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VOLTERRA INTEGRAL INCLUSIONS IN BANACH SPACES

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ABSTRACT. We prove the existence of solutions for Volterra integral inclusions governed by a nonconvex valued multifunction, with values in a separable Banach space. Then we examine some special kinds of solutions (like extremal solutions and quasitrajectories). In doing that we also prove some other results of independent interests (like a generalization of a result of Kato on weak convergence in the Lebesgue-Bochner spaces L^p_X and a set valued version of Fatou's lemma).

1. Introduction. In several areas of applied mathematics, like control theory, mathematical economics, mechanics, etc., we encounter problems that involve various types of ambiguity, indeterminacy or uncertainty (which in particular includes the impossibility of a comprehensive description of the dynamics of the system). The evolution of such systems is then described by a multivalued equation (differential or integral). In recent years a number of papers have appeared concerning integral inclusions. In particular we mention the works Ragimhanov [**20**], Lyapin [**15**], Cuong [**5**], Angell [**1**] and the recent work of the author [**19**]. The first two are treating inclusions of the Hammerstein type, while the last three treat general convex valued Urysohn inclusions of the Volterra type (Angell [**1**] examines Volterra integral inclusions with delays). Integral inclusions (as well as differential inclusions) arise naturally in control theory when the method of deparametrization is applied and in the study of feedback control (see Aubin-Cellina [**2**]). A different application of integral inclusions can be found in two recent papers of Glashoff and Prekels [**9**], [**10**], who consider problems related to thermostatic regulation in which the heating devices controlling the temperature of the system are governed by a relay switch.

The purpose of this note is to investigate the existence of solutions for nonconvex valued Volterra integral inclusions defined in a separable Banach space and to examine some properties of the solutions.

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2. Preliminaries. Let (Ω, Σ, μ) be a complete, σ-finite measure space and X a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex)}\},$$
$$P_{(w)k(c)}(X) = \{A \subseteq X : \text{ nonempty, } (w) - \text{compact, (convex)}\}.$$

For $A \in 2^X \backslash \{\emptyset\}$, the norm $|A|$ of A and the support function $\sigma_A(\cdot)$ of A are defined by

$$|A| = \sup_{x \in A} ||x|| \text{ and } \sigma_A(x^*) = \sup_{x \in A}(x^*, x)$$

where $x^* \in X^*$ the topological dual of X. Also, by ∂A, we will denote the boundary of A and, by ext A, the extreme points of A. A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if it satisfies any of the following three equivalent statements.

(i) $GrG = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X.

(ii) $\omega \to d_{F(\omega)}(x) = \inf_{z \in F(\omega)} ||x - z||$ is measurable for all $x \in X$.

(iii) There exists $\{f_n(\cdot)\}_{n \geq 1}$ measurable functions such that, for all $\omega \in \Omega$, $F(\omega) = \text{cl}\,\{f_n(\omega)\}_{n \geq 1}$ (Castaing's representation).

Any multifunction (not necessarily closed valued) satisfying (i) is said to be graph measurable.

For more details the reader can look at Castaing-Valadier [**4**], Himmelberg [**12**] and Wagner [**22**].

For any multifunction $F : \Omega \to 2^X \backslash \{\emptyset\}$, let

$$S_F^1 = \{f(\cdot) \in L_X^1 \Omega \inf_{x \in F(w)} ||x|| \, f(w) \in F(w) \in F(\omega)\mu - \text{ a.e.}\}.$$

If $F(\cdot)$ is measurable, then S_F^1 is nonempty if and only if $\inf_{x \in F(\omega)} ||x|| \in L_+^1$. Also if $F(\cdot)$ is closed valued, then it is easy to see that S_F^1 is strongly closed in $L_X^1(\Omega)$. Using this set we can define a set valued integral for $F(\cdot)$ as

$$\int_\Omega F(\omega) d\mu(\omega) = \left\{ \int_\Omega f(\omega) d\mu(\omega) : f(\cdot) \in S_F^1 \right\},$$

where $\int_\Omega f(\omega)d\mu(\omega)$ is defined in the sense of Bochner. This multivalued integral is known as Aumann's integral. We will say that $F : \Omega \to 2^X \backslash \{\emptyset\}$ is integrably bounded if it is measurable and $|F(\cdot)|$is an L^1_+-function.

Suppose that Y, Z are Hausdorff topological spaces and $F : Y \to 2^Z \backslash \{\emptyset\}$. We say that $F(\cdot)$ is upper semicontinuous (u.s.c) (resp. lower semi-continuous (l.s.c.)) if, for all $V \subseteq Z$ open, the set $\{y \in Y : F(y) \subseteq V\}$ (resp. $\{y \in Y : F(y) \cap V \neq \emptyset\}$) is open.

Also if $\{A_n\}_{n\geq 1}$ is a sequence of nonempty subsets of X, we write $w - \overline{\lim}_{n\to\infty} A_n = \{x \in X : x = w - \lim_{n\to\infty} x_k, x_k \in A_{n_k}\}$ and $s \underline{\lim}_{n\to\infty} A_n = \{x \in X : x = s - \lim_{n\to\infty} x_n, x_n \in A_n\}$. We say that the A_n's converge to A in the Kuratowski-Mosco sense (denoted by $A_n \xrightarrow{K-M} A$) if and only if $w - \overline{\lim}_{n\to\infty} A_n = A = s - \underline{\lim}_{n\to\infty} A_n$. For details we refer to Mosco [**17**] and Salinetti-Wets [**21**].

