that the $(-\kappa)$ consistency conditions given by

$$(4.2.12) \qquad \mathbf{v}_j^T(\mathbf{y}_m - (\mathbf{B}_m \boldsymbol{\Sigma}_{-\kappa} + \mathbf{K}_m^C)\mathbf{x}_n) = 0,$$

for $j = 1(1)(-\kappa)$ are satisfied, where the vectors $\mathbf{v}_j$ are defined by (4.1.15). This condition may be checked *a posteriori* once the solution $\mathbf{x}_n$ has been obtained from (4.2.11).

This concludes the discussion of the discretization of the complete equation for the collocation method, for all values of the index κ. As in the Galerkin method, we evaluate only integrals which are improper. The slightly greater complexity of the algebraic equations in this case compared with those in the Galerkin method is offset by the need to evaluate only single integrals and not double integrals. Only in the evaluation of the fundamental function Z do we need to evaluate a Cauchy principal value integral. It now remains to consider the

convergence of these approximate methods and this will be done in the next section.

5.   Convergence of the algorithms.   For both the methods described in §3 and §4 the approximate solution ψ_n satisfies an integral equation of the form

$$(5.1) \qquad\qquad A\psi_n + \mathcal{K}_m\psi_n = \boldsymbol{\Phi}_m.$$

In the Galerkin method of §3 both $\mathcal{K}_m$ and $\boldsymbol{\Phi}_m$ are obtained from truncated Chebyshev series expansions whereas for the collocation method of §4 they are obtained from Lagrange interpolation at the zeros of T_m. Compare the solution of (5.1) with that of

$$(5.2) \qquad\qquad A\psi + \mathcal{K}\psi = f.$$

When $\kappa > 0$ assume that both ψ and ψ_n satisfy the same κ additional conditions which have to be imposed for uniqueness. Define

$$(5.3) \qquad\qquad \varepsilon_n = \psi - \psi_n,$$

then from (5.1) and (5.2) we have, on subtraction, that

$$(5.4) \qquad\quad A\varepsilon_n + \mathcal{K}_m\varepsilon_n = (f - \boldsymbol{\Phi}_m) + (\mathcal{K} - \mathcal{K}_m)\psi.$$

On regularizing this equation with the operator $\hat{A}^I$ we have that ε_n satisfies

$$(5.5) \qquad (I + \hat{A}^I\mathcal{K}_m)\varepsilon_n = \hat{\mathbf{A}}^i(f - \boldsymbol{\Phi}_m) + \hat{A}^I(\mathcal{K} - \mathcal{K}_m)\psi.$$

Assume that $A, \mathcal{K}$ and $\mathcal{K}_m$ are linear operators on a normed space X into a normed space Y, with $\mathcal{K}$ and $\mathcal{K}_m$ being compact. Furthermore, assume that $\hat{A}^I$ is a bounded linear operator on Y into X. Consequently $\hat{A}^I\mathcal{K}_m$ will be compact so that (5.5) is a Fredholm integral equation. Later we shall choose appropriate spaces X and Y. We assume that $(I + \hat{A}^I\mathcal{K})^I$ exists and is bounded on X into itself and we first consider the existence of $(I + \hat{A}^I\mathcal{K}_m)^I$. Now

$$(5.6) \qquad I + \hat{A}^I\mathcal{K}_m = (I + \hat{A}^I\mathcal{K})\{I + (I + \hat{A}^I\mathcal{K})^I\hat{A}^I(\mathcal{K}_m - \mathcal{K})\}$$

If we write the matrix in { } as $I + B_m$ and if $||B_m|| < 1$ for $m > m_0$ then it is well known that $(I + B_m)^I$ exists for $m > m_0$ and furthermore that $||(I + B_m)^I|| < 1/(1 - ||B_m||)$. Consequently, if

$$(5.7) \qquad ||(I + \hat{A}^I \mathcal{K})^I \hat{A}^I (\mathcal{K}_m - \mathcal{K})|| < 1 \quad \text{for} \quad m > m_0$$

then $(I + \hat{A}^I \mathcal{K}_m)^I$ exists and there exists $c > 0$, independent of m such that

$$(5.8) \qquad ||(I + \hat{A}^I \mathcal{K}_m)^I|| \leq c \,\forall\, m > m_0.$$

Now (5.7) will be satisfied if $||\mathcal{K}_m - \mathcal{K}|| \to 0$ as $m \to \infty$. From (5.5) we have that

$$(5.9) \qquad ||\varepsilon_n|| \leq c\{|\hat{A}^I(f - \boldsymbol{\Phi}_m)|| + ||\hat{A}^I(\mathcal{K} - \mathcal{K}_m)\psi||\},$$

provided $m > m_0$.

