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THE DISCRETE GALERKIN METHOD FOR NONLINEAR INTEGRAL EQUATIONS (*)

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ABSTRACT. Let K be a completely continuous nonlinear integral operator, and consider solving $x = K(x)$ by Galerkin's method. This can be written as $x_n = P_n K(x_n)$, P_n an orthogonal projection; the iterated Galerkin solution is defined by $\tilde{x}_n = K(x_n)$. We give a general framework and error analysis for the numerical method that results from replacing all integrals in Galerkin's method with numerical integrals. A special high order formula is given for integral equations arising from solving nonlinear two-point boundary value problems.

1. Introduction. Consider the problem of solving the nonlinear Urysohn integral equation

$$x(t) = \int_\Omega K(t, s, x(s))ds, \quad t \in \Omega. \tag{1.1}$$

Denoting this equation by

$$x = K(x), \tag{1.2}$$

we assume that K is a completely continuous operator from an open set $D \subset L^\infty(\Omega)$ into $C(\Omega)$, with Ω a set in $\mathbf{R}^m$, some $m \geq 1$. We will analyze the use of the discretized Galerkin method to solve for the fixed points x^* of K.

Let S_h denote a finite dimensional approximating subspace of $L^\infty(\Omega)$, with h the discretization parameter. The Galerkin method for solving (1.2) is to find the element $x_h \in S_h$ for which

$$(x_h, \psi) = (K(x_h), \psi), \text{ all } \psi \in S_h. \tag{1.3}$$

This is a well-analyzed method with a large literature; for example, see Krasnoselskii (1964), Krasnoselskii-Vainikko, et al. (1972), and

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Krasnoselskii-Zabreiko (1984). Recently, we have given a more detailed analysis of the Galerkin method, in [**6**], and some of those results will be referred to in the following.

Assuming S_h is also a subspace of $L^2(\Omega)$, let P_h be the orthogonal projection of $L^2(\Omega)$ onto S_h. Then (1.3) can be rewritten as

$$x_h = P_h \mathcal{K}(x_h), \ x_h \in L^\infty(\Omega). \tag{1.4}$$

After obtaining the approximation x_h to the desired solution x^*, define

$$\tilde{x}_h = \mathcal{K}(x_h), \tag{1.5}$$

which is called the *iterated Galerkin solution.* For the approximating properties of S_h, we assume

$$P_h x \to x, \text{ as } h \to 0, \tag{1.6}$$

for all $x \in C(\Omega)$.

Assuming $[I - \mathcal{K}'(x^*)]^{-1}$ exists and is bounded, we can show the existence of x_h for all sufficiently small h, along with its convergence to x^*. In particular, it can be shown that

$$\| x^* - x_h \| \leq c \| x^* - P_h x \| \tag{1.7}$$

$$\begin{aligned} \| x^* - \tilde{x}_h \| \leq c \| x^* - P_h x \| \\ \cdot \text{Max} \{ \| x^* - P_h x \|, \| [I - P_h] \mathcal{K}'(x^*)^* \| \} \end{aligned} \tag{1.8}$$

where c is a generic constant. This shows the superconvergence of $\tilde{x}_h$ to x^*, as compared to that of x_h to x^*. More details on the resulting speed of convergence are given in [**6**].

The numerical scheme (1.3) is implemented by letting ψ run through a basis of S_h. The resulting nonlinear system will involve many integrals, both inner products and the integral operator $\mathcal{K}$. When these are approximated numerically, a new numerical method results. We will analyze that method, to see when the results (1.7)-(1.8) are still valid for the solutions obtained from the discretized nonlinear system.

In the next section we will present an abstract framework within which the discrete Galerkin methods can be analyzed. §3 contains some

general error estimates for integral equations of the form (1.1), which can be used to obtain the actual rates of convergence of the discrete Galerkin and the discrete iterated Galerkin methods for a large class of problems. In §4 we obtain more specific results for the case where Ω is a compact one-dimensional interval and the kernel $K(t, s, u)$ belongs to the class $\mathcal{G}_2(\alpha, \gamma)$ defined in [**6**]. Finally in §5, we give some numerical examples which illustrate the theory.

We note that the results of this paper generalize some previous results obtained in the linear case ([**4**] and [**7**]). Although the present paper is self-contained, we will often refer for proofs and details to the above-mentioned works.

2. Discrete inner products and projections. As we have mentioned in the introduction, the implementation of the Galerkin method requires the computation of many integrals connected both with the inner product $L^2(\Omega)$ and with the integral operator. The numerical integration rules used to this effect may be different. In this section we discuss only the problems related to the computation of the inner product and the corresponding projection operator. For each discretization parameter $h > 0$ we introduce a numerical quadrature formula of the form

$$\int_\Omega f(t)d\sigma(t) \doteq \sum_{j=1}^{R_h} w_{j,h} f(t_{j,h}). \tag{2.1}$$

Here f belongs to a certain space of piecewise continuous functions $\tilde{C}_h \subset L^\infty(\Omega)$ that is supposed to contain both $C(\Omega)$ and S_h. In what follows, the subscript h will usually be dropped from $R_h, w_{j,h}, t_{j,h}$, although implicitly understood. Applying (2.1) to the inner product of $L^2(\Omega)$, we have a discrete inner product

$$(f, g)_h := \sum_{j=1}^{R} w_j f(t_j) g(t_j). \tag{2.2}$$

This discrete inner product is an example of an indefinite inner product. For an abstract theory of indefinite inner product spaces, the interested reader may consult [**8**]. However, all properties of (2.2) needed for our

purposes are proved directly in [4]. We will use the notation introduce in the latter work.

Let $\{\varphi_1, \varphi_2, \ldots, \varphi_{N_h}\}$ be a basis of S_h and let us consider the $N \times R$ matrix

$$\Phi = [\varphi_i(t_j)],\ 1 \leq i \leq N,\ 1 \leq j \leq R. \tag{2.3}$$

For all $h > 0$, assume that

[H1] $$R \geq N$$

[H2] $$\text{Rank}\,(\Phi) = N$$

[H3] $$w_j > 0,\ 1 \leq j \leq R$$

[H4] $$\varphi \in S_h \text{ and } \varphi \neq 0 \Rightarrow (\varphi, \varphi)_h > 0.$$

Under the above hypotheses there is a unique linear operator $Q_h : \tilde{C}_h \to S_h$ defined by

$$(Q_h z, \varphi)_h = (z, \varphi)_h,\ \text{all } \varphi \in S_h. \tag{2.4}$$

It is easily seen that the linear operator Q_h is a projection (i.e., $Q_h^2 = Q_h$) and that it satisfies

$$(Q_h x, y)_h = (x, Q_h y)_h,\ \text{all } x, y \in \tilde{C}_h. \tag{2.5}$$

If $R = N$, then $Q_h x$ is simply the element of S_h that interpolates x at the nodes $\{t_j\}$. In order to obtain a representation for Q_h in the general case, we need some additional notation. Introduce the diagonal matrix

$$W = \text{diag}\,[w_1, \ldots, w_R]. \tag{2.6}$$

and the vector function $\underline{\varphi} : \Omega \to \mathbf{R}^N$

$$\underline{\varphi}(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T. \tag{2.7}$$

Also we will associate with any function $x \in \tilde{C}_h$ the vector

$$\underline{x} = [x(t_1), \ldots, x(t_R)]^T. \tag{2.8}$$

Then it is straightforward to prove that

$$(Q_h x)(t) = \underline{\varphi}(t)^T [(\Phi W \Phi^T)^{-1} \Phi W] \underline{x} \tag{2.9}$$

for any function $x \in \tilde{C}_h$ and all $t \in \Omega$.

In the Introduction we have considered the orthogonal projection $P_h : L^2(\Omega) \to S_h$. From now on, it will be called the "continuous projection" onto S_h, in contrast to the "discrete projection" Q_h. We assume that

[H5] $$\| P_h x - x \|_\infty \to 0, \text{ as } h \to 0, \text{ all } x \in C(\Omega),$$

[H6] $$\sup_{h>0} \| Q_h \|_\infty < \infty.$$

The above assumptions ensure the fact that

$$\| x - Q_h x \|_\infty \leq c \| x - P_h x \|_\infty, \quad x \in C(\Omega). \tag{2.10}$$

For a proof and other details, see [**4**]. The same reference contains some sufficient conditions under which [H1]-[H6] are satisfied, as well as some examples satisfying those conditions. In the remainder of this section we will give one such example which then is used in §4.