Let $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}$. Recall that $w : \Delta \times \mathbf{R} \to \mathbf{R}_+$ is a Kamke function if it is a Caratheodory function, $w(t, s, 0) = 0$ and $u \equiv 0$ is the only solution of the problem $u(t) \leq \int_0^t w(t, s, u(s))ds, u(0) = 0$. Finally, by $\gamma(\cdot)$, we will denote the Hausdorff measure of noncompactness (see Kisielewicz [**14**]), by $h(\cdot, \cdot)$, the Hausdorff distance on $2^X \backslash \{\emptyset\}$, and, by X_w, the Banach space X with the weak topology.

3. Existence results. In this section we study the question of existence of solutions for the following integral inclusion of Volterra type.

$$(*) \qquad x(t) \in p(t) + \int_0^t K(t, s)F(s, x)(s))ds, \ t \in T$$

where $F(\cdot, \cdot)$ is a multifunction. By a solution of $(*)$ we understand a continuous function $x : T \to X$ such that there exists $f : T \to X$ measurable, $f(s) \in F(s, x(s))$ a.e., $\int_0^t K(t, s)f(s)ds$ exists and $x(t) = p(t) + \int_0^t K(t, s)f(s)ds$ for all $t \in T$. Until now all the existence theory for $(*)$ was developed under the assumption that $F(\cdot, \cdot)$ was a convex valued multifunction (see [**1, 5, 15, 19, 20**]). Our existence theorems concern nonconvex F's and we believe that this makes them more interesting and useful.

Throughout this paper (unless otherwise stated) X will be separable Banach space and $T = [0, b]$ a bounded, closed interval in $\mathbf{R}_+$. Let

$\Delta = \{(s,t) \in T \times T : 0 \leq s \leq t \leq b\}$ and $\mathcal{L}(X)$=bounded linear operators from X into itself.

THEOREM 3.1. *If* 1. $F : T \times X \to P_k(X)$ *is a multifunction such that*

(i) *for all* $x \in X, F(\cdot, x)$ *is measurable and* $|F(t,x)| \leq \psi(t)$ *a.e. with* $\psi(\cdot) \in L^1_+$,

(ii) *for all* $t \in T, F(t,\cdot)$ *is Hausdorff continuous;*

2. $K : \Delta \to \mathcal{L}(X)$ *and* $||K(t,s)|| \leq L$ *for all* $(t,s) \in \Delta$;

3. $\lim_{t'-t \to 0^+} [\int_t^{t'} ||K(t',s)||\psi(s)ds + \int_0^t ||K(t',s) - K(t,s)||\psi(s)ds] = 0$ *for* t' *or* t *fixed;*

4. $\gamma(K(t,s)F(s,B)) \leq w(t,s,\gamma(B))$ *for all* $0 \leq s \leq t \leq b$ *and* $B \subseteq X$ *nonempty bounded and where* $w(\cdot,\cdot,\cdot)$ *is a Kamke function; and*

5. $p(\cdot) \in C_X(T)$,

then $(*)$ *admits a solution.*

PROOF. From Theorem 3.1 of Kisielewicz [**14**] we know that there exists a continuous function $u : X \to L^1_X$ such that $u(x)(t) \in F(t,x)$ for all $x \in X$ and almost all $t \in T$. Set $f(t,x) = u(x)(t)$. Clearly $f(\cdot,\cdot)$ is a Caratheodory function. Now consider the single valued integral equation $x(t) = p(t) + \int_0^t K(t,s)f(s,x(s))ds$. Let $\Phi(\cdot)$ denote the nonlinear integral operator acting on $C_X(T)$ and defined by: $\Phi(x)(t) = p(t) + \int_0^t K(t,s)f(s,x(s))ds$. Let $x_n(\cdot) \xrightarrow{C_X} x(\cdot)$. Then we have

$$||\Phi(x_n) - \Phi(x)||_\infty \leq L \cdot \int_0^b ||f(s,x_n(s)) - f(s,x(s))||ds \to 0$$

as $n \to \infty$. So $\Phi(\cdot)$ is continuous.

Now consider the classical Caratheodory approximations

$$x_n(t) = \begin{cases} p(t) & \text{for } 0 \leq t \leq 1/n \\ p(t) + \int_0^{t-1/n} K(t-1/n,s)f(s,x_n(s))ds, & \text{for } 1/n \leq t \leq b \end{cases}.$$

Then we have

$$||x_n(t) - \Phi(x_n)(t)|| = ||\Phi(x_n)(t-1/n) - \Phi(x_n)(t)||, \text{ for } 1/n \leq t \leq b$$

and

$$||x_n(t) - \Phi(x_n)(t)|| = ||\int_0^t K(t,s)f(s,x_n(s))ds||$$
$$\leq \int_0^t ||K(t,s)||\psi(s)ds, \text{ for } 0 \leq t \leq 1/n.$$

Thus we get that $||x_n - \Phi(x_n)||_\infty \to 0$ as $n \to \infty$ and this shows that $\{(I - \Phi(x_n)\}_{n\geq 1}$ is relatively compact in $C_X(T)$. Let $M = \{x_n(\cdot)\}_{n\geq 1}$ and $M(t) = \{x_n(t)\}_{n\geq 1}$, $t \in T$. Note that $M \subseteq (I - \Phi)(M) + \Phi(M) \Rightarrow \gamma(M) \leq \gamma((I - \Phi)(M)) + \gamma(\Phi(M)) = \gamma(\Phi(M))$. Also, using Lemma 2.2 of Kisielewicz [**14**], we have

$$\gamma(\Phi(M(t))) \leq \int_0^t \gamma(K(t,s)f(s,M(s)))ds,$$

implying

$$\gamma(\Phi(M(t))) \leq \int_0^t w(t,s,\gamma(M(s)))ds$$

and hence

$$\gamma(M(t)) \leq \int_0^t w(t,s,\gamma(M(s)))ds.$$

Since $w(\cdot,\cdot,\cdot)$ is a Kamke function we have that $\gamma(M(t)) = 0$ for all $t \in T$. On the other hand it is easy to see that M is equicontinuous. So $\gamma(M) = \sup_{t\in T} \gamma(M(t)) = 0$. Hence M is relatively compact in $C_X(T)$. By passing to a subsequence if necessary we may assume that $x_n(\cdot) \xrightarrow{C_X} x(\cdot)$ as $n \to \infty$. Then we have

$$||x - \Phi(x)||_\infty \leq ||x - x_n||_\infty + ||x_n - \Phi(x_n)||_\infty + ||\Phi(x_n) - \Phi(x)||_\infty.$$

