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RANDOM FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND WITH DEGENERATE KERNELS II. BOUNDS FOR PROBABILITIES

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ABSTRACT. In Part I of this series of papers [**10**] general limit theorems were proved for the distribution of the solution of random Fredholm integral equations with degenerate kernels. In this paper we consider bounds and estimates for corresponding probabilities and give some examples to illustrate the obtained results.

1. Introduction. In Part I of this series of papers we considered a sequence of random integral equations

$$\int_0^1 K_n(t,s,\omega)x_n(s,\omega)ds - \tilde{\lambda}_n x_n(t,\omega) = b_n(t,\omega), \qquad (E_n)$$
$$\tilde{\lambda}_n \neq 0, \quad n = 1,2,\ldots$$

with random degenerate kernels

$$(S_n) \qquad K_n(t,s,\omega) = \sum_{i=1}^{n} \alpha_{in}(t,\omega)\beta_{in}(s,\omega).$$

For the further statements we will use the notation of Part I. In particular, for simplification we also omit the variable ω. Under appropriate conditions the limit distribution of the sequence $\{x_n(t)\}$ was determined in [**10**]. For this end we have used a sequence of approximating processes $w_n(t)$ whose limit distribution can be easily calculated.

Of course, in this connection one must investigate the accuracy of the approximation or, at least, estimates for it. In this direction we will consider two problems for which we will derive explicit bounds. First, we will give bounds (estimates) for probabilities of the form

$$(\mathrm{P}_1) \qquad \mathrm{P}(x_n(t_0) \in G_0), \quad G_0 \ - \ \text{suitable subset of } \mathbf{R}^1,$$

approximately by terms of the approximating process $w_n(t_0)$, and, second, for

$$(\mathrm{P}_2) \qquad \mathrm{P}(x_n(t_2) \in G_2 / x_n(t_1) \in G_1), \ G_i \ - \ \text{suitable subsets of } \mathbf{R}^1,$$

in a similar manner as above.

The problems (P_1) and (P_2) may arise in practical situations, for example in the theory of reliability, where G_1 could denote the set of dangerous states of a system and G_2 could be the set of states causing a failure of the system.

2. Bounds and estimates. In this section we will consider the problems (P_1) and (P_2). First, a general inequality is proved in §2.1 using a basic result in [**8**], cf. also [**6, 7**]. In §2.2 and §2.3 we will apply this result to get solutions of (P_1) and (P_2), respectively. Since we consider the integral equations (E_n) with random degenerate kernels (S_n) only for a fixed n, we will omit the index n, in general.

2.1. *A general result.* In this section we present an inequality for the probability $\mathrm{P}(\mathbf{x}(\mathbf{t}_r) \in G_1)$, where

$$\mathbf{x}(\mathbf{t}_r) : = (x(t_1), x(t_2), \ldots, x(t_r))^T, \qquad x(t) \ - \text{solution of } (E_n), \ G_1 \ - \ \text{a Borel-set of } \mathbf{R}^r,$$

and

$$t_i \in T_r, \ T_r := \{t_1, \ldots, t_r\}, \quad t_i \in [0,1], \quad t_i \neq t_j \text{ for } i \neq j.$$

Therefore, we introduce the vector

$$\mathbf{y}(\mathbf{t}_r) := (y(t_1), \ldots, y(t_r))^T, \qquad y(t) = (x(t) - \tilde{x}(t))/\tilde{s}(t)$$

where $\tilde{x}(t)$ is a solution of the unperturbed equation (E_n) and $\tilde{s}(t)(> 0)$ is the standard deviation of a suitably chosen process, see Part I, and a corresponding set

$$(1) \qquad G_2 := \{\mathbf{z} \in \mathbf{R}^r : \mathbf{z} = \mathrm{Diag}\ (\tilde{s}^{-1}(t_i))(\mathbf{u} - \tilde{\mathbf{x}}(\mathbf{t}_r)), \mathbf{u} \in G_1\}.$$

Here, we note that, in the general case, the unperturbed solution $\tilde{x}(t)$ is not the expected value of the random solution $x(t)$. The investigation of the arising difference belongs to the so-called average problem which was considered by some authors, cf. [**4**]. Furthermore, $K_\eta(\mathbf{z})$ denotes the open ball in $\mathbf{R}^r$ with center $\mathbf{z}$ and radius η, i.e.,

$$K_\eta(\mathbf{z}) = \{\mathbf{u} \in \mathbf{R}^r : ||\mathbf{u} - \mathbf{z}|| < \eta\},$$

$$G^0(\eta, G_2) := \cup_{\mathbf{z}\in G_2} K_\eta(\mathbf{z}) \tag{2a}$$

$$G^u(\eta, G_2) := \mathbf{R}^r \setminus \cup_{\mathbf{z}\notin G_2} K_\eta(\mathbf{z}), \tag{2b}$$

and

$$g(\eta) := \mathrm{P}(||\mathbf{w}(\mathbf{t}_r) - \mathbf{y}(\mathbf{t}_r)|| \geq \eta) \tag{3}$$

with

$$w(t) = -\frac{1}{\tilde{\lambda}\tilde{s}(t)}\Big(\varphi(t) - \boldsymbol{\delta}^T(t)\tilde{\mathbf{c}} + \tilde{\boldsymbol{\alpha}}^T(t)\tilde{B}^{-1}(D\tilde{\mathbf{c}} - \mathbf{f})\Big) \tag{4}$$

$$(\tilde{s}(t) > 0 \text{ such that } D^2 w(t) = 1).$$

After these preparations we can prove

THEOREM 1. *Let the random integral equation* (E_n) *be given with the random degenerate kernel* (S_n)*. Further, let* G_1 *be a Borel set of* $\mathbf{R}^r$*. Then, for the solution process* $x(t)$*, the following inequalities are valid:*

$$\mathrm{P}(\mathbf{x}(\mathbf{t}_r) \in G_1) \leq \inf_{\eta>0}\{\mathrm{P}(\mathbf{w}(\mathbf{t}_r) \in G^0(\eta, G_2)) + g(\eta)\} \tag{5a}$$

$$\mathrm{P}(\mathbf{x}(\mathbf{t}_r) \in G_1) \geq \sup_{\eta>0}\{\mathrm{P}(\mathbf{w}(\mathbf{t}_r) \in G^u(\eta, G_2)) - g(\eta)\} \tag{5b}$$

where G^0, G^u *and* g *are defined by* (2a), (2b) *and* (3).