Single variable results. Let $\Omega = [a, b]$ be a finite interval and let $\Delta^{(n)}$ be a partition of this interval of the form

$$a = \tau_0^{(n)} < \tau_1^{(n)} < \cdots < T_{m_n}^{(n)} = b. \tag{2.11}$$

Let us define

$$h_i^{(n)} = \tau_i^{(n)} - \tau_{i-1}^{(n)}, \quad h^{(n)} = \max h_i^{(n)},$$

$$q^{(n)} = \max_{1 \leq i,j \leq m_n} \frac{h_i^{(n)}}{h_j^{(n)}}, \tag{2.12}$$

and suppose that the sequence of partitions $\{\Delta^{(n)}\}$ is quasi-uniform in the sense that

$$\lim_{n\to\infty} m_n = \infty, \quad \sup_n q^{(n)} < \infty. \tag{2.13}$$

Under this assumption we have

$$\lim_{n\to\infty} h^{(n)} = 0. \tag{2.14}$$

In what follows we will measure the convergence rates of different approximation schemes in terms of the parameter $h = h^{(n)}$. For notational convenience we will drop the index n and we will write $h \to 0$ instead of $n \to \infty$. The elements of the partition $\Delta = \Delta^{(n)}$ will be denoted simply

$$\begin{cases} \Delta = \{\tau_0, \tau_1, \dots, \tau_m\}, \ \tau_i = \tau_i^{(n)}, \ \Delta_i = [\tau_{i-1}, \tau_i] \\ h_i = h_i^{(n)}, m = m_n, \ q = q^{(n)}, \ h = h^{(n)} \end{cases} \tag{2.15}$$

We will denote C_Δ^ν the space of all piecewise continuous functions $g : [a,b] \to \mathbf{R}$ such that $g\,|_{\Delta_i} \in C^\nu(\Delta_i)$, $i = 1, 2, \dots, m$. Clearly $C_\Delta = C_\Delta^0$ is a closed subspace of $L^\infty[a,b]$ which contains the space of all continuous functions defined on $[a,b]$. Let r be a nonnegative integer and let $\mathcal{P}_{r,\Delta}$ denote the subspace of C_Δ composed of all functions that are polynomials of degree $\le r$ on each of the subintervals Δ_i.

We now put the above into the general framework considered at the beginning of this section. Let

$$\Omega = [a,b], \ S_h = \mathcal{P}_{r,\Delta}, \ \tilde{C}_h = C_\Delta. \tag{2.16}$$

We have $N = m(r+1) = \dim S_h$. The continuous projection $P_h : L^2[a,b] \to S_h$,

$$(P_h f, \varphi) = (f, \varphi), \text{ all } \varphi \in S_h. \tag{2.17}$$

In our case this means that for any polynomial ψ of degree $\le r$, we must have

$$(P_h f, \psi)_i = (f, \psi)_i, \quad i = 1, 2, \dots, m \tag{2.18}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Delta_i)$, i.e., $(u,v)_i = \int_{\tau_{i-1}}^{\tau_i} u(t)v(t)dt$. In order to define a discrete projection $Q_h : \tilde{C}_h \to S_h$, we first construct a numerical integration formula of the form

$$\int_a^b f(t)dt \doteq \sum_{j=1}^{R} w_j f(t_j). \tag{2.19}$$

We start with a numerical quadrature rule on $[0,1]$,

$$\int_0^1 g(t)dt \doteq \sum_{j=1}^{p} \hat{w}_j g(\hat{t}_j). \tag{2.20}$$

It is assumed to have the degree of precision d, such that

$$d \geq p-1 \geq r, \tag{2.21}$$

and positive weights

$$\hat{w}_j > 0, \quad j = 1,2,\ldots,p. \tag{2.22}$$

Then using the partition Δ, we define a numerical integration formula on $[a,b]$ by

$$\int_a^b f(t)dt \doteq \sum_{i=1}^{m} h_i \sum_{j=1}^{p} \hat{w}_j f(\tau_{i-1} + h_i\hat{t}_j). \tag{2.23}$$

This can be identified with (2.19) by setting

$$R = mp, \; w_{(i-1)p+j} = h_i\hat{w}_j, \; t_{(i-1)p+j} = \tau_{i-1} + h_i\hat{t}_j \tag{2.24}$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,p$. Following [7] we consider the following sets of integers,

$$J_i = \{(i-1)p + j; \; j = 1,\ldots,p\}, \; i = 1,\ldots,m \tag{2.25}$$

which allow us to rewrite (2.23) as

$$\int_a^b f(t)dt \doteq \sum_{i=1}^{m} \sum_{j\in J_i} w_j f(t_j). \tag{2.26}$$

With this numerical integration method, we define a discrete scalar product, as in (2.2), and then a discrete projection $Q_h : \tilde{C}_h \to S_h$, as in (2.4). As in [7], we can state

PROPOSITION 2.1. *Let $P_h, (\cdot, \cdot)_h$ and Q_h be defined as above, and suppose that* (2.13), (2.21), (2.22) *hold. Then* [H1]-[H6] *are satisfied for any basis $\varphi_1, \ldots, \varphi_N, N = mp$, of $S_h = \mathcal{P}_{r,\Delta}$.*

PROPOSITION 2.2. *Suppose that the hypothesis of Proposition* 2.1 *is satisfied with $p > r+1$. Let $z \in C^{\nu-1}(\Delta_i)$ be given such that $z^{(\nu-1)}$ is absolutely continuous on Δ_i, $i = 1, \ldots, m$ and $z^{(\nu)} \in L^\infty[a,b]$. Then*

$$\Big[\sum_{j\in J_i} w_j \mid z(t_j) - (Q_h z)(t_j) \mid^2\Big]^{1/2} \leq \frac{\| z^{(h)} \|_\infty}{\mu!} h_i^{\mu+\frac{1}{2}}$$

where

$$\mu = \min\{\nu, r+1\}.$$

We note that the conclusion of the above proposition is trivially verified in case $p = r+1$ when $z(t_j) = (Q_h z)(t_j)$, $j = 1, \ldots, R$.

PROPOSITION 2.3. *Under the hypothesis of Proposition* 2.1,

$$\| z - Q_h z \|_\infty = O(h^\mu).$$

3. Nonlinear discrete Galerkin methods. Let us consider again the nonlinear integral equation (1.1). Suppose we have chosen a sequence of finite dimensional spaces $\{S_h\}$ and some numerical quadrature formulas of the form (2.1) such that conditions [H1]-[H6] are satisfied. The standard way of constructing a discrete nonlinear operator $\mathcal{K}_h$ to approximate the continuous nonlinear operator $\mathcal{K}$ from (1.1) is to use the same numerical integration formula (2.1) and define

$$[\mathcal{K}_h(x)](t) = \sum_{i=1}^{m} w_j K(t, t_j, x(t_j)), \quad t \in \Omega. \tag{3.1}$$

In some cases it is advantageous to use a quadrature formula different from (2.1) in constructing the discrete operator K. Such a formula will be considered in §4.

Once the discrete inner product $(\cdot,\cdot)_h$ and the discrete operator K_h have been defined, then the corresponding discrete Galerkin approximation to the solution of (1.1) is an element $z_h \in S_h$ of the form

$$z_h = \sum_{j=1}^{N} \xi_j \varphi_j.$$

The set $\{\varphi_1, \ldots, \varphi_n\}$ is a basis of S_h, and the coefficients $\xi_1, \ldots, \xi_N$ are obtained solving the nonlinear system

$$\sum_{j=1}^{N} \xi_j (\varphi_j, \varphi_i)_h = (K_h(\sum_{j=1}^{N} \xi_j \varphi_j), \varphi_i)_h. \tag{3.2}$$

The iterated discrete Galerkin solution is

$$\tilde{z}_h(t) = K_h(\sum_{j=1}^{N} \xi_j \varphi_j)(t), \quad t \in \Omega. \tag{3.3}$$

LEMMA 3.1. *The discrete Galerkin method* (3.2) *is equivalent to solving*

$$z_h = Q_h K_h(z_h) \tag{3.4}$$

while the iterated Galerkin solution (3.3) *satisfies*

$$\tilde{z}_h = K_h(Q_h \tilde{z}_h) \tag{3.5}$$

where Q_h is the discrete projection induced by the discrete inner product $(\cdot\,,\cdot)_h$ as described in §2.

PROOF. Equation (3.4) implies (1) $z_h \in \text{Range}\,(Q_h) = S_h$, and (2) z_h satisfies

$$(z_h, \psi)_h = (K_h(z_h), \psi)_h, \text{ all } \psi \in S_h.$$

Letting $\psi = \varphi_1, \ldots, \varphi_N$, we obtain (3.2). The argument is easily reversed as well.

From (3.3) for $\tilde{z}_h$,

$$\tilde{z}_h = \mathcal{K}_h(z_h) \tag{3.6}$$

$$Q_h\tilde{z}_h = Q_h\mathcal{K}_h(z_h) = z_h. \tag{3.7}$$

Substituting into (3.6), we obtain (3.5). □

The analysis of (3.5) and $\tilde{z}_h$ will follow closely the framework of Atkinson (1973), which gave a general theory for collectively compact families of approximating operators. From this, we assume the following hypotheses for $\{\mathcal{K}_h\}$.

[A1] $\mathcal{K}$ and $\mathcal{K}_h$, $h > 0$, are completely continuous nonlinear operators from the open set $D \subset L^\infty(\Omega)$ into $C(\Omega)$. [In the case that $S_h \subset C(\Omega)$, the domain space $L^\infty(\Omega)$ can be replaced by $C(\Omega)$.]