But $||x_n - x||_\infty \to 0$ and as we have already seen $||x_n - \Phi(x_n)||_\infty \to 0$. Furthermore since $\Phi(\cdot)$ is continuous we have that $||\Phi(x_n) - \Phi(x)||_\infty \to 0$. Finally we get that $||x - \Phi(x)||_\infty = 0 \Rightarrow x(t) = p(t) + \int_0^t K(t,s)f(s,x(s))ds$, and since $f(\cdot\,,\cdot)$ was a selector of $F(\cdot\,,\cdot)$ we conclude that $x(\cdot)$ solves $(*)$. □

By requiring that the Kamke function $w(t,s,x)$ is increasing in the x-variable, we can have the following stronger version of Theorem

3.1, in which the multifunction $F(\cdot,\cdot)$ satisfies a more general growth condition. For the next result we will assume that $F : T \times X \to P_k(X)$ satisfies (i′) and (ii), where

(i′) *for all* $x \in X, F(\cdot, x)$ *is measurable and* $|F(t,x)| \le a(t)||x|| + b(t)$ *a.e. with* $a(\cdot), b(\cdot) \in L^1_+$,

while $K : \Delta \to \mathcal{L}(X)$ satisfies (2) and

(3′) $\lim_{t'-t\to 0^+} \left(\int_t^{t'} ||K(t',s)||(a(s)+b(s))ds + \int_0^t ||K(t',s)-K(t,s)|| (a(s)+b(s))ds \right) = 0$

for t' *or* t *fixed.*

The rest of the hypotheses remain the same, with the addition that $w(t,s,x)$ is increasing in x.

THEOREM 3.2. *If hypotheses* 1(i′)(ii), (2), (3′), (4) *and* (5) *are satisfied then* $(*)$ *admits a solution.*

PROOF. Let $M = Ke^{L||a||_1}$ where $K = ||p||_\infty + L||b||_1$. Consider the new multifunction $F : T \times X \to P_k(X)$ defined by:

$$\hat{F}(t,x) = \begin{cases} F(t,x), & \text{if } ||x|| \le M \\ F(t, \frac{Mx}{||x||}), & \text{if } ||x|| > M \end{cases}$$

First note that, for all $x \in X$, we have that $|\hat{F}(t,x)| \le a(t)M + b(t) = \psi(t)$ a.e. and $\psi(\cdot) \in L^1_+$. Also $\hat{F}(t,\cdot) =$(For $)(t,\cdot)$, where $r(\cdot)$ is the M-radial contraction. But recall that $r(\cdot)$ is Lipschitz. Hence $\hat{F}(t,\cdot) =$(For)$(t,\cdot)$ is Hausdorff continuous. Clearly for all $x \in X, \hat{F}(\cdot,x)$ is measurable. Also note that, for $B \subseteq X$ nonempty and bounded, we have $\hat{F}(t,B) = F(t,r(B))$. So

$$\gamma(K(t,s)\hat{F}(s,B)) = \gamma(K(t,s)F(s,r(B))) \le w(t,s,\gamma(r(B))).$$

But $r(B) \subseteq \overline{\text{conv}}(\{0\} \cup B) \Rightarrow \gamma(\overline{\text{conv}}(\{0\} \cup B)) = \gamma(\{0\} \cup B) = \gamma(B)$. Exploiting the monotonicity of $w(t,s,\cdot)$ we finally get that

$$\gamma(K(t,s)\hat{F}(s,B)) \le w(t,s,\gamma(B)), \quad \text{for all } (t,s) \in \Delta.$$

So we see that $\hat{F}(\cdot,\cdot)$ is a multifunction that satisfies all the hypotheses of Theorem 3.1. Applying that result we deduce that there exists $x(\cdot) \in C_X(T)$ such that $x(t) \in p(t)+\int_0^t K(t,s)\hat{F}(s,x(s))ds$ for all $t \in T$. Then

$$||x(t)|| \leq ||p(t)|| + \int_0^t ||K(t,s)||[a(s)||x(s)|| + b(s)]ds,$$

hence

$$||x(t)|| \leq ||p||_\infty + L||b||_1 + L\int_0^t a(s)||x(s)||ds = K + L\int_0^t a(s)||x(s)||ds.$$

Applying Gronwall's inequality we get that

$$||x(t)|| \leq Ke^{L||a||_1} = M$$

so that

$$x(\cdot) \text{ solves } (*).$$

The next existence result involves nonconvex valued lower semicontinuous multifunction. Such multifunctions appear often in control theory, in particular in connection with the bang-bang principle.