PROOF. Obviously, $\mathrm{P}(\mathbf{x}(\mathbf{t}_r) \in G_1) = \mathrm{P}(\mathbf{y}(\mathbf{t}_r) \in G_2)$. On the other hand we have, for arbitrary $\eta > 0$,

$$\begin{aligned}\mathrm{P}(\mathbf{y}(\mathbf{t}_r) \in G_2) &= \mathrm{P}(\{\mathbf{y}(\mathbf{t}_r) \in G_2\} \cap \{||\mathbf{w}(\mathbf{t}_r) - \mathbf{y}(\mathbf{t}_r)|| < \eta\}) \\ &\quad + \mathrm{P}(\{\mathbf{y}(\mathbf{t}_r) \in G_2\} \cap \{||\mathbf{w}(\mathbf{t}_r) - \mathbf{y}(\mathbf{t}_r)|| \geq \eta\}) \\ &\leq P(\mathbf{w}(\mathbf{t}_r) \in G^0(\eta, G_2)) + g(\eta).\end{aligned}$$

Thus, the inequality (5a) is proved. A similar proof can be given for (5b). □

REMARK 1. For the practical application of the inequalities (5a) and (5b) it is necessary to know:

1. exactly or approximately the distribution of $\mathbf{w}(\mathbf{t}_r)$;

2. simple (applicable) description of the sets $G^0(\eta, G_2)$ and $G^u(\eta, G_2)$, at least for special classes of sets G_2;

3. bounds of the value $g(\eta)$.

For the first task we refer to [**10**]. In that paper conditions were given for the distribution of $\mathbf{w}(\mathbf{t}_r)$ to be approximate Gaussian (cf. Theorem 2, Example 1 and 2 in [**10**]).

The second task was solved for parallelotops G_2 in [**8**], cf. also [**6**, Hilfssatz 2]. In [**8**] it was shown that $G^u(\eta, G_2)$ is again an easily describable parallelotop and that $G^0(\eta, G_2)$ can be approximated very well by a parallelotope, see also §2.2 and 2.3 and the paper [**11**].

The third problem is addressed in the following sections.

2.2. *Bounds for the one-dimensional distribution.* In this section we will give bounds for the probability $\mathrm{P}(x(t_0) \in G_{01})$ where G_{01} is a Borel-set of $\mathbf{R}^1$. Then, according to (1), the corresponding set G_{02} is

$$G_{02} = \left\{ z \in \mathbf{R}^1 : z = \frac{u - \tilde{x}(t_0)}{\tilde{s}(t_0)},\ u \in G_{01} \right\},$$

and, from Theorem 1, we immediately get

THEOREM 2. *For the solution $x(t)$ of the equation (E_n) with the kernels (S_n) the following statement is valid:*

$$\text{(6a)} \qquad \mathrm{P}(x(t_0) \in G_{01}) \leq \inf_{\eta>0} \{\mathrm{P}(w(t_0) \in G^0(\eta, G_{02})) + g(\eta)\}$$

$$\text{(6b)} \qquad \mathrm{P}(x(t_0) \in G_{01} \geq \sup_{\eta>0} \{\mathrm{P}(w(t_0) \in G^u(\eta, G_{02})) - g(\eta\}),$$

where

$$G^0(\eta, G_{02}) = \cup_{z \in G_{02}} K_\eta(z), \qquad K_\eta(z) = (z - \eta, z + \eta)$$

$$G^u(\eta, G_{02}) = \mathbf{R} \setminus \cup_{z \notin G_{02}} K_\eta(z)$$

and

$$g(\eta) = \mathrm{P}(|w(t_0) - y(t_0)| \geq \eta).$$

We want to point out the fact that $g(\eta)$ may depend also on t_0. The following lemma gives a representation of the sets G^0 and G^u for the case that G_{01} is an interval (open or closed).

LEMMA 1. *If G_{01} is an interval (open or closed, respectively), then G_{02} is also an interval (open or closed, respectively). For example: If $G_{01} = (a_1, b_1)$, then $G_{02} = (a_2, b_2)$ with*

$$a_2 = (a_1 - \tilde{x}(t_0))/\tilde{s}(t_0), \qquad b_2 = (b_1 - \tilde{x}(t_0))/\tilde{s}(t_0);$$

the representations

$$G^0(\eta, G_{02}) = (a_2 - \eta, b_2 + \eta)$$

$$G^u(\eta, G_{02}) = [a_2 + \eta, b_2 - \eta]$$

are possible; in particular, the set $G^0(\eta, G_{02})$ is always an open set and $G^u(\eta, G_{02})$ is always a closed set.

REMARK 2. Using (2a) and (2b) we see easily that the topological statements concerning the sets G^0 and G^u are also true in the general case.

REMARK 3. Under appropriate conditions it was proved that $w(t_0) \in N(0,1)$ (in **[9]**) or $w(t_0) \to_{(d)} z \in N(0,1)$ (in **[10]**).

Consequently, in these cases the main problem is in finding (upper) bounds for $g(\eta)$.

According to the Remark 3 we will now give an estimate for the difference

(7)
$$y(t_0) - w(t_0) = \frac{1}{\tilde{\lambda}\tilde{s}(t_0)}[(\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}D^0 - \boldsymbol{\delta}^T(t_0)) \cdot (I + \tilde{B}^{-1}D^0)^{-1} \cdot \tilde{B}^{-1}(D^0\tilde{\mathbf{c}} - \mathbf{f}^0) - \tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}(R\tilde{\mathbf{c}} - \mathbf{r})].$$

Equation (7) can be obtained from Part I (formulas (5), (6a), (6b)).