Now consider appropriate spaces X and Y. Choose Y to be the Hölder space of order ν. If $g \in Y$ then we define $||g||_Y$ by

$$(5.10) \qquad ||g||_Y = \max\left\{||g||_\infty, \; \sup_{t_1 \neq t_2} \frac{|g(t_1) - g(t_2)|}{|t_1 - t_2|^\nu}\right\}.$$

Here $||g||_\infty = \max_{-1 \leq t \leq 1} |g(t)|$ as usual, and in the supremum we assume that t_1, t_2 take all values in $[-1, 1]$. For X choose the space of continuous functions on $[-1, 1]$ with uniform norm so that $X = C[-1, 1]$. It can be shown (Appendix, Theorem A2) that $\hat{A}^I$ is a bounded linear operator on $H_\nu[-1, 1]$ into $C[-1, 1]$. Furthermore, suppose that $f^{(r)} \in H_\nu[-1, 1]$; that is, the rth order derivative of f is Hölder continuous of order ν on $[-1, 1]$ where $r = 0$ corresponds to the function itself. Recall that, originally, we required $\phi \in \mathcal{H}(p_1, p_2)$ and we choose $Z \in \mathcal{H}(p_1, p_2)$ and $1/Z \in \mathcal{H}(q_1, q_2)$ where $1/p_i + 1/q_i = 1$ for $i = 1, 2$. Suppose $q = \min(q_1, q_2)$. Again it can be shown (see Appendix, Theorem A6) that if $f^{(r)} \in H_\nu[-1, 1]$, where $1/q < \nu \leq 1$, and if $F = \hat{A}^I f$, then $F^{(r)} \in H_{\nu - 1/q}[-1, 1]$.

Now consider $f - \boldsymbol{\Phi}_m$. For both the Galerkin and collocation cases it can be shown that

$$(5.11) \qquad ||f - \boldsymbol{\Phi}_m||_\infty \leq (m - 1)^{-(r+\nu)}(c_1 + c_2 \log(m - 1)).$$

In the former case, when $\boldsymbol{\Phi}_m$ is the truncated Chebyshev series, the result follows from Theorem 3.3 of [9] and the observation that if $f^{(r)} \in H_\nu[-1,1]$ then $E_{m-1}(f) \leq c(m-1)^{-(r+\nu)}$ (see, for example, [8, Theorem 1.5]). For the collocation method when $\boldsymbol{\Phi}_m$ is the Lagrange interpolation polynomial of degree $\leq (m-1)$ to f, the result follows from Theorems 1.1 and 1.2 of [9] together with the aforementioned result for $E_{m-1}(f)$.

Let us now obtain an estimate for $||\hat{A}^I(f - \boldsymbol{\Phi}_m)||_\infty$. If we write

$$(5.12) \qquad\qquad e_m = f - \boldsymbol{\Phi}_m$$

then from (2.9) and (2.19)

$$(5.13) \qquad\qquad \hat{A}^I e_m = e_m \hat{A}^I 1 - I_m(t)$$

where

$$(5.14) \qquad I_m(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{b(\tau)}{r(\tau)Z(\tau)} \left[\frac{e_m(\tau) - e_m(t)}{\tau - t} \right] d\tau.$$

Suppose m is so large that $(t - 1/m^2, t + 1/m^2) \subset (-1,1)$. We shall write

$$(5.15) \qquad I_m(t) = I_m^{(1)}(t) + I_m^{(2)}(t) + I_m^{(3)}(t)$$

where the interval of integration in (5.14) is broken up into the three sub-intervals $(-1, t - 1/m^2)$, $(t - 1/m^2, t + 1/m^2)$ and $(t + 1/m^2, 1)$ respectively. From (5.11)
(5.16)

$$|I_m^{(1)}(t)| \leq 2(c_1 + c_2 \log(m-1))(m-1)^{-(r+\nu)} \int_{-1}^{t-1/m^2} |b/rZ|(t-\tau)^{-1} d\tau.$$