THEOREM 3.3. *If* 1. $F : T \times X \to P_f(X)$ *is a multifunction such that*

(i) $F(\cdot,\cdot)$ *is graph measurable and for all* $x \in X, F(t,x) \subseteq G(t)$ *a.e. where* $G : T \to P_{wck}(X)$ *is integrably bounded,*

(ii) *for all* $t \in T, F(t,\cdot)$ *is l.s.c. from* X_w *into* X*;*

2. $K : \Delta \to \mathcal{L}(X)$ *and* $||K(t,s)|| \leq L$ *for all* $(t,s) \in \Delta$*;*

3. $\lim_{t'-t\to 0^+}(\int_0^{t'} ||K(t',s) - K(t,s)||\,|G(S)|ds + \int_t^{t'} ||K(t',s)||\,|G(s)|\,ds) = 0$ *for fixed* t' *or* t*; and*

4. $p(\cdot) \in C_X(T)$,
then $(*)$ *admits a solution.*

PROOF. Consider the set

$$W = \left\{ z(\cdot) \in C_X(T) : z(t) = p(t) + \int_0^t K(t,s)g(s)ds \right.$$
$$\left. \text{for some } g \in S_G^1 \text{ and for } t \in T \right\}.$$

We claim that W is a compact subset of $C_{X_w}(T)$. Let $t',t \in T, t' > t$.

$$\begin{aligned}||x(t') - x(t)|| &\leq ||p(t') - p(t)|| + || \int_0^{t'} K(t',s)g(s)ds - \int_0^t K(t,s)g(s)ds|| \\ &\leq ||p(t') - p(t)|| + \int_t^{t'} ||K(t',s)||\,|G(s)|ds + \int_0^t ||K(t',s) \\ &\quad - K(t,s)||\,|G(s)|ds.\end{aligned}$$

Using hypotheses (3) and (4) we get that $||x(t')-x(t)|| \to 0$ as $t'-t \to 0^+$ uniformly in $x(\cdot) \in W$. Thus W is equicontinuous and a fortiori w-equicontinuous. Next we will show that W is closed in $C_{X_w}(T)$. For that purpose let $x_a(\cdot) \xrightarrow{C_{X_w}} x(\cdot), x_a(\cdot) \in W$. Then by definition we have that, for all $t \in T, x_a(t) = p(t) + \int_0^t K(t,s)g_a(s)ds$. But $\{g_a(\cdot)\} \subseteq S_G^1$ and the latter is w-compact in $L_X^1(T)$ (see Proposition 3.1 of [**18**]). So by passing to a subnet if necessary we may assume that $g_a(\cdot) \xrightarrow{w-L_X^1} g(\cdot) \in S_G^1$. Hence $\int_0^t K(t,s)g_a(s)ds \xrightarrow{w} \int_0^t K(t,s)g(s)ds$ so that $x_a(t) \to p(t) + \int_0^t K(t,s)g(s)ds$. But we already know that $x_a(t) \xrightarrow{w} x(t)$. Therefore $x(t) = p(t) + \int_0^t K(t,s)g(s)ds$ and $x(\cdot) \in W$, implying W is closed in $C_{X_w}(T)$. So until now we have that W is a closed, equicontinuous subset of $C_{X_w}(T)$. Furthermore, for all $z(\cdot) \in W$ and all $t \in T$, we have $z(t) \in p(t) + \int_0^t K(t,s)G(s)ds \in P_{wkc}(X)$. Thus invoking the Arzela-Ascoli theorem we deduce that W is a compact subset of $C_{X_w}(T)$.

Next note that $L_X^1(T)$ is separable and so S_G^1, being a w-compact subset of $L_X^1(T)$, is metrizable with the weak topology (see Dunford-Schwartz [**7**; Theorem 3, p. 434]). Since W is isomorphic to $\{p(\cdot)\} \times S_G^1$, we conclude that W is also metrizable.

Now consider the multifunctions $R : W \to P_f(L_X^1)$ defined by $R(x) = S^1_{F(\cdot,x(\cdot))}$. Our claim is that $R(\cdot)$ is a l.s.c. multifunction from W with the C_{X_w}-topology into L_X^1 with the strong topology. From the theory of multifunctions we know that it suffices to show that, for all $x_n(\cdot) \to x(\cdot)$ in $W, R(x) \subseteq s-\underline{\lim}\, R(x_n)$. So let $f(\cdot) \in R(x) = S^1_{F(\cdot,x(\cdot))}$. Consider the following multifunctions $P_n(t) = \{z \in F(t,x_n(t)) : ||z - f(t)|| \leq d_{F(t,x_n(t))}(f(t)) + 1/n\}$. Because $F(\cdot,\cdot)$ is graph measurable, $t \to d_{F(t,x_n(t))}(f(t))$ is measurable and so $GrP_n \in \Sigma \times B(X)$. Applying Aumann's selection theorem (see [**12**. Theorem 5.2]) we get $f_n : T \to X$ measurable such that $f_n(t) \in P_n(t)$

for all $t \in T$. Then $||f_n(t) - f(t)|| \leq d_{F(t,x_n(t))}(f(t)) + 1/n$. But since $F(t, \cdot)$ is l.s.c. from X_w into X, we have $f(t, x(t)) \subseteq s - \underline{\lim}\, F(t, x_n(t))$ for all $t \in T$ and so $d_{F(t,x_n(t))}(f(t)) \to 0$ as $n \to \infty$. Hence $||f_n(t) - f(t)|| \to 0$ so that $f_n(\cdot) \overset{s-L^1_X}{\to} f(\cdot)$. Thus $f(\cdot) \in s - \underline{\lim}\, R(x_n)$ and $R(\cdot)$ is l.s.c. as claimed. Now apply Theorem 3.1 of Fryszkowski [**8**] to get $r : W \to L^1_X$ continuous such that, for all $x(\cdot) \in W, r(x) \in R(x)$. Set $\lambda(x)(t) = p(t) + \int_0^t K(t,s)r(x)ds$. It is easy to see that $\lambda(x)(\cdot) \in W$. So $\lambda : W \to W$. We claim that it is continuous. So let $x_n(\cdot) \overset{C_{X_w}}{\to} x(\cdot)$, i.e., $r(x_n) \overset{s-L^1_X}{\to} r(x)$. Then

$$\begin{aligned} ||\lambda(x_n)(t) - \lambda(x)(t)|| &= || \int_0^t K(t,s)r(x_n)(s)ds - \int_0^t K(t,s)r(x)(s)ds|| \\ &\leq \int_0^t ||K(t,s)||\, ||r(x_n)(s) - r(x)(s)||ds \\ &\leq L \int_0^b ||r(x_n)(s) - r(x)(s)||ds. \end{aligned}$$