[A2] $\{\mathcal{K}_h \mid h > 0\}$ is a collectively compact family on D, i.e., for every bounded set $B \subset D$, the closure of

$$\bigcup_{h>0} \mathcal{K}_h(B)$$

is compact in $C(\Omega)$

[A3] $\mathcal{K}_h$ is pointwise convergent to $\mathcal{K}$ on D, i.e., for all $x \in D$,

$$\mathcal{K}_h(x) \to \mathcal{K}(x) \text{ as } h \to 0.$$

[A4] At each $x \in D$, $\{\mathcal{K}_h\}$ is an equicontinuous family.

Examples of such families $\{\mathcal{K}_h\}$ are given in Atkinson (1973). For $K(t,s,u)$ continuous, the definition (3.1) satisfies [A1]-[A4]. For a satisfactory definition of point evaluation for functions in $L^\infty(\Omega)$, see [**5**, §2].

We begin the analysis of $\tilde{z}_h$ by examining the operators

$$\mathcal{K}_hQ_h : x \to \mathcal{K}_h(Q_hx),\ x \in \tilde{D} \tag{3.8}$$

with $\tilde{D} = D \cap C(\Omega)$.

LEMMA 3.2. *Assume* [H1]-[H6] *for the approximating subspaces* $\{S_h\}$ *and the discrete product* $(\cdot, \cdot)_h$. *Further assume* K *and* $\{K_h\}$ *satisfy* [A1]-[A4]. *Then* $\{K_h Q_h\}$ *also satisfies* [A1]-[A4], *from* $\tilde{D}$ *into* $C(\Omega)$.

PROOF. The derivations of [A1]-[A4] are straightforward, and we include only the proof of [A3]. From [A4] for $\{K_h\}$, we have that for each $x \in D$ there exists a real-valued function $\varepsilon_x(r)$ for which

$$\| K_h(x) - K_h(y) \|_\infty \leq \varepsilon_x(\| x - y \|_\infty), \quad h > 0 \tag{3.9}$$

$$\varepsilon_x(r) \to 0 \text{ as } r \to 0. \tag{3.10}$$

Then for $K_h Q_h$,

$$\begin{aligned} \| K(x) - K_h(Q_h x) \| &\leq \| K(x) - K_h(x) \| + \| K_h(x) - K_h(Q_h x) \| \\ &\leq \| K(x) - K_h(x) \| + \varepsilon_x(\| x - Q_h x \|). \end{aligned}$$

Use (3.10) and [A3] for $\{K_h\}$ to prove [A3] for $\{K_h Q_h\}$.

The analysis of $\{\tilde{z}_h\}$ in (3.5) can now follow that given in [**7**]. Rather than reproduce that complete theory, we consider the most important case. Assume$\{K_h\}$ satisfies

[A5] Let $x^* \in \tilde{D}$ be a fixed point of K, and let $B(x^*, r) \subset C(\Omega)$ denote a ball of radius r about x^*. Then for some $r > 0$, K and $K_h, h > 0$, are twice differentiable on $B(x^*, r)$ and

$$\| K''(x) \|, \| K_h''(x) \| \leq M < \infty, \quad h > 0, \quad x \in B(x^*, r).$$

[Note: This will imply [A4] for $\{K_h\}$ on $B(x^*, r)$.]

LEMMA 3.3. *The second derivatives of* $K_h Q_h$ *will also satisfy [A5] on the ball* $B(x^*, r)$.

PROOF. Let $\mathcal{L}_h(x) \equiv K_h(Q_h x)$. Then

$$\begin{aligned} \mathcal{L}_h'(x)\varphi &= K_h'(Q_h x) Q_h \varphi \\ \mathcal{L}_h''(x)(\varphi, \psi) &= K_h''(Q_h x)(Q_h \varphi, Q_h \psi). \end{aligned} \tag{3.11}$$

Apply [A5] for $\{\mathcal{K}_h\}$ and [H6] for $\{Q_h\}$ to finish the proof. □

THEOREM 3.4. *Assume that* [H1]-[H6] *and* [A1]-[A5] *are satisfied. Let* x^* *be a fixed point of* $\mathcal{K}$, *and assume* 1 *is not an eigenvalue of* $\mathcal{K}'(x^*)$. *Then there is a neighborhood* $B(x^*, r)$ *and an* $h_0 > 0$ *such that for* $0 < h \leq h_0$, *the equation* $x = \mathcal{K}_h(Q_h x)$ *has a unique solution* $\tilde{z}_h$ *inside* $B(x^*, r)$. *In addition,*

$$\| x^* - \tilde{z}_h \|_\infty \leq c \| \mathcal{K}(x^*) - \mathcal{K}_h(Q_h x^*) \|_\infty \tag{3.12}$$

with some $c > 0$. [*The letter c denotes a generic positive constant.*]

PROOF. This is immediate from applying [**1**,Theorem 4] to the family $\{\mathcal{K}_h Q_h\}$. □

COROLLARY 3.5. *For each* $0 < h \leq h_0$,

$$\begin{aligned} \| x^* - \tilde{z}_h \|_\infty \leq c\text{Max}\{ & \| \mathcal{K}(x^*) - \mathcal{K}_h(x^*) \|_\infty, \\ & \| x^* - Q_h x^* \|_\infty^2, \\ & \| \mathcal{K}_h'(x^*)[x^* - Q_h x^*] \|_\infty \} \end{aligned} \tag{3.13}$$

PROOF. Take bounds in the expansion

$$\begin{aligned} &\mathcal{K}(x^*) - \mathcal{K}_h(Q_h x^*) = \\ &\quad [\mathcal{K}(x^*) - \mathcal{K}_h(x^*)] + \mathcal{K}_h'(x^*)[x^* - Q_h x^*] + O(\| x^* - Q_h x^* \|^2). \end{aligned}$$

The family $\{\mathcal{K}_h''(x^*)\}$ is uniformly bounded using [A5]. □

Error bounds for the discrete Galerkin solution are obtained by using (3.7). We have

$$\begin{cases} x^* - z_h = x^* - Q_h \tilde{z}_h = x^* - Q_h x^* + Q_h(x^* - \tilde{z}_h) \\ \| x^* - z_h \| \leq \| x^* - Q_h x^* \| + [\sup \| Q_h \|] \| x^* - \tilde{z}_h \| . \end{cases} \tag{3.14}$$

Actual rates of convergence are then easily obtained using (3.13).

Urysohn Operators. We will now consider more specific approximations $\mathcal{K}_h$. For the Urysohn operator (1.1), assume $K(t, s, u)$ is twice

continuously differentiable with respect to u. Define K_h by (3.1). Then easily

$$K'(x)\varphi(t) = \int_\Omega K_u(t,s,x(s))\varphi(s)ds, \quad t \in \Omega \tag{3.15}$$

$$K_h'(x)\varphi(t) = \sum_{j=1}^{R} w_j K_u(t,t_j,x(t_j))\varphi(t_j) \tag{3.16}$$

with $K_u \equiv \partial K/\partial u$. The weights w_j are to be the same as in (2.2) for the discrete inner product. K'' and K_h'' are defined similarly.

Examine the terms on the right side of (3.13). The term $\| K(x^*) - K_h(x^*) \|_\infty$ is simply a numerical integration error. The term $\| x^* - Qx^* \|_\infty^2$ involves the approximation properties S_h, and it can be bounded from (2.10). The third term can be treated in the same way as in [**4**, Lemma 5.4]. Let

$$\ell_t^*(s) = \ell^*(t,s) = K_u(t,s,x^*(s)).$$

Then

(3.17)

$$\begin{aligned} K_h'(x^*)[x^* - Q_h x^*](t) &= (\ell_t^*, (I-Q_h)x^*)_h = ((I-Q_h)\ell_t^*, (I-Q_h)x^*)_h \\ &\quad | K_h'(x^*)[I-Q_h]x^*(t)\xi | \\ &\leq \| (I-Q_h)\ell_t^* \|_{h,1} \cdot \| (I-Q_h)x^* \|_{h,\infty} \end{aligned}$$

where

$$\| f \|_{h,1} = \sum_{j=1}^{R} w_j | f(t_j) |, \quad \| f \|_{h,\infty} = \operatorname{Max}_{1\leq j\leq R} | f(t_j) | .$$

For smooth $\ell(t,s)$, the term $\| (I-Q_h)\ell_t^* \|_{h,1}$ will be of the same order as $\| (I-Q_h)x^* \|$; and then the error bound (3.13) will reduce to

$$\| x^* - \tilde{z}_h \| \leq c \operatorname{Max} \{ \| K(x^*) - K_h(x^*) \|_\infty, \| x^* - Q_h x^* \|_\infty^2 \}. \tag{3.18}$$

More detailed results will require additional assumptions about S_h and the numerical integration scheme used in defining K_h and $(\cdot, \cdot)_h$. We

have not stated these results more formally because they are mainly a guide to obtaining the actual rates of convergence.

An especially interesting case of the iterated discrete Galerkin solution is when $N = R$.