Now $|b/rZ| \in L_q$ where $q = \min(q_1, q_2)$ so that on using Hölder's inequality we find

$$(5.17) \qquad |I_m^{(1)}(t)| \leq (c_1 + c_2 \log(m-1))m^{-(r+\nu-2/q)}.$$

(Note that constants c, c_1, c_2 do not necessarily take the same value from one equation to the next). A similar argument shows that the

same upper bound holds for $|I_m^{(3)}(t)|$. It remains to consider $I_m^{(2)}(t)$. By Kalandiya's Lemma [5], we have

$$(5.18) \quad |e_m(\tau) - e_m(t)| \le (c_1 + c_2 \log(m-1))(m-1)^{-(r+\nu-2\mu)}|\tau - t|^\mu$$

where $0 < \mu < \nu/2$. Substituting this into the integral for $I_m^{(2)}$ and using Hölder's inequality we find, provided $\mu > 1/q$, that

$$(5.19) \qquad |I_m^{(2)}(t)| \le (c_1 + c_2 \log(m-1))(m-1)^{-(r+\nu-2/q)}.$$

Combining (5.17) and (5.19) we have

$$(5.20) \quad ||\hat{A}^I(f - \Phi_m)||_\infty \le (c_1 + c_2 \log(m-1)) \cdot (m-1)^{-(r+\nu-2/q)},$$

provided that $f^{(r)} \in H_\nu[-1,1]$ where $2/q < \nu \le 1$ for $r = 0,1,2,\ldots$.

If we assume that $\partial^r k/\partial t^r$ is uniformly Hölder continuous of order ν for all $\tau \in [-1,1]$ on $[-1,1]$ then a similar estimate for $||\hat{A}^I(\mathcal{K} - \mathcal{K}_m)\psi||_\infty$ holds in both cases. Thus, from (5.9), under the assumed conditions we have

$$(5.21) \qquad ||\varepsilon_n||_\infty \le (c_1 + c_2 \log(m-1)) \cdot (m-1)^{-(r+\nu-2/q)}.$$

This may be considered to be an excellent result. For $\psi^{(r)} \in H_{\nu-1/q}[-1,1]$ and, as a consequence of Jackson's theorem [8], we have

$$(5.22) \qquad\qquad E_n(\psi) \le cn^{-(r+\nu-1/q)}.$$

The rate of convergence of ψ_n to ψ as given by (5.21) is worse than the best possible result for polynomial approximation to ψ by only the factor $n^{1/q} \log n$. Again, as p approaches $1, 1/q$ approaches zero and the factor approaches $\log n$.

Consider mean convergence which, as can be seen below, can be considered under more general conditions than those given above. For this analysis return to the original equations (1.1)-(1.4). Suppose we choose $p_1 = p_2 = p$ and we want $\phi \in \mathcal{H}(p,p), 1 < p < \infty$ where we impose the L_p norm onto this space. The approximate methods of §3 and §4 yield an approximation ψ_n which gives us ϕ_n where $\phi_n = (Z/r)\psi_n$. Thus ϕ_n satisfies the singular integral equation

$$(5.23) \qquad\qquad M\phi_n + K_m\phi_n = \Phi_m$$

where $\boldsymbol{\Phi}_m$ denotes the approximation to f and

$$(5.24) \qquad K_m\phi = \int_{-1}^{1} k_m(t,\tau)\phi(\tau)d\tau$$

where k_m is the similar approximation to k considered as a function of the first variable t. Proceeding as before:

$$(5.25) \qquad (I + \hat{M}^I K_m)(\phi - \phi_n) = \hat{M}^I(f - \boldsymbol{\Phi}_m) + \hat{M}^I(K - K_m)\phi.$$

Consider in more detail the operator $\hat{M}^I$, (see (2.9)). Recall that a, b and r are Hölder continuous on $[-1,1]$ and r never vanishes there. Now Z is such that on $(-1,1)$ we have

$$(5.26) \qquad |Z| = (1-t)^\alpha(1+t)^\beta|\boldsymbol{\Omega}|$$

where $|\boldsymbol{\Omega}|$ does not vanish on $[-1,1]$ and α and β satisfy

$$(5.27) \qquad -1/p < \alpha, \beta < 1/q.$$

By Minkowski's inequality if $f \in L_p$, then from (2.9) we have

$$(5.28) \qquad ||\hat{M}^I f||_p \le c\{||\phi||_p + ||ZT(bf/rZ)||_p\}.$$

By Khevedelidze's Theorem [6] we have, for $1 < p < \infty$,

$$(5.29) \qquad ||ZT(bf/rZ)||_p \le c||f||_p.$$

Consequently, $\hat{M}^I$ is a bounded linear operator on $\mathcal{H}(p,p)$ into itself. Arguing as we did at the commencement of this section, if $\lim_{m\to\infty} ||K - K_m||_p = 0$ then, for large enough m, $(I + \hat{M}^I K_m)^I$ will exist and be uniformly bounded on $\mathcal{H}(p,p)$ into itself. Thus we find from (5.25),

$$(5.30) \qquad ||\phi - \phi_n||_p \le c\{||f - \boldsymbol{\Phi}_m||_p + ||K - K_m||_p\}.$$