Hence $||\lambda(x_n) - \lambda(x)||_\infty \to 0$ as $n \to \infty$ and $\lambda(\cdot)$ is continuous on W.

Apply Tichonoff's fixed point theorem to get $\hat{x}(\cdot) \in W$ such that $\hat{x} = \lambda(\hat{x})$. Then $\hat{x}(t) = p(t) = \int_0^t K(t,s)r(\hat{x})(s)ds \in p(t) + \int_0^t K(t,s)F(s,\hat{x}(s))ds$, implying $\hat{x}(\cdot)$ solves $(*)$.

4. Properties of the solutions. In [**23**] Wazewski introduced the important notion of quasitrajectory, a weak solution to certain differential inclusion. It can be shown that the limit functions of some sequences of trajectories with "bang-bang" controls are quasitrajectories. Such functions are known in control theory as "sliding regimes". Here we extend the concept of quasitrajectory to integral inclusions. So we say that a continuous function $x : T \to X$ is a quasitrajectory if and only if there exists a sequence $\{x_n(\cdot)\}_{n\geq 1}$ such that $x_n(t) = p(t) + \int_0^t K(t,s)f_n(s)ds, \{f_n\}_{n\geq 1}$ is relatively w-compact in $L^1_X, x_n(t) \overset{s}{\to} x(t)$ for all $t \in T$ and $d_{F(t,x_n(t))}(f_n(t)) \to 0$ a.e. as $n \to \infty$. Also we say that $x(\cdot) \in C_X(T)$ is a generalized solution of $(*)$ if $x(t) \in p(t) + \text{cl} \int_0^t K(t,s)F(s,x(s))ds$ for all $t \in T$.

The next result provides a relation between quasitrajectories and generalized solution (or solutions for the convex valued or finite dimensional cases). But first we need some auxiliary results, which are in fact important in their own. The first is a result concerning the $w - \overline{\lim}$ of a sequence of nonempty sets in a Banach space X.

PROPOSITION 4.1. *If, for all* $n \geq 1, A_n \subseteq G$ *and* $A_n, G \in P_{wk}(x)$, *then, for all* $x^* \in X^*, \overline{\lim}\, \sigma_{A_n}(x^*) \leq \sigma_{w-\overline{\lim}\, A_n}(x^*)$.

PROOF. Fix $x^* \in X^*$. Note that because the sets are w-compact, for every $n \geq 1$ there exists $x_n \in A_n$ such that $(x^*, x_n) = \sigma_{A_n}(x^*)$. Let $\{x_k\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$ such that $(x^*, x_k) \to \overline{\lim}\, \sigma_{A_n}(x^*)$. Also, since $\{x_k\}_{k\geq 1} \subseteq G$, and invoking the Eberlein-Smulian theorem, we may assume that $x_k \xrightarrow{w} x$. Then successively, $x \in w - \overline{\lim}\, A_n, (x^*, x) \leq \sigma_{w-\overline{\lim}\, A_n}(x^*), \overline{\lim}\, \sigma_{A_n}(x^*) \leq \sigma_{w-\overline{\lim}\, A_n}(x^*)$.
□

The next result generalizes significantly an earlier result of Kato [**13**] and can have important applications in various areas of applied mathematics. We emphasize that X need not be reflexive with X^* uniformly convex, p can be 1 and $f_n(\cdot)$ are not uniformly bounded. So let (Ω, Σ, μ) be a σ-finite measure space, X any Banach space and $p \geq 1$ $p < \infty$.

PROPOSITION 4.2. *If* $f_n(\cdot) \xrightarrow{w-L^p_X} f(\cdot)$ $f_n(w) \in G(w)\mu-$ *a.e. with* $G(w) \in P_{wk}(x)\mu-$ *a.e., then* $f(\omega) \in \overline{\text{conv}}\, w - \overline{\lim}\{f_n(\omega)\}_{n\geq 1}\mu$*-a.e.*

PROOF. From Mazur's lemma we have that, for all $k \geq 1, f(\omega) \in \overline{\text{conv}}\, \cup_{n\geq k} f_n(\omega)\mu$ a.e. Let $x^* \in X^*$. Then we have

$$\begin{aligned}(x^*, f(\omega)) &\leq \sigma_{\overline{\text{conv}} \cup_{n\geq k} f_n(\omega)}(x^*) \\ &= \sigma_{\cup_{n\geq k} f_n(\omega)}(x^*) = \sup_{n\geq k}(x^*, f_n(\omega))\mu - \text{ a.e.}\end{aligned}$$

and

$$(x^*, f(\omega)) \leq \overline{\lim}(x^*, f_n(\omega)) = \overline{\lim}\, \sigma_{\{f_n(\omega)\}}(x^*)\mu - \text{ a.e.}$$

Using Proposition 4.1 we get that

$$\begin{aligned}(x^*, f(\omega)) \leq \sigma_{w-\overline{\lim}\{f_n(\omega)\}_{n\geq 1}}(x^*)\mu - \text{ a.e. }, f(\omega) \\ \in \overline{\text{conv}} w - \overline{\lim}\{f_n(\omega)\}_{n\geq 1}\mu - \text{ a.e.}\end{aligned}$$

This leads us to the following, set valued version of Fatou's lemma. Now (Ω, Σ, μ) is a nonatomic, σ-finite measure space and X a separable Banach space.