LEMMA 2. *For $g(\eta)$, the following bound holds:*

(8)
$$\begin{aligned} g(\eta) \le \inf_{0<q<1} \Bigg\{ & \mathrm{P}(||\tilde{B}^{-1}D^0|| \ge q) + \frac{1}{\eta(1-q)|\tilde{\lambda}|\tilde{s}(t_0)} \\ & \cdot \min\Bigg\{ \sqrt{\mathrm{E}||\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}D^0||^2 + \Sigma_i \tilde{K}_{\delta_i}(t_0,t_0) - 2\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}\mathrm{E}(D^0\boldsymbol{\delta}(t_0))} \\ & \cdot \sqrt{\mathrm{Sp}\tilde{K} + \mathrm{E}||\tilde{B}^{-1}(R\tilde{\mathbf{c}} - \mathbf{r})||^2 + 2(\tilde{B}^{T-1}\tilde{B}^{-1}; \mathrm{E}[(R\tilde{\mathbf{c}} - \mathbf{r})(D\tilde{\mathbf{c}} - \mathbf{f})^T])} \\ & + \mathrm{E}|\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}(R\tilde{\mathbf{c}} - \mathbf{r})|(1-q);\ \left(q||\tilde{\boldsymbol{\alpha}}(t_0)|| + \sqrt{\Sigma_i \tilde{K}_{\delta_i}(t_0,t_0)}\right) \\ & \cdot \left(q||\tilde{\mathbf{c}}|| + \sqrt{\mathrm{E}||\tilde{B}^{-1}\mathbf{f}^0||^2}\right) + (1-q) \cdot \Bigg[q||\tilde{\boldsymbol{\alpha}}(t_0)|| \cdot ||\tilde{\mathbf{c}}|| \\ & + \sqrt{\tilde{\boldsymbol{\alpha}}^T(t_0)(\tilde{K} - \tilde{B}^{-1}\tilde{K}_{\mathbf{f}}\tilde{B}^{T-1} + 2\tilde{B}^{-1}\mathrm{E}(D\tilde{\mathbf{c}}\mathbf{f}^T) \cdot \tilde{B}^{T-1})\tilde{\boldsymbol{\alpha}}(t_0)} \\ & + \mathrm{E}||\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}\mathbf{r}|| \Bigg] \Bigg\} \Bigg\} \end{aligned}$$

The proof of Lemma 2 is carried out by means of (7) and Markov's and Schwarz's inequality. Therefore, we omit the details.

If some additional assumptions are satisfied, then certain expressions arising in (8) can be simplified, as is shown in the following remarks.

REMARK 4. If φ and $\boldsymbol{\epsilon}$ are uncorrelated, then

$$\tilde{K}_{\mathbf{f}} = \int_0^1 \int_0^1 \left[\tilde{\boldsymbol{\beta}}(t)\tilde{\boldsymbol{\beta}}^T(s)\tilde{K}_{\varphi}(t,s) + \tilde{b}(t)\tilde{b}(s)\Big(\tilde{K}_{\epsilon_i,\epsilon_k}(t,s)\Big)_{i,k}\right] dt\, ds.$$

If, additionally, $\boldsymbol{\delta}$ and $\varphi, \boldsymbol{\epsilon}$ and $\delta_i, i = 1(1)n$, are uncorrelated, then

$$\mathrm{E}(D\tilde{\mathbf{c}}\mathbf{f}^T) = \int_0^1 \int_0^1 \tilde{b}(s)\tilde{\boldsymbol{\alpha}}^T(t)\tilde{\mathbf{c}}\Big(\tilde{K}_{\epsilon_i,\epsilon_k}(t,s)\Big)_{i,k} dt\, ds.$$

REMARK 5. If $\boldsymbol{\epsilon}(s)$ and $\delta_i(t)$, $i = 1(1)n$, are independent for arbitrary s and t, then

$$\mathrm{E}(D^0\boldsymbol{\delta}(t_0)) = \sum_i \int_0^1 \tilde{\boldsymbol{\beta}}(t)\tilde{K}_{\delta_i}(t,t_0)\, dt;$$

if $\varphi, \boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ are independent, then

$$\mathrm{E}(R\tilde{\mathbf{c}} - \mathbf{r})(D\tilde{\mathbf{c}} - \mathbf{f})^T = 0.$$

Using Lemma 2 and Theorem 2 we get immediately

COROLLARY 1. *If in the integral equation only the right-hand side is random, i.e.,* $\epsilon_i(t) = 0$, $\delta_i(t) = 0$, *then, under the additional condition that* G_{02} *is a continuity-set of the measure* $P_{w(t_0)}$, *the inequalities* (6a) *and* (6b) *become equalities, more precisely,*

$$\mathrm{P}(x(t_0) \in G_{01}) = \mathrm{P}(w(t_0) \in G_{02}). \tag{9}$$

For the proof we only want to point out the fact that, in the above mentioned case, the value $g(\eta)$ is equal to 0 for arbitrary $\eta > 0$.

Also for the special cases considered in [**10, 1, 2**] the bound (8) for $g(\eta)$ can be simplified.

COROLLARY 2. *If $\epsilon_i(s) = 0$, $i = 1(1)n$, then, from (8), it follows that*

$$g(\eta) \le \inf_{0<q<1} \Bigg\{ \mathrm{P}(||\tilde{B}^{-1}D|| \ge q) + \frac{1}{\eta(1-q)|\tilde{\lambda}|\tilde{s}(t_0)}$$
$$\cdot \min\Bigg\{ \sqrt{\mathrm{E}||\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}D||^2 + \Sigma_i \tilde{K}_{\delta_i}(t_0,t_0) - 2\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}\mathrm{E}(D\boldsymbol{\delta}(t_0)) \cdot \mathrm{Sp}\tilde{K}};$$
$$\left(q||\tilde{\boldsymbol{\alpha}}(t_0)|| + \sqrt{\Sigma_i \tilde{K}_{\delta_i}(t_0,t_0)} \right) \left(q||\tilde{\mathbf{c}}|| + \sqrt{\mathrm{E}||\tilde{B}^{-1}\mathbf{f}||^2} \right) \Bigg\} \Bigg\},$$

with

$$\mathrm{E}||\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}D||^2 = \sum_i \int_0^1 \int_0^1 \tilde{\gamma}(t_0,s)\tilde{\gamma}(t_0,u)\tilde{K}_{\delta_i}(s,u)\, ds\, du$$

$$\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}\mathrm{E}(D\boldsymbol{\delta}(t_0)) = \sum_i \int_0^1 \tilde{\gamma}(t_0,s)\tilde{K}_{\delta_i}(s,t_0)\, ds,$$

$$\begin{aligned}\mathrm{Sp}\tilde{K} = \int_0^1 \int_0^1 \tilde{\boldsymbol{\beta}}^T(s)\tilde{B}^{T-1}\tilde{B}^{-1}\tilde{\boldsymbol{\beta}}(u) \Bigg(& \sum_{i,j} \tilde{c}_i\tilde{c}_j\tilde{K}_{\delta_i,\delta_j}(s,u) + \tilde{K}_{\varphi}(s,u) \\ & + \sum_i \tilde{c}_i[\tilde{K}_{\delta_i,\varphi}(s,u) \\ & + \tilde{K}_{\delta_i,\varphi}(u,s)] \Bigg)\, ds\, du,\end{aligned}$$

$$\mathrm{E}||\tilde{B}^{-1}\mathbf{f}||^2 = \int_0^1 \int_0^1 \tilde{\boldsymbol{\beta}}^T(s)\tilde{B}^{T-1}\tilde{B}^{-1}\tilde{\boldsymbol{\beta}}(u)\tilde{K}_{\varphi}(s,u)\, ds\, du.$$

