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CONTROLLING THE DIMENSIONS OF FORMAL FIBERS OF A UNIQUE FACTORIZATION DOMAIN AT THE HEIGHT ONE PRIME IDEALS

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ABSTRACT. Let T be a complete local (Noetherian) equidimensional ring with maximal ideal $\mathfrak{m}$ such that the Krull dimension of T is at least two and the depth of T is at least two. Suppose that no integer of T is a zerodivisor and that $|T| = |T/\mathfrak{m}|$. Let d and t be integers such that $1 \leq d \leq \dim T - 1$, $0 \leq t \leq \dim T - 1$ and $d - 1 \leq t$. Assume that, for every $\mathfrak{p} \in \operatorname{Ass} T$, $\operatorname{ht} \mathfrak{p} \leq d - 1$ and that if z is a regular element of T and $Q \in \operatorname{Ass}(T/zT)$, then $\operatorname{ht} Q \leq d$. We construct a local unique factorization domain A such that the completion of A is T and such that the dimension of the formal fiber ring at every height one prime ideal of A is $d - 1$ and the dimension of the formal fiber ring of A at (0) is t.

1. Introduction. Let A be a local (Noetherian) ring, and let T be the completion of A with respect to its maximal ideal. One fruitful way to study the relationship between A and T is to examine the relationship between $\operatorname{Spec} A$ and $\operatorname{Spec} T$, which we can do by looking at the formal fibers of A at its prime ideals. We define the formal fiber ring of A at $\mathfrak{p} \in \operatorname{Spec} A$ as $T \otimes_A \kappa(\mathfrak{p})$, and we define the formal fiber of A at $\mathfrak{p}$ as $\operatorname{Spec}(T \otimes_A \kappa(\mathfrak{p}))$, where $\kappa(\mathfrak{p})$ is the residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. There is a one-to-one correspondence between elements of the formal fiber of A at $\mathfrak{p}$ and the elements of the inverse image of $\mathfrak{p}$ under the mapping

$$\varphi : \operatorname{Spec} T \longrightarrow \operatorname{Spec} A,$$

defined by $\varphi(\mathfrak{q}) = \mathfrak{q} \cap A$. Due to this correspondence, the formal fibers of A encode information about the relationship between the

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prime ideals of T and the prime ideals of A. Note that, since T is a faithfully flat extension of A, the map φ is surjective. Using Matsumura's notation in [**6**], we denote the Krull dimension of the formal fiber ring of A at the prime ideal $\mathfrak{p}$ by $\alpha(A, \mathfrak{p})$. The supremum of $\alpha(A, \mathfrak{p})$ over all prime ideals $\mathfrak{p}$ of A is denoted by $\alpha(A)$. When A is an integral domain, we refer to the formal fiber ring of A at the prime ideal (0) as the generic formal fiber ring, and the formal fiber of A at (0) as the generic formal fiber.

Matsumura proved [**6**] that, given two elements $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Spec} A$, if $\mathfrak{p} \subseteq \mathfrak{p}'$, then $\alpha(A, \mathfrak{p}) \geq \alpha(A, \mathfrak{p}')$. Additionally, he proved that, given a ring of positive dimension n, the maximum possible value of $\alpha(A)$ is $n - 1$. Furthermore, if A is a ring of essentially finite type over a field, Matsumura showed that $\alpha(A) = n - 1$, and that $\alpha(A, \mathfrak{p}) = n - 1 - \operatorname{ht} \mathfrak{p}$. In other words, for rings of essentially finite type over a field, the dimensions of the formal fiber rings are completely understood. However, for a general ring, not much is known about how the inequality $\alpha(A, \mathfrak{p}) \geq \alpha(A, \mathfrak{p}')$ behaves when varying the heights of $\mathfrak{p}$ and $\mathfrak{p}'$. We are specifically interested in what happens when A is an integral domain, $\mathfrak{p} = (0)$, and $\mathfrak{p}'$ is a height one prime ideal of A. In particular, Heinzer, Rotthaus and Sally informally posed the following question:

Question 1.1. *Let A be an excellent local ring with $\alpha(A) > 0$, and let Δ be the set*

$$\{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{ht} \mathfrak{p} = 1 \text{ and } \alpha(A, \mathfrak{p}) = \alpha(A)\}.$$

Is Δ a finite set?