PROPOSITION 4.3. *If* $F_n : \Omega \to 2^X \setminus \{\emptyset\}$ *are graph measurable and, for all* $n \geq 1$, $F_n(\omega) \subseteq G(\omega)\mu$*-a.e. where* $G : \Omega \to P_{wkc}(X)$ *is integrably bounded, then* $w - \overline{\lim} \int_\Omega (\omega) d\mu(\omega) \subseteq \int_\Omega \overline{\text{conv}}\, w - \overline{\lim} F_n(\omega) d\mu(\omega)$. *Furthermore if* $\omega \to w - \overline{\lim} F_n(\omega)$ *is graph measurable, then* $w - \overline{\lim} \int_\Omega F_n(\omega) d\mu(\omega) \subseteq \int_\Omega w - \overline{\lim} F_n(\omega) d\mu(\omega)$.

PROOF. Let $x \in w - \overline{\lim} \int_\Omega F_n(\omega) d\mu(\omega)$. Then we can find $x_k \in \int_\Omega F_{n_k}(\omega) d\mu(\omega)$ such that $x_k \xrightarrow{w} x$. From the definition of the Aumann integral we know that $x_k = \int_\Omega f_k(\omega) d\mu(\omega)$ where $f_k(\cdot) \in S^1_{F_{n_k}}$. Since $S^1_{F_{n_k}} \subseteq S^1_G$ and the latter is w-compact (see [**18**]), thanks to the Eberlein-Smulian theorem we can assume that $f_k(\cdot) \xrightarrow{w-L^1_X} f(\cdot)$. Hence $x = \int_\Omega f(\omega) d\mu(\omega)$. But from Proposition 4.2 we know that $f(\omega) \in \overline{\text{conv}}\, w - \overline{\lim} \{f_n(\omega)\}_{n \geq 1} \subseteq \overline{\text{conv}}\, w - \overline{\lim} F_n(\omega) \mu$-a.e., hence $x \in \int_\Omega \overline{\text{conv}}\, w - \overline{\lim} F_n(\omega) d\mu(\omega)$, so that $w - \overline{\lim} \int_\Omega F_n(\omega) d\mu(\omega) \subseteq \int_\Omega \overline{\text{conv}}\, w - \overline{\lim} F_n(\omega) d\mu(\omega)$. Finally if $\omega \to w - \overline{\lim} F_n(\omega)$ is graph measurable, then, because of the nonatomicity of $\mu(\cdot)$, we have that $\int_\Omega \overline{\text{conv}}\, w - \overline{\lim} F_n(\omega) d\mu(\omega) = \text{cl} \int_\Omega w - \overline{\lim} F_n(\omega) d\mu(\omega)$.

THEOREM 4.1. *Let* 1. $F : T \times X \to P_f(X)$ *be a multifunction such that*

(i) $F(\cdot, \cdot)$ *is graph measurable and for all* $x \in X, F(t, x) \subseteq G(t)$ *a.e. where* $G : T \to P_{wkc}(X)$ *is integrably bounded and*

(ii) *for all* $t \in T, F(t, \cdot)$ *is u.s.c. from* X_w *into* X_w;

2. $K : \Delta \to \mathcal{L}(X)$ *and for all* $t \in T, ||K(t, \cdot)|| \in L^\infty_+$; *and*

3. *for all* $t \in T, (s, x) \to K(t, s)F(t, x)$ *is graph measurable on* $\{0 \leq s \leq t\} \times X$.

Then every quasitrajectory is a generalized solution of $(*)$. *If, in addition,* $F(\cdot, \cdot)$*is convex valued or* X *is finite dimensional (in which cases hypothesis* (3) *is automatically satisfied) then every quasitrajectory is a solution of* $(*)$.

PROOF. From Egoroff's theorem we know that we can find measurable sets $T_k \subseteq T$ such that $T' = \cup_{k\geq 1} T_k, \lambda(T\backslash T') = 0$ (λ =Lebesgue measure on $\mathbf{R}$), $T_k \cap T_\ell = \emptyset$ for $k \neq \ell$, and $d_{F(t,x_n(t))}(f_n(t)) \to 0$ uniformly on $T_k, k \geq 1$. Fix $k \geq 1$, otherwise arbitrary. Then, given $\varepsilon > 0$, we can find $n_0(\varepsilon)$ such that, for $n \geq n_0(\varepsilon)$, we have $f_n(t) \in F(t, x_n(t)) + B_1$ for all $t \in T_k$, where B_1 is the unit ball in X. Let $s < t'$. Then

$$K(t',t)f_n(t) \in K(t',t)F(t,x_n(t)) + \varepsilon K(t',t)B_1, \quad t \in T_k, \quad t \leq t'$$

implies

$$\begin{aligned}\int_{[s,t']\cap T_k} K(t',t)f_n(t)dt \in \int_{[s,t']\cap T_k} & K(t',t)F(t,x_n(t))dt \\ & + \int_{[s,t']\cap T_k} \in K(t',t)B_1 dt.\end{aligned}$$