THEOREM 3.6. *Assume $K_h(x)$ depends on $x(t)$ at only $t = t_1, \dots, t_R$. Further assume that $N = R$, $h > 0$. Then*

$$\tilde{z}_h = K_h(\tilde{z}_h), \tag{3.19}$$

and thus $\tilde{z}_h$ is independent of the choice of S_h. Then the error bound for $\tilde{z}_h$ is of the simpler form

$$\| x^* - \tilde{z}_h \|_\infty \leq C \| K(x^*) - K_h(x^*) \|_\infty . \tag{3.20}$$

PROOF. From the comment following (2.5), $N = R$ implies $Q_h x$ interpolates x at $t = t_1, \dots, t_R$. In (3.5), $K_h(Q_h \tilde{z}_h)$ depends on $Q_h \tilde{z}_h(t)$ at $t = t_1, \dots, t_R$; and by the interpolating property, these are simply $\tilde{z}_h(t_1), \dots, \tilde{z}_h(t_R)$. Thus $K_h(Q_h \tilde{z}_h) = K_h(\tilde{z}_h)$. The error bound is immediate from (3.12).

4. Discrete Galerkin methods for equations with kernels of class $\mathcal{G}_2(\alpha, \gamma)$. In this section we will give a more detailed analysis in the one dimensional case. Accordingly, Ω will be a closed and bounded interval of the real axis, so that (1.1) can be written in the form

$$x(t) = \int_a^b K(t, s, x(s)) ds. \tag{4.1}$$

We will assume that the kernel K belongs to the class $\mathcal{G}_1(\alpha, \gamma)$, defined in [**6**] where α and γ are two integers such that $\alpha \geq \gamma, \alpha \geq 0, \gamma \geq -1$. This means that K has the following properties:

[G1] The partial derivative

$$\ell(t, s, u) = \frac{\partial K(t, s, u)}{\partial u} \tag{4.2}$$

exists for all $(t, s, u) \in \Psi \equiv [a, b] \times [a, b] \times \mathbf{R}$.

[G2] Define

$$\Psi_1 = \{(t,s,u) \mid a \le s \le t \le b,\ u \in \mathbf{R}\}$$
$$\Psi_2 = \{(t,s,u) \mid a \le t \le s \le b,\ u \in \mathbf{R}\}.$$

There are functions $\ell_i \in C^\alpha(\Psi_i), i = 1,2$, with

$$\ell(t,s,u) = \begin{cases} \ell_1(t,s,u), & (t,s,u) \in \Psi_1,\ t \neq s \\ \ell_2(t,s,u), & (t,s,u) \in \Psi_2. \end{cases}$$

[G3] If $\gamma \ge 0$, then $\ell \in C^\gamma(\Psi)$. If $\gamma = -1$, then ℓ may have a discontinuity of the first kind along the line $s = t$.

Under the above assumptions it is clear that there are two functions $K_i \in C^\alpha(\Psi_i), i = 1,2$ such that

$$K(t,s,u) = \begin{cases} K_1(t,s,u), & (t,s,u) \in \Psi_1,\ t \neq s \\ K_2(t,s,u), & (t,s,u) \in \Psi_2. \end{cases}$$

If the kernel K satisfies [G1]-[G2] as well as

[G4]

$$\frac{\partial^2 K_i}{\partial u^2} \in C(\Psi_i),\ i = 1,2, \tag{4.4}$$

then we say that K belongs to the class $\mathcal{G}_2(\alpha,\gamma)$. If $\alpha \ge 2$ then obviously $\mathcal{G}_1(\alpha,\gamma) = \mathcal{G}_2(\alpha,\gamma)$.

Let us denote by $\mathcal{K}$ the nonlinear operator defined by the right hand side of (4.1):

$$(\mathcal{K}x)(t) = \int_a^b K(t,s,x(s))ds. \tag{4.5}$$

In [**6**] we have proved the following result:

THEOREM 4.1. *Suppose that Urysohn's operator* (4.5) *has a kernel* $K \in \mathcal{G}_1(\alpha,\gamma)$. *Then*

(a) $\mathcal{K}$ *is a completely continuous operator from* $L^\infty[a,b]$ *into* $C^\nu[a,b]$ *for* $\nu = 0,1,\ldots,\gamma_1$, *with* $\gamma_1 \equiv Min\{\gamma+1, \alpha\}$.

(b) *For $\alpha > 1$, K is a continuous operator from $C^{\nu}[a,b]$ into $C^{\nu+1}[a,b]$, $\nu = 0, 1, \ldots, \alpha - 1$.*

As an immediate consequence we have

COROLLARY 4.2. *Under the hypothesis of Theorem* 4.1 *suppose that K has a fixed point x^*. Then $x^* \in C^{\alpha}[a,b]$.*

Now let Δ be a partition of the interval $[a,b]$ as defined in the last part of §2, let $S_h = \mathcal{P}_{r,\Delta}$ and let P_h (resp., Q_h) be the continuous (resp., the discrete) projection onto S_h considered there. In accordance with the notation introduced in §1, let $x_h, \tilde{x}_h$ denote the Galerkin and iterated Galerkin solutions of (4.1) corresponding to the projection P_h. In [**6**] we have proved that under appropriate hypothesis the orders of convergence of these solutions are the same as those obtained by Chatelin-Lebbar (1984) in the linear case:

THEOREM 4.3. *Assume $K \in \mathcal{G}_2(\alpha, \gamma)$, and let x^* be a fixed point of the Urysohn operator* (4.5) *with 1 not an eigenvalue of $K'(x^*)$. Also, suppose that the hypothesis of Proposition 1 is satisfied. Then for all sufficiently small $h > 0$, the equations*

$$x_h = P_h K(x_h), \quad \tilde{x}_h = K(P_h \tilde{x}_h) \tag{4.6}$$

have solutions $x_h, \tilde{x}_h$ which satisfy

$$\| x_h - x^* \|_{\infty} = O(h^{\beta}) \tag{4.7}$$

$$\| \tilde{x}_h - x^* \|_{\infty} = O(h^{\beta+\beta_2}) \tag{4.8}$$

$$\max_{t \in \Delta} | \tilde{x}_h(t) - x^*(t) | = O(h^{2\beta}) \tag{4.9}$$

where

$$\beta = \operatorname{Min}\{\alpha, r+1\}, \quad \beta_2 = \operatorname{Min}\{\alpha, r+1, \gamma+2\}. \tag{4.10}$$

Let us consider now a quadrature formula of the form (2.19)-(2.26), the corresponding discrete inner product $(\cdot\,,\cdot)_h$, and the discrete projection Q_h induced by it. Also let us denote, by K_h, the standard numerical approximation of the operator (4.5) constructed with this quadrature rule:

$$(K_h x)(t) = \sum_{j=1}^{R} w_j K(t, t_j, x(t_j)). \tag{4.11}$$

The following result is then easily proved.

PROPOSITION 4.4. *Assume that the hypotheses of Proposition* 2.1 *and Theorem* 4.3 *are satisfied with* $\gamma \geq 0$, *and let* K_h *be the discrete operator defined by* (4.11). *Then for any* $z \in C_\Delta^\nu[a,b]$,

$$\| (K_h - K)(z) \|_\infty = O(h^{\omega_\nu}) \tag{4.12}$$

$$\max_{t\in\Delta} | [(K_h - K)(z)](t) | = O(h^{\delta_\nu}) \tag{4.13}$$

where

$$\delta_\nu = \min\{\alpha, \nu, d+1\},\ \omega_\nu = \min\{\delta_\nu, \gamma+2\}. \tag{4.14}$$

Moreover, conditions [A1]-[A5] *from* §3 *are satisfied.*

As observed in §3, the Frechét derivative of K, resp. K_h, at x^* can be represented by the formulae

$$[K'(x^*)\varphi](t) = \int_a^b \ell^*(t,s)\varphi(s)ds, \tag{4.15}$$

respectively

$$[K_h'(x^*)\varphi](t) = \sum_{j=1}^{R} w_j \ell^*(t, t_j)\varphi(t_j), \tag{4.16}$$

where

$$\ell^*(t,s) = K_u(t,s,x^*(s)). \tag{4.17}$$

Under the hypothesis of Theorem 4.3, the function ℓ^* belongs to the class $\mathcal{G}(\alpha,\gamma)$ defined in Chatelin-Lebbar (1984).