COROLLARY 3. *If $\delta_i(t) = 0$, $i = 1(1)n$, and $\varphi(t) = 0$, then (8) yields*

$$g(\eta) \le \inf_{0<q<1} \Bigg\{ \mathrm{P}(||\tilde{B}^{-1}D|| \ge q) + \frac{1}{\eta(1-q)|\tilde{\lambda}|\tilde{s}(t_0)}$$
$$\cdot \min\Bigg\{ \sqrt{\mathrm{E}||\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}D||^2}$$
$$\cdot \sqrt{\mathrm{Sp}\tilde{K}}, q||\tilde{\boldsymbol{\alpha}}(t_0)|| \left(q||\tilde{\mathbf{c}}|| + \sqrt{\mathrm{E}||\tilde{B}^{-1}\mathbf{f}||^2} \right) \Bigg\} \Bigg\},$$

with

$$\begin{aligned}
&\mathrm{E}||\tilde{\boldsymbol{\alpha}}^T(t_0)\tilde{B}^{-1}D||^2 \\
&= \tilde{\boldsymbol{\rho}}^T(t_0)\int_0^1\int_0^1 \tilde{\boldsymbol{\alpha}}^T(s)\tilde{\boldsymbol{\alpha}}(u)\Big(\tilde{K}_{\epsilon_i,\epsilon_k}(s,u)\Big)_{i,k} ds\; du\; \tilde{\boldsymbol{\rho}}(t_0),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Sp}\tilde{K} &= \Big(\tilde{B}^{T-1}\tilde{B}^{-1}, \int_0^1\int_0^1 [\tilde{\boldsymbol{\alpha}}^T(s)\tilde{\mathbf{c}}\tilde{\boldsymbol{\alpha}}^T(u)\tilde{\mathbf{c}} - 2\tilde{\boldsymbol{\alpha}}^T(s)\tilde{\mathbf{c}}\tilde{b}(u) \\
&\qquad + \tilde{b}(s)\tilde{b}(u)]\cdot[\tilde{K}_{\epsilon_i,\epsilon_k}(s,u)]_{i,k}\; ds\; du\Big),
\end{aligned}$$

$$\mathrm{E}||\tilde{B}^{-1}\mathbf{f}||^2 = \Big(\tilde{B}^{T-1}\tilde{B}^{-1}, \int_0^1\int_0^1 \tilde{b}(s)\tilde{b}(u)\Big(\tilde{K}_{\epsilon_i,\epsilon_k}(s,u)\Big)_{i,k} ds\; du\Big).$$

Likewise in the special cases with only one random function in the kernel (S_n) we get sharper bounds using a result of Geary in [**5**]. We quote it.

LEMMA 3. *Let* $(p,q)^T$ *be a normally distributed random vector with* $p \in N(\mu_p,\sigma_p^2)$, $q \in N(\mu_q,\sigma_q^2)$, $\sigma_{pq} := \mathrm{cov}(p,q)$. *Then, for*

$$z := \frac{p}{q} \text{ with } \mu_q > 0,$$

we have

$$\Phi(t(\tilde{z}_2)) - \Phi(t(\tilde{z}_1)) \le \mathrm{P}(\tilde{z}_1 < z < \tilde{z}_2),$$
$$\mathrm{P}(\tilde{z}_1 < z < \tilde{z}_2) \le \Phi(t(\tilde{z}_2)) - \Phi(t(\tilde{z}_1)) + \epsilon$$

with

$$\epsilon = 2(1 - \Phi(\mu_q/\sigma_q))$$

and

$$t(z) := \frac{\mu_q z - \mu_p}{\sqrt{\sigma_p^2 - 2\sigma_{pq}z + \sigma_q^2 z^2}}.$$

As announced above we study the special cases

$$\text{(SC1)} \qquad K_n(t,s) = \sum_{i=1}^n \tilde{\alpha}_i(t)\tilde{\beta}_i(s) + \delta(t)\tilde{\beta}_n(s),$$

(SC2) $$K_n(t,s) = \sum_{i=1}^{n} \tilde{\alpha}_i(t)\tilde{\beta}_i(s) + \tilde{\alpha}_n(t)\epsilon(s).$$

Applying a well-known rank-1-perturbation theorem and (1), (2), (3a), (3b) in [**10**], we obtain, after a long but easy calculation, the following results.

LEMMA 4. *Given* (E_n) *and* (S_n) *with* $b(t) = \tilde{b}(t)$, $K_n(t,s)$ *as in* (SC1), *let the matrix* $\tilde{B} := (\tilde{A} - \tilde{\lambda} I)$ *be regular with* $\tilde{B}^{-1} := (\tilde{\boldsymbol{\pi}}_{\cdot k})$ *and* $\tilde{B}^{-1}\mathbf{b} =: \tilde{\mathbf{c}} = (\tilde{c}_i)$. *Then*

$$\tilde{x}(t) = -\frac{1}{\tilde{\lambda}}[\tilde{b}(t) - \tilde{\boldsymbol{\alpha}}^T(t)\tilde{\mathbf{c}}]$$

and, for $z(t) := x(t) - \tilde{x}(t)$,

$$z(t) = \tilde{\lambda}^{-1}\tilde{c}_n \cdot \frac{1}{g}\Big(\delta(t) - \tilde{\boldsymbol{\alpha}}^T(t)\sum_{k=1}^{n} \tilde{\boldsymbol{\pi}}_{\cdot k} \cdot u_k\Big),$$

where

$$g = 1 + \sum_{k=1}^{n} \tilde{\pi}_{nk} u_k, \qquad (u_k) = \mathbf{u} = \int_0^1 \tilde{\boldsymbol{\beta}}(s)\delta(s)\, ds.$$