Recalling that $\{f_n(\cdot)\}_{n\geq 1}$ is relatively w-compact in L^1_X, we may assume that $f_n(\cdot) \stackrel{w-L^1_X}{\to} f(\cdot) \in S^1_G$. Then $\int_{[s,t']\cap T_k} K(t',t)f_n(t)dt \stackrel{w}{\to} \int_{[s,t']\cap T_k} K(t',t)f(t)dt$ and so $\int_{[s,t']\cap T_k} K(t',t)f(t)dt \in w - \overline{\lim} \int_{[s,t']\cap T_k} K(t',t) F(t,x_n(t))dt + \varepsilon \int_{[s,t']\cap T_k} K(t',t)B_1\, dt$, for all $\varepsilon > 0$. Let $\varepsilon \downarrow 0$; we get that $\int_{[s,t']\cap T_k} K(t',t)f(t)dt \in w - \overline{\lim} \int_{[s,t']\cap T_k} K(t',t)F(t,x_n(t))dt$. From Proposition 4.2 we have that

$$\begin{aligned} w - \overline{\lim} & \int_{[s,t']\cap T_k} K(t',t)F(t,x_n(t))dt \\ \subseteq & \int_{[s,t']\cap T_k} \overline{\text{conv}}\, w - \overline{\lim}\, K(t',t)F(t,x_n(t))dt \end{aligned}$$

while, from Proposition 4.1 and the fact that $F(t,\cdot)$ is u.s.c. from X_w into X_w, we have that $\overline{\text{conv}}\, w - \overline{\lim}\, K(t',t)F(t,x_n(t)) \subseteq \overline{\text{conv}}\, K(t',t) F\ (t,x(t))$. So

$$\begin{aligned}\int_{[s,t']\cap T_k} K(t',t)f(t)dt \in & \int_{[s,t']\cap T_k} \overline{\text{conv}}\, K(t',t)F(t,x(t))dt \\ & = \text{cl} \int_{[s,t']\cap T_k} K(t',t)F(t,x(t))dt.\end{aligned}$$

But recall that the Aumann integral is a set valued measure and $k \geq 1$ was arbitrary. So we find successively

$$\int_s^{t'} K(t',t)f(t)dt \in \mathrm{cl} \int_s^{t'} K(t',t)F(t,x(t))dt,$$

$$p(t') + \int_0^{t'} K(t',t)f(t)dt \in p(t') + \mathrm{cl} \int_0^{t'} K(t',t)F(t,x(t))dt,$$

$$x(t') \in p(t') + \mathrm{cl} \int_0^{t'} K(t',t)F(t,x(t))dt$$

for all $t' \in T$, and so $x(\cdot)$ is a generalized solution. Finally, if $F(\cdot,\cdot)$ is convex valued or X is finite dimensional, then $\int_0^{t'} K(t',t)F(t,x(t))dt$ is closed (see [**18, 11**]) and so $x(\cdot)$ is a solution of $(*)$.

Another special kind of trajectories are the bang-bang solutions. Motivated from control theory we will call a solution $x(\cdot)$ of $(*)$, a "bang-bang solution" (or "extremal solution") if and only if, for all $t \in T$,

$$x(t) \in p(t) = \int_0^t K(t,s)\partial F(t,x(s))ds.$$

We will prove that the family of this solutions is closed under pointwise convergence. But first we need the following auxiliary result. Let X be any Banach space.

PROPOSITION 4.4. *If* $\{B_n\}_{n\geq 1} \in P_{fc}(X)$ *are bounded,* $B_n \xrightarrow{h} B$ *and* $K \in \mathcal{L}(X)$*, then* $K(B_n) \xrightarrow{h} K(B)$.

PROOF. Using Hörmander's formula we have that

$$\begin{aligned} h(K(B_n),K(B)) &= \sup_{\|x^*\|\leq 1} |\sigma_{K(B_n)}(x^*) - \sigma_{K(B)}(x^*)| \\ &= \sup_{\|x^*\|\leq 1} \left| \|K^*\|\sigma_{B_n}\left(\frac{K^*x^*}{\|K^*\|}\right) - \|K^*\|\sigma_B\left(\frac{K^*x^*}{\|K^*\|}\right)\right| \\ &\leq \|K^*\|h(B_n,B) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now we are ready for our stability result concerning bang-bang solutions. Here X is a separable Banach space. □

THEOREM 4.2. *Let* 1. $F : T \times X \to P_{fc}(X)$ *be a multifunction with bounded values such that*

(i) *for all* $x \in X, F(\cdot, x)$ *is measurable and* $|F(t,x)| \leq \varphi(t)$ *a.e. with* $\varphi(\cdot) \in L^1_+$

(ii) *for all* $t \in T, F(t, \cdot)$ *is h-continuous;*

2. $K : \Delta \to \mathcal{L}(X)$ *and for all* $t \in T, ||K(t, \cdot)|| \in L^\infty_+$; *and*

3. $\{x_n(\cdot)\}_{n\geq 1}$ *are bang-bang solutions and* $x_n(t) \xrightarrow{s} x(t)$ *for all* $t \in T$.

Then $x(\cdot)$ *is a bang-bang solution, too.*

PROOF. By hypothesis, for every $n \geq 1$ and every $t \in T$, we have that

$$x_n(t) \in p(t) + \int_0^t K(t,s)\partial F(s, x_n(s))ds.$$

Also it is well known (see for example DeBlasi-Pianigiani [**6**]) that since $F(s, \cdot)$ is h-continuous, $\partial F(s, \cdot)$ is h-continuous too. So we have that $\partial F(s, x_n(s)) \xrightarrow{h} \partial F(s, x(s))$ for all $s \in T$. Then

$$h\left(\int_0^t K(t,s)\partial F(s,x_n(s))ds, \int_0^t K(t,s)\partial F(s,x(s))ds\right)$$
$$\leq \int_0^t h(K(t,s)\partial F(s,x_n(s)), K(t,s)\partial F(s,x(s)))ds$$

and using Proposition 4.4, we have that

$$\lim_{n\to\infty} h\Big(\int_0^t K(t,s)\partial F(s,x_n(s))ds, \int_0^t K(t,s)\partial F(s,x(s))ds\Big) = 0.$$

But $x_n(t) - p(t) \in \int_0^t K(t,s)\partial F(s, x_n(s))ds$ and $x_n(t) - p(t) \xrightarrow{s} x(t) - p(t)$. So $x(t) \in p(t) + \int_0^t K(t,s)\partial F(s,x(s)) \Rightarrow x(\cdot)$ is a bang-bang solution, too.