To exhibit convergence, consider each term on the right hand side separately. Consider the collocation method only of §4. As we have already observed in this case $\boldsymbol{\Phi}_m = L_{m-1}(f)$, the Lagrange interpolation polynomial of degree $\le (m-1)$ to f based on the zeros of T_m. A well

known theorem of Erdös and Feldheim (see, for example, [**9**, p. 49])
shows that for all $p > 0$ and for all $f \in C[-1, 1]$

$$(5.31) \qquad \lim_{m \to \infty} \int_{-1}^{1} |f(\tau) - L_{m-1}(f; \tau)|^p (1 - \tau^2)^{-1/2} d\tau = 0.$$

Since $1 \le (1 - \tau^2)^{-1/2}$ on $(-1, 1)$ it follows that

$$(5.32) \qquad \qquad \lim_{m \to \infty} ||f - \boldsymbol{\Phi}_m||_p = 0.$$

whenever f is continuous on $[-1, 1]$ and certainly for all $p \in (1, \infty)$.
For the second term of (5.30) an upper bound is given by

$$(5.33) \quad ||K - K_m||_p \le \left[\int_{-1}^{1} dt \left\{ \int_{-1}^{1} |k(t, \tau) - k_m(t, \tau)|^q d\tau \right\}^{p/q} \right]^{1/p},$$

see, for example, [**4**, Theorem 3.4.10]. Again, if k, considered as a
function of t, is continuous on $[-1, 1]$ then as above,

$$(5.34) \qquad \qquad \lim_{m \to \infty} ||K - K_m||_p = 0.$$

Under these conditions we have convergence in the p norm of ϕ_n to ϕ.
It would appear that the conditions on f and k could be generalized so
that they are piecewise continuous on $[-1, 1]$.

   For the Galerkin case a similar argument shows convergence in the 2-
norm when f and k, considered as a function of t, are square integrable.

   6. Conclusion. This analysis of global methods, based on poly-
nomial approximation, for the approximate solution of singular inte-
gral equations has addressed all the problems raised in §1. The use
of Chebyshev polynomials of the first kind in the range space has led
to simple discretizations of the singular integral operator so that the
structure of the solution of the discrete equations has been readily
exhibited. The orthogonality property of the Chebyshev polynomials
with respect to summation provided a key element of the analysis of the
discrete equations in the collocation case. The analysis of convergence
has also been straightforward in both cases and again has made good

use of the key position that Chebyshev polynomials hold among all the orthogonal polynomials.

As remarked elsewhere, the algorithm depends more on the evaluation of improper integrals than of Cauchy principal value integrals. It is only in the evaluation of the fundamental function Z that a Cauchy principal value integral needs to be evaluated either exactly or approximately.

No restriction is now placed on the Hölder continuous functions a and b. These may be complex-valued. Furthermore, the analysis no longer restricts b to being a polynomial which was required in other algorithms. The approximate methods described here now sit comfortably with the theory of singular integral equations, which required no such restrictions. Again note that the methods are valid no matter what the value of the index turns out to be.

Some further work remains to be done. In particular, note two things. The first is that we have not considered any numerical results; these must await another paper. The second is that we have not given any *a posteriori* estimates of error. These should follow quite readily from the analysis in §5 but do perhaps more naturally belong to an implementation paper.

One final point may be mentioned. The algorithm depended, among other things, on the fact that the operators A and $\hat{A}^I$ mapped polynomials into polynomials. A similar analysis readily shows that they also map meromorphic functions into meromorphic functions. This raises the interesting possibility of approximating to f (and k) by rational functions, but this also will have to await a further paper.