We first note that, if A is a Noetherian integral domain, then $\alpha(A) = \alpha(A, (0))$. Given Matsumura's results, one might expect that, for an integral domain A, the dimension of the formal fiber ring of A at "most" height one prime ideals is strictly less than the dimension of the formal fiber ring of A at the zero ideal. In other words, one might expect that the set Δ is "small," so the above question is a very natural one to ask.

In [**1**], Boocher, Daub and Loepp showed that the set Δ need not be finite by constructing an excellent local unique factorization domain A for which Δ is countably infinite. However, the UFD that

they construct has uncountably many height one prime ideals; thus, the set Δ for their ring is "small" in the sense that it does not contain "most" of the height one prime ideals of A. Based on this result, one might think it unlikely that there exists a ring such that Δ is an uncountable set and even less likely that a ring exists such that Δ contains *all* of its height one prime ideals. In this paper, we construct a non-excellent unique factorization domain A for which this property holds; that is, every height one prime ideal of A is in Δ. In fact, we generalize the result so that, if d and t are integers such that $1 \leq d \leq \dim T - 1$, $0 \leq t \leq \dim T - 1$, $d - 1 \leq t$, then our A satisfies the property that every height one prime ideal $\mathfrak{p}$ of A has a formal fiber ring of dimension $d-1$, and the generic formal fiber ring of A has dimension t. In other words, we show that the relationship between the dimensions of the formal fiber rings of a ring at its height one prime ideals and the dimension of the generic formal fiber ring of the ring is restricted only by the inequality given by Matsumura, that is, if $\mathfrak{p}$ is a height one prime ideal of a ring A, then $\alpha(A, \mathfrak{p}) \leq \alpha(A, (0))$; however, for any nonnegative integer less than or equal to the dimension of the generic formal fiber ring, there exists a ring such that the dimension of the formal fiber ring at each height one prime ideal is equal to this integer.

Our main result is given in Theorem 4.1. Let T be a complete local equidimensional ring with maximal ideal $\mathfrak{m}$, and suppose that the Krull dimension of T is at least two. Also, suppose that $|T| = |T/\mathfrak{m}|$, no integer is a zerodivisor in T and T has depth greater than one. Let d and t be integers with $1 \leq d \leq \dim T - 1$, $0 \leq t \leq \dim T - 1$, $d - 1 \leq t$. Suppose that $\operatorname{ht} \mathfrak{p} \leq d - 1$ for every $\mathfrak{p} \in \operatorname{Ass} T$ and that, if z is a regular element of T and $Q \in \operatorname{Ass}(T/zT)$, then $\operatorname{ht} Q \leq d$. We show that there exists a local unique factorization domain A whose completion is T such that $\alpha(A, (0)) = t$ and, for every $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht} \mathfrak{p} = 1$, $\alpha(A, \mathfrak{p}) = d - 1$, that is to say, with relatively weak conditions on T, we can find a local UFD A where the dimensions of the formal fiber rings of A at all height one prime ideals are equal. Moreover, we can control the dimension of the generic formal fiber ring of A, and the dimension of the formal fiber rings of A at its height one prime ideals can be any nonnegative integer less than or equal to the dimension of the generic formal fiber ring of A. In particular, if we set $d - 1 = t$, then we can find a UFD A such that Δ is the set of *all* height one prime ideals of

A, which, in our case, will be an uncountable set. Furthermore, in this case, we have the ability to set the dimension of the formal fiber rings at our height one prime ideals and the generic formal fiber ring to be any integer value between 0 and $\dim T - 2$.

In **[3]**, Heitmann found necessary and sufficient conditions for a complete local ring to be the completion of a unique factorization domain. For our ring A to exist, then, our ring T must satisfy those conditions. In particular, this means that T must have depth greater than one and satisfy the condition that no integer of T is a zerodivisor. We impose the additional condition that $\dim T \geq 2$ to avoid any trivial examples. The remaining conditions we put on T are relatively weak. We base our construction on that employed by Heitmann **[3]**, modifying and adding lemmas so that we can control the dimensions of the formal fiber rings of A at (0) and at the height one prime ideals. We employ Lemma 2.1, which gives necessary and sufficient conditions for a quasi-local subring of a complete local ring T to be Noetherian and have completion T. In particular, if T is a complete local ring with maximal ideal $\mathfrak{m}$ and A is a quasi-local subring of T with maximal ideal $\mathfrak{m} \cap A$, then A is Noetherian and has completion isomorphic to T if the map