If the random process $\delta(t)$ is Gaussian, we get immediately

COROLLARY 4. *Let* $\delta(t)$ *be a Gaussian process with* $\mathrm{E}\delta(t) = 0$. *Then, for* (SC1), *the one-dimensional distribution of* $z(t)$ *can be calculated approximately by means of Lemma* 3. *Toward this end we write*

$$z(t) = \frac{p(t)}{q}, \qquad q = g \quad (g \textit{ from Lemma } 4)$$

and take into account that $(p(t), q)^T$ *is normally distributed. The needed quantities of Lemma* 3 *are*

$$\mathrm{E}p(t) = 0$$

$$\sigma^2_{p(t)} = \tilde{\lambda}^{-2}\tilde{c}_n^2[\tilde{K}_\delta(t,t) - 2\tilde{\boldsymbol{\alpha}}^T(t)\tilde{B}^{-1}\tilde{\mathbf{u}}_0(t) + \tilde{\boldsymbol{\alpha}}^T(t)\tilde{B}^{-1}\tilde{U}\tilde{B}^{T-1}\tilde{\boldsymbol{\alpha}}(t)]$$

$$\mathrm{E}q = 1, \qquad \sigma_q^2 = \tilde{\boldsymbol{\pi}}_{n\cdot}^T\tilde{U}\tilde{\boldsymbol{\pi}}_{n\cdot},$$

$$\mathrm{cov}(p(t), q) = \tilde{\lambda}^{-1}\tilde{c}_n(\tilde{\boldsymbol{\pi}}_{n\cdot}^T\tilde{\mathbf{u}}_0(t) - \tilde{\boldsymbol{\alpha}}^T(t)\tilde{B}^{-1}\tilde{U}\tilde{\boldsymbol{\pi}}_{n\cdot})$$

with

$$\tilde{\mathbf{u}}_0(t) := \int_0^1 \tilde{\boldsymbol{\beta}}(s)\tilde{K}_\delta(t,s)\, ds,$$

$$\tilde{U} := \int_0^1\int_0^1 \tilde{\boldsymbol{\beta}}(t)\tilde{\boldsymbol{\beta}}^T(s)\tilde{K}_\delta(t,s)\, dt\, ds.$$

In the same way we derive the following formulas for (SC2) (notation as in Lemma 4).

LEMMA 5. *Given* (E_n) *and* (S_n) *with*

$$b(t) = \tilde{b}(t), \qquad K_n(t,s) \text{ as in (SC2)},$$

we have

$$z(t) = \tilde{a}(t)\cdot\frac{1}{g}\cdot\int_0^1 \epsilon(s)\tilde{x}(s)\, ds,$$

where

$$\tilde{a}(t) = -\tilde{\boldsymbol{\alpha}}^T(t)\tilde{\boldsymbol{\pi}}_{\cdot n}, \qquad g = 1 - \int_0^1 \epsilon(s)\tilde{a}(s)\, ds.$$

COROLLARY 5. *Let* $\epsilon(t)$ *be a Gaussian process with* $\mathrm{E}\epsilon(t) = 0$. *Then, for* (SC2), *the one-dimensional distribution of* $z(t)$ *can be computed approximately with the help of Lemma* 3. (*We notice that* $z(t) = p(t)/q$, $q = g$ (g *from Lemma* 5) *and* $(p(t),q)^T$ *is Gaussian.*) *The required quantities of Lemma* 3 *are*

$$\mathrm{E}p(t) = 0$$

$$\sigma^2_{p(t)} = \int_0^1\int_0^1 \tilde{K}_\epsilon(r,s)\tilde{x}(s)\tilde{x}(r)\, dr\, ds\cdot\tilde{a}^2(t)$$

$$\mathrm{E}q = 1, \qquad \sigma_q^2 = \int_0^1\int_0^1 \tilde{K}_\epsilon(r,s)\tilde{a}(s)\tilde{a}(r)\, dr\, ds,$$

$$\mathrm{cov}(p(t), q) = -\tilde{a}(t) \cdot \int_0^1 \int_0^1 \tilde{K}_\epsilon(r, s)\tilde{a}(s)\tilde{x}(r)\, dr\, ds.$$

2.3. *Bounds for conditional probabilities.* In this section we will give only some hints for construction of bounds of conditional probabilities

$$\mathrm{P}(x(t_2) \in G_2/x(t_1) \in G_1),$$

where $x(t)$ is a solution process of (E_n), using results of §2.2 and §2.1.

Taking the definition of the conditional probability

$$\begin{aligned} &\mathrm{P}(x(t_2) \in G_2/x(t_1) \in G_1) \\ &= \mathrm{P}(\{x(t_1) \in G_1\} \cap \{x(t_2) \in G_2\}) : \mathrm{P}(x(t_1) \in G_1) \end{aligned} \tag{10}$$

we will estimate the dividend on the right-hand side of (10) by means of the general inequalities (5a), (5b) and the divisor shall be estimated by (6a), (6b). Therefore, we introduce the following set:

$$\begin{aligned} G_2^{12} = &\left\{z \in \mathbf{R} : z = \frac{u - \tilde{x}(t_1)}{\tilde{s}(t_1)},\ u \in G_1\right\} \\ &\times \left\{z \in \mathbf{R} : z = \frac{u - \tilde{x}(t_2)}{\tilde{s}(t_2)},\ u \in G_2\right\}. \end{aligned}$$