The next result is also closely related to the notion of bang-bang control and was also motivated by control problems.

We consider the following two integral inclusions:

$$(**) \qquad x(t) \in p(t) + \int_0^t K(t,s)f(s,x(s))L(s)ds$$

and

$$(**_e) \qquad x(t) \in p(t) + \int_0^t K(t,s)f(s,x(s))\operatorname{ext} L(s)ds.$$

Let S be the solution set of $(**)$ and S_e the solution set of $(**_e)$. An important problem is to find the relation between those two sets. The next theorem provides an answer to this question. Assume that X is finite dimensional.

THEOREM 4.3. *If* 1. $p(\cdot) \in C_X(T)$,

2. $f : T \times X \to \mathbf{R}$ *is measurable in* t, *locally Lipschitz in* x *and* L^1*-bounded,*

3. $L : T \to P_{fc}(X)$ *is integrably bounded,*

4. $K : \Delta \to \mathcal{L}(X)$ *and for all* $t \in T \|K(t,\cdot)\| \in L_+^\infty$,

5. *for all* $0 \le s \le t \le t' \le b$ *and all* $x \in X$,

$$|K(t,s)f(s,x)L(s)| \le m(t,s) \quad \text{with } \sup_{t\in T}\int_0^t m(t,s) \le +\infty$$

and $|(K(t',s) - K(t,s)f(s,x)L(s)| \le n(t',t,s)$, *where*

$$\lim_{t'-t\to 0^+}\left[\int_t^{t'} m(t',s)ds + \int_0^t n(t',t,s)ds\right] = 0 \text{ for } t' \text{ or } t \text{ fixed,}$$

then S_e *is dense in* S.

PROOF. From Benamara [**3**], we know that $t \to \operatorname{ext} L(t)$ has a measurable graph. So $S^1_{\operatorname{ext} L} \neq \emptyset$. Let $\ell(\cdot) \in S^1_{\operatorname{ext} L}$ and consider the point valued integral equation

$$x(t) = p(t) + \int_0^t K(t,s)f(s,x(s))\ell(s)ds.$$

From Theorem 1.1 (p. 87) of Miller [**16**] we know that the above equation admits a unique solution $x_{\mathcal{L}}(\cdot)$. Consider the map $\ell(\cdot) \to x_{\mathcal{L}}(\cdot)$. Recall that $S^1_{\operatorname{ext} L} = \operatorname{ext} S^1_L$ is w-dense in S^1_L (see [**4**]). So it suffices to show that $\ell(\cdot) \to x_\ell(\cdot)$ is continuous from (S^1_L, w) into $C_X(T)$.

For that purpose let $\{\ell_n, \ell\}_{n\geq 1} \subseteq S^1_L$ be such that $\ell_n \overset{w-L^1_X}{\rightarrow} \ell$. Let $\{x_n(\cdot), x(\cdot)\}_{n\geq 1}$ be the corresponding solutions of the integral equation. Then

$$x_n(t) = p(t) + \int_0^t K(t,s)f(s, x_n(s))\ell_n(s)ds.$$

First note that

$$\sup_{n\geq 1} ||x_n(t)|| \leq ||p||_\infty + \sup_{t\in T} \int_0^t m(t,s)ds < +\infty,$$

hence $\{x_n(\cdot)\}_{n\geq 1}$ is equibounded.

Also, for $t', t \in T, t < t'$, we have

$$\begin{aligned} ||x_n(t') - x_n(t)|| \leq & ||p(t') - p(t)|| \\ & + \int_0^t ||(K(t',s) - K(t,s))f(s, x_n(s))\mathcal{L}_n(s)||ds \\ & + \int_t^{t'} ||K(t',s)f(s, x_n(s))\mathcal{L}_n(s)||ds \\ & \leq ||p(t') - p(t)|| + \int_0^t n(t',t,s)ds + \int_t^{t'} m(t,s)ds \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ and $\{x_n(\cdot)\}_{n\geq 1}$is equicontinuous.

Invoking the Arzela-Ascoli theorem we deduce that $\{x_n(\cdot)\}_{n\geq 1}$ is relatively compact in $C_X(T)$. So by passing to a subsequence if necessary we may assume that $x_n(\cdot) \overset{C_X(T)}{\rightarrow} \hat{x}(\cdot)$. Also, since $\ell_n(\cdot) \overset{w-L^1_X}{\rightarrow} \ell(\cdot)$ and $f(s, x_n(s)) \rightarrow f(s, x(s))$, we have that $x_n(t) = p(t) + \int_0^t K(t,s)f(s, x_n(s))$ $l_n(s)ds \rightarrow p(t) + \int_0^t K(t,s)f(s,x(s))\ell(s)ds = x(t)$. Hence $x(t) = \hat{x}(t)$ for all $t \in T$ so that $\ell(\cdot) \rightarrow x_\ell(\cdot)$ is continuous, and S_e is dense in S for the $C_X(T)$-topology. □

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