$$A \longrightarrow T/\mathfrak{m}^2$$

is onto and every finitely generated ideal $\mathfrak{a}$ of A is closed; that is, $\mathfrak{a}T \cap A = \mathfrak{a}$. To ensure that our subring A satisfies these conditions and has the desired formal fibers, we construct a strictly ascending chain of subrings of T starting with a localization of the prime subring of T. We then adjoin elements of T in succession to make the above map onto, to close up finitely generated ideals and to manipulate the dimensions of the formal fiber rings. We argue that the union of the subrings in this chain is our desired local UFD A.

In this paper, all rings are commutative with unity. We define a quasi-local ring as any ring with exactly one maximal ideal and a local ring as a quasi-local Noetherian ring. We use $(T, \mathfrak{m})$ to refer to a local ring T with maximal ideal $\mathfrak{m}$ and $\operatorname{Spec}_k T$ to refer to the set of prime ideals of T of height k. $\widehat{T}$ will denote the completion of T in the $\mathfrak{m}$-adic topology.

2. Preliminaries. Before beginning our construction, we present, without proof, several lemmas which we use throughout the construc-

tion. The next lemma will be used to show that the completion of A is indeed T.

Lemma 2.1 ([**4,** Proposition 1]). *If $(A, \mathfrak{m} \cap A)$ is a quasi-local subring of a complete local ring $(T, \mathfrak{m})$, $A \to T/\mathfrak{m}^2$ is onto and $\mathfrak{a}T \cap A = \mathfrak{a}$ for every finitely generated ideal $\mathfrak{a}$ of A, then A is Noetherian, and the natural homomorphism $\widehat{A} \to T$ is an isomorphism.*

We use the following lemma to find prime ideals of T of height d to include in the formal fibers at the height one prime ideals of A.

Lemma 2.2 ([**2,** Lemma 2.3]). *Let $(T, \mathfrak{m})$ be a complete local ring of dimension at least one. Let $\mathfrak{p}$ be a nonmaximal prime ideal of T. Then, $|T/\mathfrak{p}| = |T| \geq 2^{\aleph_0}$.*

Throughout our construction, we adjoin elements of T to our intermediate subrings. The next two lemmas ensure that we can adjoin the necessary elements of T while also preserving the desired properties of our subrings.

Lemma 2.3 ([**3,** Lemma 2]). *Let $(T, \mathfrak{m})$ be a complete local ring, and let $D \subset T$. Suppose that $C \subset \operatorname{Spec} T$ such that $\mathfrak{m} \notin C$, and suppose that $\mathfrak{a}$ is an ideal of T such that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in C$. If C and D are countable, then*

$$\mathfrak{a} \not\subseteq \bigcup \{t + \mathfrak{p} \mid t \in D,\ \mathfrak{p} \in C\}.$$

Lemma 2.4 ([**3,** Lemma 3]). *Let $(T, \mathfrak{m})$ be a local ring, and let $D \subset T$. Suppose that $C \subset \operatorname{Spec} T$ and that $\mathfrak{a}$ is an ideal of T such that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in C$. If $|C \times D| < |T/\mathfrak{m}|$, then $\mathfrak{a} \not\subseteq \bigcup \{t + \mathfrak{p} \mid t \in D,\ \mathfrak{p} \in C\}$.*

3. The construction. Here, we describe the construction of our local UFD A. We begin with the definition of a Z_d-subring of T. If we can maintain Z_d-subrings throughout the construction, then the formal fiber rings of A will be of the desired dimension.

Definition 3.1. Let $(T, \mathfrak{m})$ be a complete local ring, let d be an integer such that $1 \leq d \leq \dim T - 1$ and let $(R, \mathfrak{m} \cap R)$ be a quasi-local unique factorization domain contained in T satisfying the following:

(i) $|R| \leq \sup(\aleph_0, |T/\mathfrak{m}|)$ with equality only if $T/\mathfrak{m}$ is countable;
(ii) $\mathfrak{q} \cap R = (0)$ for all $\mathfrak{q} \in \operatorname{Ass} T$;
(iii) if $t \in T$ is regular and $\mathfrak{q} \in \operatorname{Ass}(T/tT)$, then $\operatorname{ht}(\mathfrak{q} \cap R) \leq 1$; and
(iv) there exists a set $\mathcal{Q}_R \subset \operatorname{Spec}_d T$ such that the map

$$f : \mathcal{Q}_R \longrightarrow \operatorname{Spec}_1 R,$$

given by $f(\mathfrak{q}) = \mathfrak{q} \cap R$, is a bijection.