Appendix

In this appendix we shall prove two results, which were important in the convergence analysis of §5, concerning properties of the operator $\hat{A}^I$. From (2.9) and (2.19) we have for any $g \in Y$

$$(A1) \qquad \hat{A}^I g = \frac{ag}{rZ} - T\left(\frac{bg}{rZ}\right).$$

If $\hat{A}^I : Y \to X$, then we shall choose X to be the space $C[-1, 1]$ of continuous functions with uniform norm and Y to be the Hölder space $H_\nu[-1, 1]$ of order ν

where if $g \in H_\nu[-1,1]$ then

$$(A2) \qquad ||g||_Y = \max\left\{||g||_\infty, \sup_{t_1 \neq t_2} \frac{|g(t_1) - g(t_2)|}{|t_1 - t_2|^\nu}\right\}.$$

Here the supremum is taken over all distinct points t_1, t_2 of $[-1,1]$ and we shall assume initially that $0 < \nu \leq 1$. Then the norm of $\hat{A}^I$ is defined by

$$(A3) \qquad ||\hat{A}^I|| = \sup_{g \in H_\nu[-1,1]} ||\hat{A}^I g||_\infty ||g||_Y,$$

and we shall first show that $\hat{A}^I$ is a bounded linear operator on Y into X. One further definition is required; if

$$(A4) \qquad w = b/rZ$$

then from the conditions imposed on Z, see (2.5), we have that $w \in \mathcal{H}(q_1, q_2)$.

THEOREM A1. *Given* $w \in \mathcal{H}(q_1, q_2)$, *where* $1 < q_1, q_2 < \infty$, *let* $q = \min(q_1, q_2)$. *Suppose* $g \in H_\nu[-1,1]$ *where* $1/q < \nu \leq 1$. *If*

$$(A5) \qquad G(t) = \int_{-1}^1 w(\tau)\left(\frac{g(\tau) - g(t)}{\tau - t}\right)d\tau, \quad t \in [-1,1],$$

then $G \in H_{\nu - 1/q}[-1,1]$.

PROOF . Choose $h > 0$ and let t_0 and $t_1 = t_0 + h$ be points of $[-1,1]$. We must discuss 3 cases:

(i) if $[t_0 - 2h, t_0 + 2h] \subset (-1,1)$ then shall write $\ell = [t_0 - 2h, t_0 + 2h]$,

(ii) if $t_1 = t_0 + h = 1$ then we shall write $\ell = [t_0 - 2h, 1]$,

(iii) if $t_0 = -1$ then we shall write $\ell = [-1, -1 + 2h]$.

In all three cases $L = (-1,1) \backslash \ell$. Now

$$G(t_1) - G(t_0) = \left[\int_L + \int_\ell\right]\left[w(\tau)\left\{\frac{g(\tau) - g(t_1)}{\tau - t_1} - \frac{g(\tau) - g(t_0)}{\tau - t_0}\right\}\right]d\tau$$

$$= I_1 + I_2.$$

Since $g \in H_\nu[-1,1]$, if A denotes the Hölder constant of g then

$$|I_2| \leq A \int_\ell w(\tau)\{|\tau - t_1|^{\nu - 1} + |\tau - t_0|^{\nu - 1}\}d\tau$$

$$= J_1 + J_2.$$

Again consider these in reverse order. Now for case (i)

$$J_2 = A \int_{t_0-2h}^{t_0+2h} w(\tau)|\tau - t_0|^{\nu-1} d\tau.$$

Since $q = \min(q_1, q_2)$ then w is q-integrable on $(-1, 1)$ so that applying Hölder's inequality to J_2 gives

$$J_2 \leq c \left(\int_{t_0-2h}^{t_0+2h} |\tau - t_0|^{p(\nu-1)} d\tau \right)^{1/p},$$

where $1/p + 1/q = 1$. Evaluating this integral gives

$$J_2 \leq ch^{\nu-1/q}, \quad \text{since} \ \ \nu - 1/q > 0.$$

Note that c denotes a generic constant whose value may be different in different places. Since we can argue similarly for the integral J_1 we find

$$|I_2| \leq ch^{\nu-1/q}.$$

Return to the integral I_1; rewrite this as

$$I_1 = (g(t_0) - g(t_1)) \int_L \frac{w(\tau) d\tau}{\tau - t_0} + h \int_L \frac{w(\tau)(g(\tau) - g(t_1))}{(\tau - t_0)(\tau - t_1)} d\tau,$$

so that

$$|I_1| \leq Ah^\nu \cdot J_3 + J_4.$$

Again, considering these integrals in reverse order

$$|J_4| \leq Ah \int_L \frac{w(\tau)|\tau - t_1|^{\nu-1}}{|\tau - t_0|} d\tau$$

$$= Ah \int_L \frac{w(\tau) d\tau}{|\tau - t_0|^{2-\nu}|1 - h/(\tau - t_0)|^{1-\nu}},$$

on recalling that $t_1 = t_0 + h$. Now on L we have, for case (i),

$$\frac{2}{3} \leq \frac{1}{1 - h/(\tau - t_0)} \leq 2,$$

so that

$$|J_4| \leq ch \left\{ \int_{-1}^{t_0-2h} \frac{w(\tau) d\tau}{(t_0 - \tau)^{2-\nu}} + \int_{t_0+2h}^{1} \frac{w(\tau) d\tau}{(\tau - t_0)^{2-\nu}} \right\}.$$