DEFINITION 4.5. A function $y : [a.b] \times [a,b] \to \mathbf{C}$ is of class $\mathcal{G}(\alpha,\gamma)$ (with $\alpha \geq \gamma, \alpha \geq 0, \gamma \geq -1$) if and only if

$$y(t,s) = \begin{cases} y_1(t,s), & a \leq s < t \leq b \\ y_2(t,s), & a \leq t \leq s \leq b \end{cases}$$

with $y_1 \in C^\alpha(\{a \leq s \leq t \leq b\}), y_2 \in C^\alpha(\{a \leq t \leq s \leq b\})$, and $y \in C^\gamma([a,b] \times [a,b])$, for $\gamma \geq 0$. In case $\gamma = -1$, y may have a discontinuity of the first kind on $\{s = t\}$. Let us denote by $\mathcal{M}$, respectively $\mathcal{M}_h$, the resolvents of the linear operators (4.15), respectively (4.16):

$$\mathcal{M} = (I - \mathcal{K}'(x^*))^{-1}\mathcal{K}'(x^*)$$
$$\mathcal{M}_h = (I - \mathcal{K}_h'(x^*))^{-1}\mathcal{K}_h'(x^*).$$

Using results proved in [**9**] and [**6, 7**], we obtain

PROPOSITION 4.5. *Under the hypothesis of Proposition* 4.4, *there is a function* $M \in \mathcal{G}(\alpha,\gamma)$ *and a function* $M_h \in C^\gamma([a,b] \times [a,b])$ *such that*

$$(\mathcal{M}x)(t) = \int_a^b M(t,s)x(s)ds \tag{4.18}$$

$$(\mathcal{M}_h x)(t) = \int_a^b M_h(t,s)x(s)ds \tag{4.19}$$

$$\sup_{s,t\in[a,b]} | M(t,s) - M_h(t,s) | = O(h^\omega) \tag{4.20}$$

$$\| M_h(Q_h - I)x^* \|_\infty = O(h^{\beta+\beta_1}) \tag{4.21}$$

$$\| (I + M_h)(K_h - K)(x^*) \|_\infty O(h^\omega) \tag{4.22}$$

where β *is given by* (4.10) *and*

$$\omega = \min\{\alpha, d+1, \gamma+2\},\ \beta_1 = \min\{\alpha, r+1, \gamma+1\}. \tag{4.23}$$

The above proposition and Proposition 2.3 are used in the proof of our main result.

THEOREM 4.6. *Assume that the hypothesis of Proposition* 4.4 *holds. Then, for all sufficiently small* $h > 0$, *the equation*

$$\tilde{z}_h = K_h(Q_h \tilde{z}_h) \tag{4.24}$$

has a solution $\tilde{z}_h$ *satisfying the following:*

(a) *If* $p = r + 1$, *then*

$$\| \tilde{z}_h - x^* \|_\infty = O(h^\omega) \tag{4.25}$$

(b) *If* $p > r + 1$, *then*

$$\| \tilde{z}_h - x^* \|_\infty = O(h^{\bar{\omega}}). \tag{4.26}$$

The exponents are defined by

$$\omega = \min\{\alpha, d+1, \gamma+2\},\ \bar{\omega} = \min\{\omega, 2r+2\}.$$

PROOF. Let us denote

$$\tilde{a} = K_h(Q_h \tilde{z}_h) - K_h(Q_h x^*),$$

$$\tilde{b} = K_h(x^*) - K(x^*),$$

$$\tilde{c} = K_h(Q_h x^*) - K_h(x^*),$$

so that

$$\tilde{z}_h - x^* = \tilde{a} + \tilde{b} + \tilde{c}. \tag{4.27}$$

We have

$$\begin{aligned}\tilde{a} &= K_h'(Q_h x^*)Q_h(\tilde{z}_h - x^*) + \tilde{a}_1 \\ &= K_h'(x^*)Q_h(\tilde{z}_h - x^*) + \tilde{a}_1 + \tilde{a}_2 \\ &= K_h'(x^*)(Q_h - I)(\tilde{z}_h - x^*) + K_h'(x^*)(\tilde{z}_n - x^*) + \tilde{a}_1 + \tilde{a}_2.\end{aligned}$$

where

$$\| \tilde{a}_1 \|_\infty = O(\| \tilde{z}_h - x^* \|_\infty^2),$$

$$\| \tilde{a}_2 \|_\infty = O(\| \tilde{z}_h - x \|_\infty \| Q_h x^* - x^* \|_\infty).$$

According to Proposition 2.3,

$$\| \tilde{a}_2 \|_\infty = \| \tilde{z}_h - x^* \| O(h^\beta).$$

Also

$$\tilde{c} = K_h'(x^*)(Q_h - I)x^* + \tilde{c}_1$$

with $\tilde{c} = \tilde{c}_1 = 0$ for $p = r + 1$ and

$$\| \tilde{c}_1 \|_\infty = O(\| (Q_h - I)x^* \|_\infty^2) = O(h^{2\beta})$$

for $p > r + 1$.

Using Corollary 4.2 and (4.12), we deduce that

$$\| \tilde{b} \|_\infty = O(h^\omega).$$

With the above notation, we can move the term $K_h'(x^*)(\tilde{z}_h - x^*)$ to the left side of the error equation (4.27) and then solve for $\tilde{z}_h - x^*$, obtaining

$$\begin{aligned}\tilde{z}_h - x^* &= M_h(Q_h - I)(\tilde{z}_h - x^*) \\ &\quad + M_h(Q_h - I)x^* + (M_h + I)(\tilde{b} + \tilde{a}_1 + \tilde{a}_2 + \tilde{c}_1).\end{aligned} \tag{4.28}$$

In case $p = r + 1$ the first two terms on the right hand side of (4.28), as well as $\tilde{c}_1$, vanish so that

$$\| \tilde{z}_h - x^* \|_\infty \leq \| M_h + I \|_\infty (\| \tilde{b} \|_\infty + \| \tilde{a}_1 \| + \| \tilde{a}_2 \|_\infty).$$

It follows that

$$\| \tilde{z}_h - x^* \|_\infty \leq \| \tilde{z}_h - x^* \|_\infty O(\| \tilde{z}_h - x^* \|_\infty + h^\beta) + O(h^\omega),$$

from which we deduce that

$$\| \tilde{z}_h - x^* \|_\infty = O(h^\omega),$$

thus proving (4.25).

In case $p > r+1$, then the first term on the right hand side of (4.28) can be majorized if we observe that, according to Proposition 2.2,

$$\begin{aligned} &|[M_h(Q_h - I)(\tilde{z}_h - x^*)](t)| \\ &\leq c \sum_{i=1}^{m} h_i^{\beta_1 + \frac{1}{2}} \Big\{ \sum_{j \in J_1} w_j \, | \, [(I - Q_h)(\tilde{z}_h - x^*)](t_j) \, |^2 \Big\}^{1/2} \\ &\leq c \, \| \tilde{z}_h - x^* \|_\infty \, h^{\beta_1}. \end{aligned}$$

The constant c does not depend on t or h.

The second term on the right side of (4.28) is majorized by (4.21) so that

$$\begin{aligned} &\|\tilde{z}_h - x^*\|_\infty \\ &\leq \| \tilde{z}_h - x^* \|_\infty O(h^{\beta_1} + h^\beta + \| \tilde{z}_h - x^* \|_\infty) + O(h^{\beta+\beta_1} + h^\omega + h^{2\beta}). \end{aligned}$$

Hence

$$\| \tilde{z}_h - x^* \|_\infty = O(h^{\beta+\beta_1} + h^\omega + h^{2\beta}) = O(h^{\overline{\omega}}),$$

which completes the proof of our theorem. □

COROLLARY 4.7. *Assume that the hypotheses of Proposition 4.4 hold. Then for all sufficiently small $h > 0$, the discrete Galerkin equation*

$$z_h = Q_h K_h(z_h)$$

has a solution z_h such that

$$\| \tilde{z}_h - x^* \|_\infty = O(h^{\beta_2}) \tag{4.29}$$

where $\beta_2 = \min\{\alpha, r+1, \gamma+2\}$.

PROOF. We have

$$z_h - x^* = Q_h(\tilde{z}_h - x^*) + (Q_h x^* - x^*).$$

By using [H6], Proposition 2.3 and Theorem 4.6, we obtain

$$\| \tilde{z}_h - x^* \|_\infty = O(h^{\overline{\omega}} + h^{\beta}) = O(h^{\beta_2}).$$

By comparing Theorem 4.3 and Corollary 4.7 it follows that if

$$\min\{\alpha, r+1\} = \min\{\gamma+2, r+1\}, \tag{4.30}$$

then the order of convergence of the continuous Galerkin method coincides with the order of convergence of the discrete Galerkin method. In particular it follows that this order of convergence can be attained for $d = p - 1 = r$, which leads to a nonlinear system of the form (3.2) with $N = mp = R$.

On the other hand, by comparing Theorems 4.3 and 4.6 we deduce that the order of the discrete iterated Galerkin method equals the order of convergence of the continuous iterated Galerkin method if (1)

$$\alpha \geq 2r+2, \ \gamma \geq 2r, \tag{4.31}$$

and (2) the quadrature rule (2.20) is chosen such that

$$d \geq 2r+1. \tag{4.32}$$

We note that if (4.31) is satisfied, then by choosing (2.20) to be the Gaussian quadrature with $p = r+1$, we have

$$d = 2r+1 = 2p-1, \tag{4.33}$$

so that (4.32) is satisfied. This leads again to a system of the form (3.2) with $N = mp = R$, which is to be solved for $\xi, \ldots, \xi_N$. Then $\tilde{z}_h$ is to be obtained from (3.3). However, as we have noted in §3, in this

case (i.e., $p = r + 1, d = 2r + 1$) $\tilde{z}_h$ can be obtained directly by solving the Nyström equation

$$\tilde{z}_h(t) = \sum_{j=1}^{N} w_j K(t, t_j, \tilde{z}_h(t_j)). \tag{4.34}$$

This is done by obtaining first $\tilde{z}(t_1), \ldots, \tilde{z}(t_N)$ as a solution of the nonlinear system

$$\tilde{z}_h(t_i) = \sum_{j=1}^{N} w_j K(t_i, t_j, \tilde{z}_h(t_j)), \quad i = 1, \ldots, N, \tag{4.35}$$

and then using (4.34) as an interpolation formula. This is clearly more efficient than solving (3.2) and then using (3.3).