THEOREM 3. *For the transition probability of the solution process $x(t)$ of (E_n) the following bounds hold:*

$$\begin{aligned} &\mathrm{P}(x(t_2) \in G_2/x(t_1) \in G_1) \\ &\leq \inf_{\rho>0}\{P(\mathbf{w}(\mathbf{t}^{12}) \in G^0(\rho, G_2^{12})) + g_2(\rho)\} : P_u, \end{aligned} \tag{11a}$$

$$\begin{aligned} &P(x(t_2) \in G_2/x(t_1) \in G_1) \\ &\geq \sup_{\rho>0}\{P(\mathbf{w}(\mathbf{t}^{12}) \in G^u(\rho, G_2^{12})) - g_2(\rho)\} : P_0, \end{aligned} \tag{11b}$$

where

$$\mathbf{w}(\mathbf{t}^{12}) = (w(t_1), w(t_2))^T$$

$$G^0(\rho, G_2^{12}) = \cup_{\mathbf{y}\in G_2^{12}} K_\rho(\mathbf{y}), \qquad K_\rho(\mathbf{y}) = \{\mathbf{z} \in \mathbf{R}^2 : ||\mathbf{z} - \mathbf{y}|| < \rho\},$$

$$G^u(\rho, G_2^{12}) = \mathbf{R}^2 \setminus \cup_{\mathbf{y} \notin G_2^{12}} K_\rho(\mathbf{y}),$$

$$g_2(\rho) := \mathrm{P}(||\mathbf{w}(\mathbf{t}^{12}) - \mathbf{y}(\mathbf{t}^{12})|| \geq \rho)$$

and P_u, P_0 *from* (6b), (6a), *respectively.*

Until now we have not fixed the used vector-norm arising in $K_\rho(\mathbf{y})$ and $g_2(\rho)$. Above all, the evaluation of G^u, G^0, and $g_2(\rho)$ depend on the chosen vector-norm, and therefore further consideration of this problem is needed; some related results are given in [**11**].

2.4. *An algorithm.* Now we will develop a framework of an algorithm for the calculation of bounds of the probability $P_0 := \mathrm{P}(x(t_0) \in G_{01})$.

ALGORITHM.
1. Calculate the unperturbed solution $\tilde{x}(t)$
   - see Part I: (1), (2), (3a), (3b) with unperturbed quantities
2. Examine special cases
   . only right-hand side is random
      - see Remark 6 below
   . (SC1) is given
      .. with Gaussian perturbation
         - use Corollary 4
      .. with other perturbation
         - use Lemma 4 or
         - go on with step 3
   . (SC2) is given
      .. with Gaussian perturbation
         - use Corollary 5
      .. with other perturbation
         - use Lemma 5 or
         - go on with step 3
3. Calculate $\tilde{s}(t_0)$
   - see Part I, §2.2
4. Compute $w(t)$
   - see (4)
5. Determine the distribution of $w(t)$
   . if the perturbative processes are Gaussian
      - see [**9**]

. general case
 - see Part I or
 - use step 4 directly
6. Determine the sets G^0 and G^u
 . if G_{01} is an interval
 - use Lemma 1
 . general case
 - use (2a), (2b); see also [**11**]
7. Determine $\mathrm{P}(w(t_0) \in G^0)$ and $\mathrm{P}(w(t_0) \in G^u)$
 - use 5^{th} and 6^{th} steps
8. Calculate $g(n)$ or bounds for it
 - use (7) directly or
 . general case
 - use Lemma 2
 . if all $\alpha_i(t)$ and $b(t)$ are deterministic
 - use Corollary 3
 . if all $\beta_i(t)$ are deterministic
 - use Corollary 2
9. Calculate bounds of P_0
 - use Theorem 2, formulas (6a), (6b).

REMARK 6. If only the right-hand side is random, then the steps 6, 7, and 8 can be omitted; in this case the wanted probability (step 9) can be calculated by means of Corollary 1.

3. Examples. In the section we will consider some examples of random integral equations where the results presented in §2 can be applied. In §3.1 we will consider the case where only the right-hand side of the equation is random. This case was also considered by other authors, cf. for example [**1**]. In §3.2 random integral equations with random kernels are investigated.

3.1 *Random right-hand side.* For the chosen examples we refer to [**2**, §3].

EXAMPLE 1. Consider the random integral equation (E_n) with

$$K(t, s) = t, \qquad \tilde{\lambda} = 1,$$

$$b(t,\omega) = -\sin t + \tilde{F} \cdot B(t,\omega), \quad \tilde{F} \in \mathbf{R}^+,$$

where

$$B(t,\omega) = \sum_{k=0}^{m} y_k \tilde{L}_{mk}(t)$$

is the Lagrange interpolation polynomial with $y_k \in N(0;1)$, independent, and $\tilde{L}_{mk}(t)$ is a polynomial of degree m with $\tilde{L}_{mk}(t) = 0$, for $t = i/m$, $i = 0(1)m$, $i \neq k$, and $\tilde{L}_{mk}(k/m) = 1$.

We shall determine the probability

(12) $\mathrm{P}(x(t_0) \in G_{01})$ with $G_{01} = (\tilde{x}(t_0) - \mu\tilde{s}(t_0), \tilde{x}(t_0) + \mu\tilde{s}(t_0))$, $\mu > 0$.

For this purpose we will apply the algorithm given in 2.4 step by step.

1. The unperturbed solution $\tilde{x}(t)$ is $\tilde{x}(t) = \sin t + 2(1 - \cos(1))t$.

3. For the calculation of $\tilde{s}(t)$ we use the representation

$$\begin{aligned}\tilde{s}^2(t) = &\int_0^1 \int_0^1 \tilde{K}_\varphi(r,s)\tilde{\gamma}(t,r)\tilde{\gamma}(t,s)\, dr\, ds + \tilde{K}_\varphi(t,t) \\ &- 2\int_0^1 \tilde{K}_\varphi(t,r)\tilde{\gamma}(t,r)\, dr\end{aligned}$$

given in Part I, §2.2. With $\varphi(t) = \tilde{F} \cdot B(t)$, $\tilde{\gamma}(t,r) = -2t$, and $\tilde{J}_{mi} = \int_0^1 \tilde{L}_{mi}(t)\, dt$ we get

$$\tilde{s}^2(t) = \tilde{F}^2 \cdot \sum_{i=0}^{m} (2t\tilde{J}_{mi} + \tilde{L}_{mi}(t))^2.$$

4./5. In order to apply Corollary 1 we need the distribution of the random process $w(t)$. An easy calculation yields

$$w(t) = -\frac{1}{\tilde{s}(t)}\left[\tilde{F} \cdot B(t) - \int_0^1 \tilde{\gamma}(t,s) \cdot \tilde{F} \cdot B(s)\, ds\right] \in N(0;1).$$

9. With that, and using Corollary 1, the wanted probability (12) is

$$\begin{aligned}\mathrm{P}(x(t_0) \in G_{01}) &= \mathrm{P}(\tilde{x}(t_0) - \mu\tilde{s}(t_0) < x(t_0) < \tilde{x}(t_0) + \mu\tilde{s}(t_0)) \\ &= 2\Phi(\mu) - 1,\end{aligned}$$

where Φ denotes the distribution function of a $N(0;1)$-variable.