Applying Hölder's inequality to each of these integrals we find, as before, that

$$|J_4| \leq ch^{\nu - 1/q}.$$

Finally we have, using Hölder's inequality, that

$$|Ah^\nu J_3| \leq ch^\nu \left\{ \left(\int_{-1}^{t_0 - 2h} (t_0 - \tau)^{-p} d\tau \right)^{1/p} \right.$$
$$\left. + \left(\int_{t_0 + 2h}^{1} (\tau - t_0)^{-p} d\tau \right)^{1/p} \right\}$$
$$\leq ch^{\nu - 1/q},$$

after some algebra. Thus $|I_1| \leq ch^{\nu - 1/q}$ and combining this with the bound for $|I_2|$ gives

$$|G(t_1) - G(t_0)| \leq ch^{\nu - 1/q}.$$

This analysis applies for case (i). Although we shall not do so here it can be shown that the same bound also applies in cases (ii) and (iii) and this establishes the theorem. $\square$

We now show that $\hat{A}^I$ is a bounded linear operator on $H_\nu[-1, 1]$ into $C[-1, 1]$.

THEOREM A2. *If* $w \in \mathcal{H}(q_1, q_2)$, $1 < q_1, q_2 < \infty$, *then* $\hat{A}^I$ *maps* $H_\nu[-1, 1]$ *into* $H_{\nu - 1/q}[-1, 1]$ *where* $q = \min(q_1, q_2)$ *and* $1/q < \nu \leq 1$. *Furthermore* $\hat{A}^I$ *is a bounded linear operator on* $H_\nu[-1, 1]$ *into* $C[-1, 1]$.

PROOF. The linearity of $\hat{A}^I$ is obvious from (A1). From (2.17) with $P \equiv 1$ we have

$$(A6) \qquad\qquad a/rZ = \text{p.p.}(X^{-1}) + T(w).$$

Substituting this into (A1) gives

$$(A7) \qquad \hat{A}^I g = g \text{ p.p.}(X^{-1}) - \frac{1}{\pi} \int_{-1}^{1} w(\tau) \left(\frac{g(\tau) - g(t)}{\tau - t} \right) d\tau.$$

Now p.p. (X^{-1}) is a polynomial so that g p.p. $(X^{-1}) \in H_\nu[-1, 1]$ and therefore in $H_{\nu - 1/q}[-1, 1]$. That the second term is in $H_{\nu - 1/q}[-1, 1]$ follows immediately from Theorem A1. Hence $\hat{A}^I$ maps $H_\nu[-1, 1]$ into $H_{\nu - 1/q}[-1, 1]$ and therefore into $C[-1, 1]$. From (A7)

$$(A8) \qquad |(\hat{A}^I g)(t)| \leq c \left\{ |g(t)| + \int_{-1}^{1} w(\tau) \left| \frac{g(\tau) - g(t)}{\tau - t} \right| d\tau \right\},$$

where c is a constant independent of both g and t. From (A2) we have that

$$(A9) \qquad\qquad \|\hat{A}^I g\|_\infty \le c.\|g\|_Y$$

where c is independent of g. This establishes the boundedness of $\hat{A}^I$. $\square$

Returning to (A5) we want to show that if g possesses a Hölder continuous derivative of order r then G also possesses a Hölder continuous derivative of order r. To do this we need to introduce some auxiliary functions.

DEFINITION. Suppose that for some $r \ge 1$, $g^{(r)} \in H_\nu[-1, 1]$ where $0 < \nu \le 1$. On $[-1, 1]$ we define

$$\begin{cases} g_0(\tau, t) = & g(\tau), \\ g_k(\tau, t) = & k \int_0^1 g^{(k)}(\eta t + (1 - \eta)\tau)\eta^{k-1} d\eta, \end{cases}$$

for $k = 1(1)r$.

We now have the following lemma

LEMMA A3. *If $g^{(r)} \in H_\nu[-1, 1]$ then*

(i) $g_k(t, t) = g^{(k)}(t), \quad k = 0(1)r$,

(ii) $\frac{\partial g_k}{\partial t} = \frac{k}{k+1} g_{k+1}, \quad k = 0(1)(r - 1)$,

(iii) $\frac{g_k(\tau, t) - g_k(t, t)}{\tau - t} = \frac{1}{k+1} g_{k+1}(\tau, t), \quad k = 0(1)(r - 1)$,

(iv) $g_r(\tau, t) \in H_\nu[-1, 1] \times H_\nu[-1, 1]$.