Then, R is called a *Z_d-subring of T with distinguished set $\mathcal{Q}_R$*. Note that if $(R, \mathfrak{m} \cap R)$ satisfies conditions (i)–(iii), but not necessarily (iv), R is an *N-subring of T*, as defined in **[3]**.

Definition 3.2. If $R \subseteq S$ are Z_d-subrings of T with distinguished sets $\mathcal{Q}_R$ and $\mathcal{Q}_S$, we say that S is an *A_+-extension of R* if:

(i) prime elements of R are prime in S;
(ii) $|S| \leq \sup(\aleph_0, |R|)$; and
(iii) $\mathcal{Q}_R \subseteq \mathcal{Q}_S$.

If $R \subseteq S$ are N-subrings of T satisfying conditions (i) and (ii), but not necessarily (iii), then S is an *A-extension of R*, as defined in **[3]**.

In order to simplify the statements of many of our subsequent lemmas, we will refer to the following assumption.

Assumption 3.3. *$(T, \mathfrak{m})$ is a complete local equidimensional ring with $2 \leq \dim T$, d is an integer such that $1 \leq d \leq \dim T - 1$ and $(R, \mathfrak{m} \cap R)$ is a Z_d-subring of T with distinguished set $\mathcal{Q}_R$.*

The next lemma is used to adjoin a coset representative of an element of $T/\mathfrak{m}^2$ to our subring R. Applying this lemma infinitely often will allow us to ensure that the map $A \to T/\mathfrak{m}^2$ is onto so that we can employ Lemma 2.1.

Lemma 3.4. *Under Assumption* 3.3, *let $t \in T$, and suppose that* $\operatorname{depth} T > 1$. *Let P be a nonmaximal prime ideal of T such that*

$P \cap R = (0)$. *Then, there exists an* A_+*-extension* S *of* R *such that*

$$t + \mathfrak{m}^2 \in \text{Image}(S \to T/\mathfrak{m}^2)$$

and $P \cap S = (0)$.

Proof. Note that this lemma is similar to **[3,** Lemma 5]. Let

$$C = \{\mathfrak{p} \in \operatorname{Spec} T \mid \mathfrak{p} \in \operatorname{Ass}(T/rT) \text{ with } 0 \neq r \in R\} \cup \operatorname{Ass} T \cup \mathcal{Q}_R \cup \{P\}.$$

Since $\operatorname{depth} T > 1$, we have that $\mathfrak{m} \notin C$, so $\mathfrak{m}^2 \not\subset \mathfrak{p}$ for every $\mathfrak{p} \in C$. For $\mathfrak{p} \in C$, let $D_{(\mathfrak{p})}$ be a set of coset representatives of the cosets $u+\mathfrak{p} \in T/\mathfrak{p}$ that make $u + t + \mathfrak{p}$ algebraic over $R/\mathfrak{p} \cap R$. Define

$$D = \bigcup_{\mathfrak{p} \in C} D_{(\mathfrak{p})}.$$

Using Lemma 2.3 if R is countable and Lemma 2.4, otherwise, we choose $x \in \mathfrak{m}^2$ such that $(x+t)+\mathfrak{p}$ is transcendental over $R/(\mathfrak{p} \cap R)$ for every $\mathfrak{p} \in C$. Define $S = R[x+t]_{\mathfrak{m} \cap R[x+t]}$; we claim that S is our desired Z_d-subring with distinguished set $\mathcal{Q}_S$, where we will define $\mathcal{Q}_S$ later.