In order to see the influence of $\tilde{F}$ and m on the standard deviation $\tilde{s}(t)$ we will calculate it for $m = 2,4,8,14$ (number of mesh-points $3,5,9,15$, respectively). The influence of $\tilde{F}$ is obvious. Therefore, the following table gives the values of $\tilde{s}(t), t = 0(0.1)1$, only for $\tilde{F} = 0.2$.

TABLE 1

m=2:	0.20	0.18	0.21	0.26	0.30	0.34	0.36	0.37	0.37	0.38	0.38
m=4:	0.20	0.26	0.26	0.22	0.23	0.25	0.26	0.29	0.37	0.39	0.31
m=8:	0.20	0.40	0.27	0.25	0.24	0.21	0.28	0.33	0.27	0.57	0.33
m=14:	0.20	2.15	0.87	0.97	1.10	1.60	1.69	2.15	2.63	0.79	2.98

These results show that, as the number of mesh-points of the interpolation polynomial increases, the standard deviation $\tilde{s}(t)$ increases essentially (in this sense the interpolation is numerically unstable).

EXAMPLE 2. The same equation is considered as in Example 1, but the random perturbation $\varphi(t)$ may be a Wiener process with covariance function $\tilde{K}_\varphi(t,s) = \min(t,s)$.

Then $w(t_0)$ is also again a $N(0;1)$-variable (cf. [**9**]), and therefore

$$\mathrm{P}(\tilde{x}(t_0) - \mu\tilde{s}(t_0) < x(t_0) < \tilde{x}(t_0) + \mu\tilde{s}(t_0)) = 2\Phi(\mu) - 1$$

with

$$\tilde{s}^2(t_0) = t_0(-2t_0^2 + \frac{16}{3}t_0 + 1), \quad 0 < t_0 \leq 1.$$

For the comparison with Example 1 we give the standard deviation $\tilde{s}(t), t = 0.1(0.1)1$:

TABLE 2

0.39	0.60	0.85	1.06	1.26	1.44	1.62	1.79	1.94	2.08

3.2. *Random kernel.* Now it is assumed that the degenerate kernel contains a random part.

EXAMPLE 3. Given is a random integral equation (E_n) with

$$K(t,s) = t + \tilde{F} \cdot B(t), \qquad \tilde{\lambda} = 1, \qquad \tilde{b}(t) = -\sin t,$$

where $B(t,\omega)$ is the same random interpolation polynomial as in Example 1. Again, bounds for the probability (12) are wanted. Toward this end we again apply the algorithm given in 2.4.

1. The unperturbed solution is again

$$\tilde{x}(t) = \sin t + t \cdot \tilde{c}_1, \qquad \tilde{c}_1 = 2(1 - \cos(1)).$$

2. (SC1) is given with $n = 1$,

$$\tilde{\alpha}_1(t) = t, \quad \tilde{\beta}_1(s) = 1, \quad \delta(t) = \tilde{F} \cdot B(t,\omega) \text{ is Gaussian.}$$

Thus, the required quantities in Corollary 4 are

$$\sigma^2_{p(t)} = \tilde{c}_1^2 \cdot \tilde{F}^2 \cdot \sum_{k=0}^{m} (\tilde{L}_{mk}(t) + 2t\tilde{J}_{mk})^2$$

$$\tilde{J}_{mk} = \int_0^1 \tilde{L}_{mk}(t)\, dt$$

$$\sigma_q^2 = 4\tilde{F}^2 \cdot \sum_{k=0}^{m} \tilde{J}_{mk}^2$$

$$\operatorname{cov}(p(t), q) = -\tilde{c}_1 \cdot \tilde{F}^2 \left[2 \cdot \sum_{k=0}^{m} \tilde{J}_{mk} \tilde{L}_{mk}(t) + 4t \cdot \sum_{k=0}^{m} \tilde{J}_{mk}^2 \right].$$

The following table gives the "standard deviation" $\tilde{s}(t)$ of a linearized version of the centered solution process $z(t)$ for $t = 0(0.1)1$ and bounds for the probability $P_0 := \mathrm{P}(x(t_0) \in G_{01})$, see (12). The last mentioned values were calculated by means of Lemma 3. We write down the lower bound p and the error e of the inequality $p \le P_0 \le p{+}e$ for $t_0 = 0(0.1)1$.

TABLE 3

m=4:							
			$\tilde{F} = 0.20$				
$\tilde{s}(t)$:	0.1839	0.2379	0.2420	0.2019	0.2074	0.2289	0.2356
	0.2682	0.3380	0.3594	0.2873			
$\mu = 1.0$:							
p :	0.6724	0.6807	0.6834	0.6852	0.6815	0.6802	0.6841
	0.6886	0.6881	0.6876	0.6845			
e: 0.0000							
$\mu = 2.0$:							
p :	0.9341	0.9299	0.9279	0.9265	0.9293	0.9302	0.9273
	0.9236	0.9240	0.9244	0.9270			
e: 0.0000							
$\mu = 3.0$:							
p :	0.9879	0.9776	0.9748	0.9731	0.9767	0.9782	0.9741
	0.9701	0.9705	0.9709	0.9737			
e: 0.0000							
			$\tilde{F} = 0.05$				
$\tilde{s}(t)$:	0.0460	0.0595	0.0605	0.0505	0.0518	0.0572	0.0589
	0.0671	0.0845	0.0898	0.0718			
$\mu = 1.0$:							
p :	0.6820	0.6826	0.6827	0.6829	0.6826	0.6825	0.6828
	0.2831	0.6831	0.6830	0.6828			
e: 0.0000							

TABLE 3 (CONTD.)