PROOF. The proofs of (i) - (iii) follow immediately from the definition. For the proof of (iv), if we assume that

$$|g^{(r)}(t_1) - g^{(r)}(t_2)| \le A_r |t_1 - t_2|^\nu$$

for any $t_1, t_2 \in [-1, 1]$ then for every $\tau_1, \tau_2, t_1, t_2 \in [-1, 1]$ we have that

$$|g_r(\tau_1, t_1) - g_r(\tau_2, t_2)| \le r A_r \int_0^1 \eta^{r-1} |\eta(t_1 - t_2) + (1 - \eta)(\tau_1 - \tau_2)|^\nu d\eta.$$

Now for $\sigma_1, \sigma_2 \ge 0$ and $0 < \nu < 1$ we have that $(\sigma_1 + \sigma_2)^\nu \le \sigma_1^\nu + \sigma_2^\nu$ so that

$$|g_r(\tau_1, t_1) - g_r(\tau_2, t_2)| \le r A_r \{ |t_1 - t_2|^\nu \int_0^1 \eta^{r-1+\nu} d\eta$$

$$+ |\tau_1 - \tau_2|^\nu \int_0^1 \eta^{r-1}(1 - \eta)^\nu d\eta \},$$

from which the result follows. □

With these results established we can now prove the following lemma.

LEMMA A4. *For $k = 0(1)r$, $\tau \neq t$ then*

$$\frac{\partial^k}{\partial t^k}\left(\frac{g(\tau) - g(t)}{\tau - t}\right) = \frac{g_k(\tau, t) - g_k(t, t)}{\tau - t}.$$

PROOF. Proceed by induction. The result is obviously true for $k = 0$; suppose it is true for $k = \ell$. Then

$$\frac{\partial^{\ell+1}}{\partial t^{\ell+1}}\left(\frac{g(\tau) - g(t)}{\tau - t}\right) = \frac{\partial}{\partial t}\left(\frac{g_\ell(\tau, t) - g_\ell(t, t)}{\tau - t}\right)$$

$$= \frac{1}{\tau - t}\left\{\left(\frac{\partial g_\ell}{\partial t} - g^{(\ell+1)}(t)\right) + \frac{g_\ell(\tau, t) - g^{(\ell)}(t)}{\tau - t}\right\} \quad \text{by Lemma A3(i)}$$

$$= \frac{1}{\tau - t}\left\{\frac{\ell}{\ell + 1}g_{\ell+1}(\tau, t) - g^{(\ell+1)}(t) + \frac{1}{\ell + 1}g_{\ell+1}(\tau, t)\right\},$$

$$\text{by Lemma A3(ii) and (iii)}$$

$$= \frac{g_{\ell+1}(\tau, t) - g_{\ell+1}(t, t)}{\tau - t}, \quad \text{on using Lemma A3(i) again.}$$

The result now follows. □

We now give a generalization of Theorem A2.

THEOREM A5. *Suppose $g^{(r)} \in H_\nu[-1, 1], 1/q < \nu \leq 1$, where $w \in H(q_1, q_2)$, $1 < q_1, q_2 < \infty$ and $q = \min\{q_1, q_2\}$. If*

$$(A10) \qquad G(t) = \int_{-1}^{1} w(\tau)\left(\frac{g(\tau) - g(t)}{\tau - t}\right)d\tau, \quad t \in [-1, 1]$$

then $G^{(r)} \in H_{\nu - 1/q}[-1, 1]$.

PROOF. From Lemma A4 we can write

$$G^{(r)}(t) = \int_{-1}^{1} w(\tau)\left(\frac{g_r(\tau, t) - g_r(t, t)}{\tau - t}\right)d\tau.$$

When $r = 0$, the result is that given by Theorem A1. When $r \geq 1$ we have, by Lemma A3(iv), that $g_r \in H_\nu[-1, 1] \times H_\nu[-1, 1]$. Following the proof of Theorem A1 we shall assume $h > 0$ and let $t_1 = t_0 + h$. Again, we have three cases depending upon whether or not t_0 or t_1 is an end point of $[-1, 1]$. Here we shall consider only case (i) where $\ell = [t_0 - 2h, t_0 + 2h]$ is in $(-1, 1)$. The other two cases follow by similar arguments. Now

$$G^{(r)}(t_1) - G^{(r)}(t_0)$$

$$= \left[\int_L + \int_\ell \right] \left[w(\tau) \left\{ \frac{g_r(\tau, t_1) - g_r(t_1, t_1)}{\tau - t_1} - \frac{g_r(\tau, t_0) - g_r(t_0, t_0)}{\tau - t_0} \right\} \right] d\tau = K_1 + K_2.$$