However, if (4.31) is not satisfied, then in general, the order of convergence of the discrete iterated Galerkin method is less than the order of convergence of the continuous Galerkin method. Numerical experiments show that we have to take p very big in order to recover the accuracy of the continuous iterated Galerkin method. This is impractical for numerical applications.

A method for Green's kernels. We will show that by considering different quadrature rules in discretizing the inner product $(\cdot, \cdot)$ and the integral operator K, we can recover the order of convergence of the continuous Galerkin method (and more!). For the discrete inner product, we consider the rule (2.20)-(2.26); while in constructing the discrete nonlinear operator, we use the quadrature rule employed in [7] for the numerical solution of linear integral equations with Green's function type kernels.

In what follows, we use the notation introduced in the above quoted paper. Thus let us assume again that (2.20) is the Gauss-Legendre quadrature rule, so that (4.33) holds. Denote

$$P = \{1, 2, \ldots, R\}, \quad T = \{t_1, t_2, \ldots, t_R\}. \tag{4.36}$$

Let q be a positive integer such that

$$p \leq q \leq 2p \tag{4.37}$$

and let us consider a family $\{\mathcal{T}_i\}_{i=1}^m$ of subsets of $\mathcal{T}$ such that

$$t_j \in \mathcal{T}_i = \{t_i^1, t_i^2, \dots, t_i^q\}, \quad \text{all } j \in J_i,$$

$$\max\{|\, t - t_j \,|; t \in \mathcal{T}_i, j \in J_j\} \le \chi h$$

where χ is a given constant (generally $\chi \le 2$).

Also, let $\ell_i^k, k = 1, \dots, q$, denote the elementary Lagrange polynomials associated with the nodes in $\mathcal{T}_i$, so that the Lagrange interpolation polynomial of a function x at these nodes can be written as

$$(\mathcal{L}_i x)(t) + \sum_{k=1}^{q} x(t_i^k)\ell_i^k(t). \tag{4.38}$$

If we introduce a family of indices $\{\mathcal{D}_i\}_{i=1}^m$ such that

$$J_i \subset \mathcal{D}_i = \{i_1, i_2, \dots, i_q\} \subset J, \; J_i = \{t_j : j \in \mathcal{D}_i\},$$

$$t_i^k = t_{i_k}, \; i = 1, \dots, m, \; k = 1, \dots, q,$$

then (4.38) can be written as

$$\mathcal{L}_i(x)(t) = \sum_{k=1}^{q} x_{i_k}\ell_i^k(t), \; x_{i_k} = x(t_i^k). \tag{4.39}$$

If q is large, the above formula is computationally inefficient, so that we will use instead the Newton form of the interpolating polynomial:

$$[\mathcal{L}_i(x)](t) = \sum_{k=1}^{q-1} x[t_i^1, \dots, t_i^{k+1}](t - t_i^1)\dots(t - t_i^k). \tag{4.40}$$

The actual evaluation is done by nested multiplication. In fact we use (4.40) to construct a discrete analogue $K_{h,q}$ of K and (4.39) to compute the Fréchet derivative of this discrete operator.

It is convenient to introduce the following functions:

$$\theta_i^{(1)}(t) = \begin{cases} 0 & \text{for } t \le \tau_{i-1} \\ t - \tau_{i-1} & \text{for } \tau_{i-1} \le t \le \tau_i \\ h_i & \text{for } \tau_i \le t \end{cases} \tag{4.41}$$

$$\theta_i^{(2)}(t) = \begin{cases} h_i & \text{for } t \le \tau_{i-1} \\ \tau_i - t & \text{for } \tau_{i-1} \le t \le \tau_i \\ 0 & \text{for } \tau_i \le t \end{cases} \tag{4.42}$$

$$\xi_{ij}^{(1)}(t) = \tau_{i-1} + \theta_i^{(1)}(t)\hat{t}_j \tag{4.43}$$

$$\xi_{ij}^{(2)}(t) = \tau_i + \theta_i^{(2)}(t)(\hat{t}_j - 1) \tag{4.44}$$

$$i = 1, \ldots, m, \quad j = 1, \ldots, p.$$

With the above notation we can define the discrete operator K_{hq} : $C[a,b] \to C[a,b]$ such that

$$(K_{h,q}x)(t) = \sum_{\sigma=1}^{2} \sum_{i=1}^{m} \theta_i^{(\sigma)}(t) \sum_{j=1}^{p} \hat{w}_j K(t, \xi_{ij}^{(\sigma)}(t), (\mathcal{L}_i x)(\xi_{ij}^{(\sigma)}(t))). \tag{4.45}$$

As a simpler computational definition for $t \in \Delta_i$ we can write

$$(K_{h,q}x)(t) = \sum_{\substack{\nu=1 \\ \nu \neq i}}^{m} \sum_{j \in J_\nu} \hat{w}_j K(t, t_j, x(t_j)) + \sum_{\sigma=1}^{2} A_i^{(\sigma)}(t) \tag{4.46}$$

where

$$A_j^{(\sigma)}(t) = \theta_i^{(\sigma)}(t) \sum_{j=1}^{P} w_i k(t, \xi_{ij}^{(\sigma)}(t), (\mathcal{L}_i x)(\xi_{ij}^{(\sigma)}(t))). \tag{4.47}$$

As in [7], we can prove the following result concerning the distance between K and $K_{h,q}$.

PROPOSITION 4.8. *Under the hypothesis of Proposition* 4.4, *suppose that* $p = r+1$ *and that* (2.20) *is the Gauss-Legendre quadrature formula (so that* $d = 2r + 1$*). Then*

$$\| (K_{h,q} - K)(x^*) \|_\infty = O(h^{q_2}) \tag{4.48}$$

$$\max_{t\in\Delta} |\, [(\mathcal{K}_{h,q} - \mathcal{K})(x^*)](t)\, | = O(h^{\alpha_1}) \tag{4.49}$$

where

$$\alpha_1 = \min\{\alpha, 2r+2\},\ q_2 = \min\{\alpha, 2r+2, q+1\}. \tag{4.50}$$

Moreover, conditions [A1]-[A5] *from* §3 *are satisfied for the family of operators* $\{\mathcal{K}, \mathcal{K}_{h,q} : h > 0\}$.

It is easily seen that for any function x, the function $\mathcal{K}_{h,q}(x)$ defined in (4.45) depends on x only via the values $x(t_1), \ldots, x(t_R)$. On the other hand, if $p = r+1$ then $Q_h x$ is the element of S_h which interpolates x at $t_1, t_2, \ldots, t_R$. Hence $\mathcal{K}_{h,q}(x) = \mathcal{K}_{h,q}(Q_h x))$, so that in this case the discrete iterated Galerkin method (3.5) reduces to the Nyström method

$$\tilde{z}_h^q = \mathcal{K}_{h,q}(\tilde{z}_h^q). \tag{4.51}$$

THEOREM 4.9. *Suppose that the hypothesis of Proposition* 4.8 *is satisfied. Then the equation* (4.51) *has a unique solution for all sufficiently small* h, *and we have*

$$\| \tilde{z}_h^q - x^* \|_\infty = O(h^{q_2}) \tag{4.52}$$

PROOF. Apply Theorem 3.6 and Proposition 4.8. □

Suppose now that α is sufficiently big, while γ is small in the sense that

$$\alpha \geq 2r+2,\ \gamma \leq r-1 \tag{4.53}$$

This is certainly true when the integral equation (4.1) is an equivalent formation of the two-point boundary value problem

$$x''(t) = f(t, x(t)),\ a < t < b,\ x(a) = x(b) = 0 \tag{4.54}$$

which leads to a kernel of the form

$$K(t, s, u) = G(t, s) f(s, u) \tag{4.55}$$

with

$$G(t,s) = \begin{cases} -(1-s)t, & t \le s \\ -(1-t)s, & s \le t \end{cases} \tag{4.56}$$

In this case, $\gamma = 0$ and (4.53) is satisfied whenever f is of class C^α. Actually, in many applications we have $\alpha = \infty, \gamma = 0$.

If (4.53) holds, then by taking $q = 2r+1 = 2p-1$ in those cases where (4.53) holds, the discrete iterated Galerkin (also Nyström) method (4.51) with $q = 2p - 1$ has a better uniform convergence rate than the continuous iterated Galerkin method. The latter is able to attain the same order (i.e., $O(h^{2r+2})$) only at break points (see (4.9)). In addition, under our assumptions the discrete iterated Galerkin method using standard Gaussian quadrature has the convergence order

$$\| \tilde{z}_h - x^* \|_\infty = O(h^{\gamma+2}) \tag{4.57}$$

(see (4.25)), so that the order cannot be increased by increasing the number of node points in each of the intervals $\Delta_1, \ldots, \Delta_m$ (i.e., by increasing $p = r + 1$).