We first show that $\mathfrak{p} \cap S = (\mathfrak{p} \cap R)S$ for any $\mathfrak{p} \in C$. Elements of $\mathfrak{p} \cap S$ are of the form uf, where $u \in S^\times$ is a unit and $f \in R[x+t]$. Since u is a unit, we have that $f \in \mathfrak{p}$. Treating f as a polynomial in $x+t$ over R, each of its coefficients must be in $\mathfrak{p} \cap R$ since $x+t+\mathfrak{p}$ is transcendental over $R/\mathfrak{p} \cap R$. Thus, $f \in (\mathfrak{p} \cap R)R[x+t]$, and so, $uf \in (\mathfrak{p} \cap R)S$. This gives us that

$$\mathfrak{p} \cap S \subseteq (\mathfrak{p} \cap R)S,$$

and the opposite containment is clear, so we have equality.

Clearly, $|S| = \sup(\aleph_0, |R|)$ as we are simply adjoining a transcendental element and localizing, and so S satisfies condition (i) of Z_d-subrings and condition (ii) of A_+-extensions. Now, let $\mathfrak{q} \in \operatorname{Ass} T$; then,

$$\mathfrak{q} \cap S = (\mathfrak{q} \cap R)S = (0),$$

and so condition (ii) of Z_d-subrings is satisfied. By the same argument, $S \cap P = (0)$.

Now, let $\mathfrak{p} \in \operatorname{Ass}(T/rT)$ for some regular $r \in T$. First, suppose that $\mathfrak{p} \cap R = (0)$. Then, in

$$S_{\mathfrak{p} \cap S} = R[x+t]_{\mathfrak{p} \cap R[x+t]},$$

all nonzero elements of R are units, and thus, $S_{\mathfrak{p}\cap S}$ is isomorphic to $k[X]$ with additional elements inverted, where k is a field and X is an indeterminate. It follows that $\dim S_{\mathfrak{p}\cap S} \leq 1$, and we have $\mathrm{ht}(\mathfrak{p} \cap S) \leq 1$. Now, suppose $\mathfrak{p} \cap R = aR$ for some $a \in R$. We know that $\mathfrak{p} \in \mathrm{Ass}(T/rT)$ if and only if $\mathfrak{p}T_{\mathfrak{p}} \in \mathrm{Ass}(T_{\mathfrak{p}}/rT_{\mathfrak{p}})$ [**5,** Theorem 6.2], and so, $\mathfrak{p}T_{\mathfrak{p}}$ consists of zerodivisors of $T_{\mathfrak{p}}/rT_{\mathfrak{p}}$. Then, $T_{\mathfrak{p}}/rT_{\mathfrak{p}}$ consists only of zerodivisors and units, and hence, has depth zero, and, since $a \in T$ is regular, $T_{\mathfrak{p}}/aT_{\mathfrak{p}}$ must also have depth zero. Then, $\mathfrak{p}T_{\mathfrak{p}}$ consists only of zerodivisors of $T_{\mathfrak{p}}/aT_{\mathfrak{p}}$, and thus,

$$\mathfrak{p}T_{\mathfrak{p}} \in \mathrm{Ass}(T_{\mathfrak{p}}/aT_{\mathfrak{p}}),$$

and we have that $\mathfrak{p} \in \mathrm{Ass}(T/aT)$. Therefore, $\mathfrak{p} \in C$; hence, we have that

$$\mathfrak{p} \cap S = (\mathfrak{p} \cap R)S = aS.$$

Since R is a UFD and $x+t$ is transcendental over R, $R[x+t]$ is a UFD as well, and furthermore, any localization of a UFD is a UFD; thus, $S = R[x+t]_{\mathfrak{m}\cap R[x+t]}$ is a UFD. Therefore, since aS is a principal ideal of S, $\mathrm{ht}(\mathfrak{p} \cap S) \leq 1$, and S satisfies condition (iii) of Z_d-subrings.