Consider first the integral K_2 over ℓ. Since

$$|g_r(\tau_1, t_1) - g_r(\tau_2, t_2)| \leq A_r \{ |\tau_1 - \tau_2|^\nu + |t_1 - t_2|^\nu \}$$

for all $(\tau_1, t_1), (\tau_2, t_2) \in [-1, 1] \times [-1, 1]$ we have that

$$|K_2| \leq A_r \int_\ell w(\tau) \{ |\tau - t_1|^{\nu-1} + |\tau - t_0|^{\nu-1} \} d\tau.$$

Using the same arguments as we did in the proof of Theorem A1 when estimating I_2 we find that
$$|K_2| \leq ch^{\nu - 1/q}.$$

Consider now K_1, the integral over L. We can rewrite this as

$$K_1 = (g_r(t_0, t_0) - g_r(t_1, t_1)) \int_L \frac{w(\tau) d\tau}{\tau - t_0} + \int_L w(\tau) \left(\frac{g_r(\tau, t_1) - g_r(\tau, t_0)}{\tau - t_0} \right) d\tau$$

$$+ h \int_L w(\tau) \left(\frac{g_r(\tau, t_1) - g_r(t_1, t_1)}{(\tau - t_0)(\tau - t_1)} \right) d\tau.$$

From the Hölder condition on g_r we find

$$|K_1| \leq 3A_r h^\nu \int_L \frac{w(\tau) d\tau}{|\tau - t_0|} + A_r h \int_L \frac{w(\tau) |\tau - t_1|^{\nu-1}}{|\tau - t_0|} d\tau.$$

Using the same techniques as we did for estimating the integrals J_3 and J_4 in Theorem A1 we find that $|K_1| \leq ch^{\nu - 1/q}$ so we have, in case (i), that

$$|G^{(r)}(t_1) - G^{(r)}(t_0)| \leq ch^{\nu - 1/q}.$$

A similar argument holds for cases (ii) and (iii) so that the theorem is proved. $\square$

We now give a generalization of Theorem A2.

THEOREM A6. *Suppose* $w \in H(q_1, q_2)$, $1 < q_1, q_2 < \infty$ *and* $q = \min\{q_1, q_2\}$. *Suppose also that for some non-negative integer* r, $g^{(r)} \in H_\nu[-1, 1]$, *where* $1/q < \nu \le 1$. *Then if* $G = \hat{A}^I g$ *we have that* $G^{(r)} \in H_{\nu-1/q}[-1, 1]$.

PROOF. If in equation (A7) we apply Theorem A5 to the appropriate terms on the right hand side, we see that $\hat{A}^I g$ is the sum of two terms each of which possesses a Hölder continuous derivative of order r. □

REFERENCES

1. C.W Clenshaw, *A note on the summation of Chebyshev series*, M.T.A.C. **9** (1955), 118-120.

2. D. Elliott, *Singular integral equations on the arc* $(-1, 1)$: *theory and approximate solution Part 1: theory*, University of Tasmania, Math. Dept., Tech Rpt. No. 218 (1987)

3. M.A. Golberg, *The numerical solution of Cauchy singular integral equations with constant coefficients*, J. Int. Equns. **9** (1985), 127-151.

4. V. Hutson and J.S. Pym, *Applications of functional analysis and operator theory*, Academic Press, London, 1980.

5. A.I. Kalandiya, *On a direct method of solution of an equation in wing theory and its application to the theory of elasticity*, Mat. Sb. **42** (1957), 249-272.

6. B.V. Khevedelidze, *Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications*, Akad. Nauk. Gruzin S.S.R. **23** (1956), 3-158.

7. N.I. Muskhelishvili, *Singular integral equations*, Noordhoff, Groningen, 1953.

8. T.J. Rivlin, *An introduction to the approximation of functions*, Blaisdell, Waltham, 1969.

9. ———— *The Chebyshev polynomials*, John Wiley, New York, 1974.

10. E. Venturino, *Recent developments in the numerical solution of singular integral equations*, J. Math. Anal. Appln. **115** (1986), 239-277.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TASMANIA, BOX 252C G.P.O., HOBART, TASMANIA, 7001, AUSTRALIA.