Operations count. Let us now consider the cost of solving the respective Nyström approximations:

$$\tilde{z}_h = K_h(\tilde{z}_h) \tag{4.58}$$

$$\tilde{z}_h^{2r+1} = K_{h,2r+1}(\tilde{z}_h^{2r+1}). \tag{4.59}$$

We have seen that, in order to solve(4.58), we have first to solve the nonlinear system (4.35) and then to use (4.34) as an interpolation formula. Similarly, (4.59) is solved by first obtaining $v_1 = \tilde{z}_h^{2r+1}(t_1), \ldots, v_N = \tilde{z}_h^{2r+1}(t_N)$ as a solution of a nonlinear system whose μ-th equation with

$$\mu = (i-1)p + s, \ i = 1, \ldots, m, \quad s = 1, \ldots, p, \tag{4.60}$$

is formed as follows:

$$\theta_{is}^1 = h_i \hat{t}_s, \ \theta_{is}^2 = h_i(1 - \hat{t}_s) \tag{4.61}$$

$$\xi_{ijs}^1 = \tau_{i-1} + \theta_{is}^1 \hat{t}_j, \ \xi_{ijs}^2 = \tau_i + \theta_{is}^2(\hat{t}_j - 1) \tag{4.62}$$

$$\ell_{ijs}^{\sigma} = [\mathcal{L}_i \tilde{z}^{2r+2}](\xi_{ijs}^{\sigma}), \quad \sigma = 1, 2 \tag{4.63}$$

$$v_\mu = \sum_{\substack{\nu=1 \\ \nu \neq i}}^{m} \sum_{j \in J_\nu} w_j K(t_\mu, t_j, v_j) + \sum_{\sigma=1}^{2} \sum_{j=1}^{p} \theta_{is}^{\sigma} \hat{w}_j K(t_\mu, \xi_{ijs}^{\sigma}, \ell_{ijs}^{\sigma}). \tag{4.64}$$

Then (4.46) can be used as an interpolation formula.

Consider now that we have fixed a partition $\Delta = \Delta_h$ with mesh size

$$h = O\left(\frac{1}{m}\right) = O\left(\frac{p}{N}\right)$$

and that we want to solve (4.58) and (4.59) for this h. If we denote

$$u_i = \tilde{z}_h(t_i), \quad u = [u_1, \ldots, u_N]^T, \tag{4.65}$$

then the nonlinear system corresponding to (4.58) can be written in the form

$$F(u) = 0. \tag{4.66}$$

The function $F : \mathbf{R}^N \to \mathbf{R}^N$ has its i-th component given by

$$F_i(u) = u_i - \sum_{j=1}^{N} w_j K(t_i, t_j, u_j).$$

Similarly the nonlinear system corresponding to (4.59) is of the form

$$H(v) = 0, \tag{4.67}$$

where the μ-th component of $H : \mathbf{R}^N \to \mathbf{R}^N$ can be written as

$$H_\mu(v) = v_\mu - \sum_{\substack{\nu=1 \\ \nu \neq i}}^{m} \sum_{j \in J_\nu} w_j K(t_\mu, t_j, v_j) - \sum_{\sigma=1}^{2} \sum_{j=1}^{p} \theta_{is}^{\sigma} \hat{w}_j K(t_\mu, \xi_{ijs}^{\sigma}, \ell_{ijs}^{\sigma}).$$

In the above formula, ℓ_{ijs}^{σ} is to be computed as in (4.60)-(4.63) with the interpolation polynomial (4.40) evaluated by nested multiplication.

It is easily seen that the computation of the vector $F(u)$ for a given $u \in \mathbf{R}^N$ implies N^2 kernel evaluations $(K(t_i, t_j, u_j), i, j = 1, \dots N)$ plus N^2 multiplications and N^2 additions, that is $O(N^2)$ arithmetic operations (we have omitted the $O(N)$ arithmetic operations needed to compute $t_1, \dots, t_N, w_1, \dots, w_N$).

On the other hand, the computation of $H(v)$ involves $N(N+P)$ kernel evaluations, $N(N+3p)$ multiplications, $N(N+3P)$ additions plus the evaluation of ℓ^σ_{ijs} for $i = 1, \dots, m; \sigma = 1, 2; j, s = 1, \dots, p$. The evaluation of the divided differences appearing in (4.40) involve $\frac{1}{2}mq^2$ divisions and $\frac{1}{2}mq^2$ additions; the nested multiplications involve $2Npq$ multiplications and $2Npq$ additions. Thus the evaluation of the ℓ_{ijs} costs about $mq^2 + 4Npq \doteq 4N(p + 2p^2)$ arithmetic operations. It follows that

$$\text{cost}\,(H(v)) = \text{cost}\,(F(u)) + pN\ \text{cost}\,(K) + O(8p^2N).$$

In case cost (K) and p are small in comparison with N, then cost $(H(v))$ and cost $(F(u))$ are about the same $(= O(N^2))$.

In solving the systems (4.66), (4.67) we will use a Newton-like method. Therefore it is important to assess the cost of evaluating the Jacobians $F'(v)$ and $H'(v)$. The ij entry of $F'(u)$ is given simply by

$$[F'(u)]_{ij} = \delta_{ij} - w_j K_u(t_i, t_j, u_j).$$

This means that

$$\text{cost}\,(F'(u)) = N^2\text{cost}\,(K_u) + N^2\text{mult.} + N^2\text{add.}$$

Consider now a μ of the form (4.60). Then:

$$\begin{aligned}
&[H'(v)]_{\mu\rho} \\
&= -w_\rho K_u(t_\mu, t_\rho, v_\rho) \text{ if } \rho \notin \mathcal{D}_i \\
&= -w_\rho K_u(t_\mu, t_\rho, v_\rho) - \sum_{j=1}^{p} \hat{w}_j \sum_{\sigma=1}^{2} \theta^\sigma_{is} \ell^k_i(\xi^\sigma_{ijs}) K_u(t_\mu, \xi^\sigma_{ijs}, \ell^\sigma_{ijs}) \\
&\qquad\qquad \text{if } \rho \in \mathcal{D}_i \backslash J_i,\ \rho = i_k, \\
&= \delta_{\mu\rho} - \sum_{j=1}^{p} \hat{w}_j \sum_{\sigma=1}^{2} \theta^\sigma_{is} \ell^k_i(\xi^\sigma_{ijs}) K_u(t_\mu, \xi^\sigma_{ijs}, \ell^\sigma_{ijs}) \\
&\qquad\qquad \text{if } \rho \in \mathcal{D}_i \cap J_i,\ \rho = i_k.
\end{aligned}$$

In the formulas above, the ℓ_i^k are the elementary Lagrange polynomials from (4.38), (4.39). The computational cost of producing the $\ell_i^k(\xi_{ijs}^\sigma)$ is about

$$8Np^2\text{mult.} + 4Np^2\text{div.}$$

If this is to be done, then we can compute the ℓ_{ijs}^σ via (4.38) with an additional

$$4Np^2\text{mult.} + 4Np^2 \text{ add.}$$

This is about the same as the cost of evaluating ℓ_{ijs}^σ directly by (4.40) with nested multiplication. Adding all arithmetic operations we come to the conclusion that

$$\text{cost}\,[H'(v)] = \text{cost}\,[F'(u)] + pN\text{cost}\,(K_u) + O(20p^2N).$$

Again, if cost (K) and p are small in comparison with N, then the cost of both $F'(u)$ and $H'(v)$ will be of order $O(N^2)$. In this case it follows that the cost of a Newton step for solving (4.66), respectively (4.67), will be dominated by the cost of solving the respective linear systems, which is of order $O(\frac{2}{3}N^3)$. Thus techniques to reduce the cost should seek to reduce the cost of solving the linear systems.

It turns out that by using a suitable multigrid technique we can reduce the cost of the linear algebra to $O(cN^2)$, reducing in the same time the cost of $F'(u)$, respectively $H'(v)$. Moreover if one needs only one iteration on the finest grid then it is possible to reduce the cost of $F(u)$,respectively $H(v)$, as well. These problems, and others involving the iterative solution of the nonlinear systems, will be addressed in a future paper.

5. Numerical examples. To illustrate the preceding results for discrete Galerkin methods we give numerical results for three integral equations. Two of these equations were used as illustrations for the earlier paper [**6**] on Galerkin methods for nonlinear integral equations.