$\mu = 2.0$:							
p :	0.9533	0.9528	0.9526	0.9525	0.9527	0.9528	0.9526
	0.9523	0.9524	0.9524	0.9526			
e: 0.0000							
$\mu = 3.0$:							
p :	0.9969	0.9961	0.9959	0.9957	0.9960	0.9962	0.9958
	0.9954	0.9954	0.9955	0.9958			
e: 0.0000							
m=14:							
			$\tilde{F} = 0.20$				
$\tilde{s}(t)$:	0.1839	1.9780	0.8038	0.8894	1.0154	1.4700	1.5589
	1.9726	2.4212	0.7301	2.7361			
$\mu = 1.0$:							
p :	0.2504	0.2874	0.2894	0.2916	0.2922	0.2930	0.2930
	0.2932	0.2932	0.2568	0.2934			
e: 0.7363							
$\mu = 2.0$:							
p :	0.2602	0.2694	0.2698	0.2703	0.2704	0.2705	0.2705
	0.2706	0.2706	0.2621	0.2706			
e: 0.7363							
$\mu = 3.0$:							
p :	0.2621	0.2662	0.2664	0.2666	0.2666	0.2667	0.2667
	0.2667	0.2667	0.2630	0.2667			
e: 0.7363							

TABLE 3 (CONTD.)

			$\tilde{F} = 0.05$				
$\tilde{s}(t)$:	0.0460	0.4945	0.2010	0.2224	0.2538	0.3675	0.3892
	0.4931	0.6053	0.1825	0.6840			
$\mu = 1.0$:							
p :	0.5780	0.7147	0.7167	0.7173	0.7173	0.7171	0.7171
	0.7171	0.7171	0.6036	0.7170			
e: 0.1779							
$\mu = 2.0$:							
p :	0.7361	0.7920	0.7914	0.7905	0.7902	0.7899	0.7898
	0.7897	0.7898	0.7532	0.7897			
e: 0.1779							
$\mu = 3.0$:							
p :	0.7809	0.8148	0.8155	0.8161	0.8162	0.8164	0.8164
	0.8164	0.8164	0.7898	0.8164			
e: 0.1779							

EXAMPLE 4. Given (E_n) with

$$K(t,s) = t \cdot (1 + \tilde{F} \cdot B(s)), \quad \tilde{\lambda} = 1, \quad \tilde{b}(t) = -\sin t,$$

$B(s,\omega)$ is the same random interpolation polynomial as in Example 1. Also, here, bounds for the probability (12) are wanted. As in Example 3 we use the algorithm from §2.4.

1. $\tilde{x}(t) = \sin t + t \cdot \tilde{c}_1$
2. (SC2) is given with $n = 1$,

$$\tilde{\alpha}_1(t) = t, \quad \tilde{\beta}_1(s) = 1, \qquad \epsilon(s) = \tilde{F} \cdot B(s,\omega) \text{ is Gaussian .}$$

Thus, the required quantities for Corollary 5 are

$$\sigma^2_{p(t)} = 4t^2 \cdot \tilde{F}^2 \cdot \sum_{k=0}^{m} \tilde{L}^2_{mk}, \qquad \tilde{L}_{mk} := \int_0^1 \tilde{L}_{mk}(s)\tilde{x}(s)\ ds,$$

$$\sigma^2_q = 4 \cdot \tilde{F}^2 \cdot \sum_{k=0}^{m} \tilde{I}^2_{mk}, \qquad \tilde{I}_{mk} := \int_0^1 s\tilde{L}_{mk}(s)\ ds$$

$$\operatorname{cov}(p(t), q) = -4t \cdot \tilde{F}^2 \cdot \sum_{k=0}^{m} (\tilde{L}_{mk} \cdot \tilde{I}_{mk}).$$

The following Table 4 contains also the "standard deviation" $\tilde{s}(t)$ of a linearized version of the centered solution process $z(t)$ for $t = 0(0.1)1$ and bounds for the probability $P_0 := \mathrm{P}(x(t_0) \in G_{01})$. The last mentioned values were likewise calculated by means of Lemma 3. But in this case there is only the necessity to write down one lower bound for all $t = 0(0.1)1$ because this bound does not depend on t as an easy examination shows.

TABLE 4

m=4:

$\tilde{F} = 0.20$

$\tilde{s}(t)$:	0.0000	0.0218	0.0436	0.0654	0.0873	0.1091	0.1309
	0.1527	0.1745	0.1963	0.2182			

	p	e
$\mu = 1.0$:	0.6861	0.0000
$\mu = 2.0$:	0.9424	0.0000
$\mu = 3.0$:	0.9864	0.0000

$\tilde{F} = 0.05$

$\tilde{s}(t)$:	0.0000	0.0055	0.0109	0.0164	0.0218	0.0273	0.0327
	0.0382	0.0436	0.0491	0.0545			

	p	e
$\mu = 1.0$:	0.6829	0.0000
$\mu = 2.0$:	0.9537	0.0000
$\mu = 3.0$:	0.9966	0.0000

m=14:

$\tilde{F} = 0.20$

$\tilde{s}(t)$:	0.0000	0.2661	0.5322	0.7983	1.0644	1.3305	1.5966
	1.8627	2.1288	2.3949	2.6610			

	p	e
$\mu = 1.0$:	0.6275	0.5101
$\mu = 2.0$:	0.5268	0.5101
$\mu = 3.0$:	0.5061	0.5101

$\tilde{F} = 0.05$

$\tilde{s}(t)$:	0.0000	0.0665	0.1330	0.1996	0.2661	0.3326	0.3991
	0.4657	0.5322	0.5987	0.6652			

	p	e
$\mu = 1.0$:	0.7122	0.0084
$\mu = 2.0$:	0.8722	0.0084
$\mu = 3.0$:	0.9197	0.0084

4. Concluding remarks. The considered examples show that the bounds developed for the one-dimensional distributions are applicable, and they are the sharper (closer) the smaller the random perturbation part is in comparison with the deterministic one. This statement is quantified in Example 3 and 4 by the ratio μ_q/σ_q. The magnitude of the random perturbation is controlled by the factor $\tilde{F}$. Further, if the arising random processes are interpolation polynomials, as described above, then, if m is equal to about 4, the interpolation has a smoothing effect while an increasing number of mesh points causes an unfavourable influence on the accuracy of the bounds. So, we got for $m = 14$ and $\tilde{F} = 0.20$ a decreasing (!) lower bound for $\mu = 1, 2, 3$. The reason for this bad behaviour is the nonlinearity of the function $t(z)$ in Lemma 3. Examples 3 and 4 were also handled without taking into consideration that special cases were given. We used the complete algorithm in 2.4 with steps 3 to 9. The bounds gained by this method were not so good, of course.

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