Our first equation is

$$x(t) = \int_0^1 \frac{ds}{t+s+x(s)} + y(t), \quad 0 \le t \le 1, \tag{5.1}$$

where y is so chosen that

$$x^*(t) = \frac{1}{t+c}, \quad c > 0$$

is a solution of (5.1). The function K is given by

$$K(t,s,u) = \frac{1}{t+s+u} + y(t).$$

For the error, we refer to Theorem 4.6. In this case, the constants α and γ of Theorem 4.6 can be chosen as large as desired. The error formulas (4.25) and (4.26) become

$$\| \tilde{z}_h - x^* \|_\infty = \begin{cases} O(h^{d+1}), & p = r+1 \\ O(h^{\overline{\omega}}), & p > r+1 \end{cases}$$

with $\overline{\omega} = \text{Min}\{d+1, 2r+2\}$. We give results for only $p = r+1$, and we use Gaussian quadrature with $r+1$ nodes on each subinterval to define the numerical integration. Then

$$\| \tilde{z}_h - x^* \|_\infty = O(h^{2r+2}). \tag{5.2}$$

The numerical results are given in Tables 1 through 4. For comparison, we also include the errors for the iterated continuous Galerkin method, taken from [**6**]. The number of nonlinear equations that must be solved is denoted by n_e.

Table 1. $x^* = 1/(t+1) : r = 1.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	4	7.30E-5		4.02E-6	
			17.8		5.13
4	8	4.10E-6		7.83E-7	
			16.6		13.3
8	16	2.48E-7		5.88E-8	
			16.1		15.4
16	32	1.53E-8		3.82E-9	

Table 2. $x^* = 1/(t+1) : r = 2.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	6	6.62E-7		1.05E-6	
			52.9		56.5
4	12	1.25E-8		1.86E-8	
			63.5		64.1
8	24	1.97E-10		2.90E-10	
			63.4		63.3
16	48	3.11E-12		4.58E-12	

Table 3. $x^* = 1/(t+1) : r = 1.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	4	1.15E-4		1.12E-2	
			19.1		9.41
4	8	6.31E-6		1.19E-3	
			15.7		13.3
8	16	3.83E-7		8.95E-5	
			16.1		14.9
16	32	2.38E-8		5.99E-6	

Table 4. $x^* = 1/(t+.1) : r = 2.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	6	1.30E-6		1.69E-3	
			75.2		21.3
4	12	1.54E-8		7.94E-5	
			67.4		36.4
8	24	2.24E-10		2.18E-6	
			74.7		49.7
16	48	2.96E-12		4.39E-8	

The values of Ratio generally agree with (5.2). For the smooth solution $x^* = 1/(t+1)$, the iterated Galerkin method is somewhat superior to the discrete iterated Galerkin method. The reverse is true for the more rapidly changing function $x^* = 1/(t+.1)$, in Tables 3 and 4. In all cases, the iterated Galerkin method was far more expensive in computation cost, due primarily to the numerical integration of the Galerkin coefficients to reasonably high accuracy. Also, the discrete iterated Galerkin method is a Nyström method in this case (since $p = r+1$); and implementing it as Nyströms method is less expensive than implementing it as a discrete iterated Galerkin method.

Our second example is

$$x(t) = \int_0^1 G(t,s)[f(s,x(s)) + z(s)]ds \tag{5.3}$$

$$G(t,s) = \begin{cases} -(1-t)s & s \leq t \\ -(1-s)t & s \geq t \end{cases} \tag{5.4}$$

with $z(s)$ so chosen that

$$x^*(t) = \frac{t(1-t)}{t+c}, \quad c > 0. \tag{5.5}$$

The integral equation (5.3) is a reformulation of the boundary value problem

$$x'' = f(t,x(t)) + z(t), \quad 0 < t < 1 \tag{5.6}$$

$$x(0) = x(1) = 0.$$

We consider the particular example

$$f(t,u) = \frac{1}{1+t+u}. \tag{5.7}$$

Referring to the discussion following Theorem 4.9, we can take $\gamma = 0$ and α arbitrarily large with equation (5.3). By letting $q = 2r+1$ in the scheme (4.51), we obtain

$$\| x^* - \tilde{z}_h^q \|_\infty = O(h^{2r+2}). \tag{5.8}$$

In contrast, the continuous iterated Galerkin method yields

$$\| x^* - \tilde{x}_h \|_\infty = O(h^{r+3}). \tag{5.9}$$

The numerical results are given in Tables 5 through 8.

Table 5. $x^* = t(1-t)/(t+2) : r = 1, q = 3.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	4	4.99E-5		1.13E-4	
			15.4		10.8
4	8	3.25E-6		1.05E-5	
			15.2		13.2
8	16	2.13E-7		7.98E-7	
			16.4		14.6
16	32	1.30E-8		5.47E-8	

Table 6. $x^* = t(1-t)/(t+2) : r = 2, q = 5.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	6	3.73E-7		2.91E-6	
			51.6		23.7
4	12	7.22E-9		1.23E-7	
			60.1		27.8
8	24	1.20E-10		4.43E-9	
			65.2		29.9
16	48	1.84E-12		1.48E-10	

Table 7. $x^* = t(1-t)/(t+.4) : r = 1, q = 3.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	4	6.74E-3		3.91E-4	
			4.6		6.8
4	8	1.48E-3		5.77E-5	
			11.2		9.4
8	16	1.31E-4		6.17E-6	
			14.1		11.9
16	32	9.31E-6		5.17E-7	
			15.6		13.8
32	64	5.96E-7		3.75E-8	

The values of Ratio are consistent with the error results in (5.8), (5.9). For $r = 1$, the continuous and discrete iterated Galerkin methods have the same order of convergence; but the continuous method is slower for $r \geq 2$. The tables also show the continuous method to be superior in the size of the error for the more badly behaved case $x^* = t(1-t)/(t+.4)$.

Table 8. $x^* = t(1-t)/(t+.4) : r = 2, q = 5.$

n	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{x}_h \|_\infty$	Ratio
2	6	7.37E-4		3.84E-5	
			14.1		12.7
4	12	5.22E-5		3.03E-6	
			37.8		18.6
8	24	1.38E-6		1.63E-7	
			52.7		23.9
16	48	2.62E-8		6.82E-9	
			61.1		27.6
32	96	4.28E-10		2.47E-10	

The third example is the two-dimensional integral equation

$$x(s,t) = \int_0^1 \int_0^1 \frac{[x(\sigma,\tau)]^2 d\tau d\sigma}{(e^\sigma + e^{-s} + c)(e^t + e^{-\tau} + c)} + y(s,t), \tag{5.10}$$

for $0 \leq s,t \leq 1$. For this equation, with

$$K(s,t,\sigma,\tau,u) = \frac{u^2}{(e^\sigma + e^{-s} + c)(e^t + e^{-\tau} + c)} + y(s,t)$$

and we give numerical results for $c = -.9$ and

$$x(s,t) = e^{s-t}, \tag{5.11}$$

with $y(s,t)$ defined accordingly.

For the numerical method, we first triangulate the domain $D = [0,1] \times [0,1]$. For $n \geq 1$, define $h = \frac{1}{n}, s_j = t_j = jh, j = 0,1,\ldots,n$, and $D_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$. Divide D_{ij} into two triangles. This yields a decomposition of D into $2n^2$ triangular subregions. On each triangle, we use polynomial approximations of degree 2, meaning there are six degrees of freedom in choosing the approximate solution $x_h(s,t)$. Based on polynomial interpolation error over triangles (see [**3**]), it is straightforward to show

$$\| x^* - x_h \|_\infty = O(h^3), \ \| x^* - \tilde{x}_h \|_\infty = O(h^6). \tag{5.12}$$

For the discrete iterated Galerkin method, the error bound (3.18) implies

$$\| x^* - \tilde{z}_h \|_\infty = O(h^\omega) \tag{5.13}$$

with $\omega = \text{Min}\{6, d+1\}$, d = degree of precision of the numerical integration over each triangle.

Two numerical integration schemes were used. Method # 1 used a six point integration scheme of [**13**]; see Rule 41 in Table 4 of that paper. Method # 2 used the seven point method, $T2:5-1$ of [**15**]. Method # 1 has degree of precision 4. However, if method # 1 is applied to

each of the two triangles in D_{ij}, then it has degree of precision 5 over each such region D_{ij}. Consequently, both methods lead to

$$\| x^* - \tilde{z}_h \|_\infty = O(h^6). \tag{5.14}$$

The numerical results for both methods 1 and 2 are given in Table 9. In the table, n_t denotes the number of triangles, and n_e is the number of nonlinear equations to be solved. The errors were approximated by the maximum of the errors at the centroids of the triangles.

Table 9. The numerical solution of (5.10): $c = -.9$.

			Method # 1		Method # 2	
n	n_t	n_e	$\| x^* - \tilde{z}_h \|_\infty$	Ratio	$\| x^* - \tilde{z}_h \|_\infty$	Ratio
1	2	12	4.97E-5		6.85E-5	
				28		37
2	8	48	1.77E-6		1.86E-6	
				45		48
4	32	192	3.91E-8		3.87E-8	

When n is doubled, then h is halved and formula (5.12) implies the error should decrease by a factor of about 64. We observe this only approximately. The results with the two cases are of comparable accuracy. We did not increase n further because the number of equations to be solved would have been too great. We do not include results for the continuous Galerkin method because the computing time needed would also have been too large. The computing time for method # 2 was approximately 1.6 times that for method # 1. Method # 1 has an equal number of integration nodes and basis functions (six per triangle), and thus it is a Nyström method. We implemented it as such, which increased its speed.

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