Note that, since $x + t$ is transcendental over R, prime elements of R are prime in S. Finally, we define the distinguished set $\mathcal{Q}_S$ for S. We first show that there is a one-to-one correspondence between height one prime ideals of R and height one prime ideals pS of S such that $pS \cap R \neq (0)$ given by $pR \mapsto pS$. If pS is a height one prime ideal of S such that $pS \cap R \neq (0)$, then there is an element $s \in S$ such that $ps \in R$. Factoring ps into primes in R, we obtain

$$ps = q_1 \cdots q_n,$$

where q_i are prime elements in R. Since prime elements in R are prime in S, we have that $q_1 \cdots q_n$ is a prime factorization of ps in S. It follows that $p = q_i u$ for some i, where u is a unit in S. Hence,

$$pS = (q_i u)S = q_i S;$$

thus, for every height one prime ideal pS of S with $pS \cap R \neq (0)$, there is a prime element q of R such that $pS = qS$, and it follows that our map is onto. Now, suppose that p and q are prime elements of R such that $pS = qS$. Then, letting $P' \in \mathrm{Ass}(T/pT)$, we have that

$$pS \cap R \subseteq P' \cap R = pR.$$

Clearly, $pR \subseteq pS \cap R$. This gives us that

$$pR = pS \cap R = qS \cap R = qR,$$

showing that our map is one-to-one. Let $\mathfrak{q}_p \in \mathcal{Q}_R$, where p is a prime element of R satisfying $\mathfrak{q}_p \cap R = pR$. Then,

$$\mathfrak{q}_p \cap S = (\mathfrak{q}_p \cap R)S = pS.$$

Hence, the map from $\mathcal{Q}_R$ to the set of height one prime ideals pS of S satisfying $pS \cap R \neq (0)$, given by $\mathfrak{q}_p \mapsto \mathfrak{q}_p \cap S = pS$, is a bijection.

Now, let p be a nonzero prime element of S such that $pS \cap R = (0)$. We claim that there is a $\mathfrak{q}_p \in \operatorname{Spec}_d T$ such that $\mathfrak{q}_p \cap S = pS$, and we show this by induction on d. If $d = 1$, let $\mathfrak{q}_p$ be a minimal prime ideal of pT. In the case $d = 2$, let $\mathfrak{p} \in \operatorname{Spec} T$ be a minimal prime ideal of pT, and consider the embedding

$$S/\mathfrak{p} \cap S \hookrightarrow T/\mathfrak{p}.$$

Since $T/\mathfrak{p}$ is a Noetherian integral domain, each nonzero element of $S/\mathfrak{p} \cap S$ is contained in finitely many height one prime ideals of $T/\mathfrak{p}$, and since, by Lemma 2.2,

$$|S/\mathfrak{p} \cap S| < |T| = |T/\mathfrak{p}|,$$

there must be some $\overline{\mathfrak{q}} \in \operatorname{Spec}_1(T/\mathfrak{p})$, whose intersection with $S/\mathfrak{p} \cap S$ is zero. Lifting to T and using the fact that T is equidimensional and catenary, we have a height two prime ideal $\mathfrak{q}$ of T containing $\mathfrak{p}$ such that $\mathfrak{q} \cap S \subseteq \mathfrak{p} \cap S$. By condition (iii) of Z_d-subrings, $\mathfrak{p} \cap S = pS$, and so, $\mathfrak{q} \cap S = pS$. Now, let $2 < d < \dim T$, and suppose that $\mathfrak{q}' \in \operatorname{Spec}_{d-1} T$ satisfies $\mathfrak{q}' \cap S = pS$. Applying the above argument to the injection

$$S/\mathfrak{q}' \cap S = S/pS \hookrightarrow T/\mathfrak{q}',$$

we can find some $\mathfrak{q}_p \in \operatorname{Spec}_d T$ containing p and satisfying $\mathfrak{q}_p \cap S \subseteq \mathfrak{q}' \cap S$, and thus, $\mathfrak{q}_p \cap S = pS$, and the induction is complete. Now, for every height one prime ideal pS of S such that $pS \cap R = (0)$, choose one $\mathfrak{q} \in \operatorname{Spec}_d T$ such that $\mathfrak{q} \cap S = pS$. We define $\mathcal{Q}_S$ to be the union of the set of these prime ideals and $\mathcal{Q}_R$. Then, S is our desired A_+-extension of R. □

Lemma 3.5. *Under Assumption* 3.3*, let* $\mathfrak{a}$ *be a finitely generated ideal of* R*, and let* $c \in \mathfrak{a}T \cap R$*. Let* P *be a nonmaximal prime ideal of* T *such*

that $P \cap R = (0)$. Then, there exists an A_+-extension S of R such that $c \in \mathfrak{a}S$ and $P \cap S = (0)$.

Proof. Note that this lemma is similar to **[3,** Lemma 4] with the change that, instead of R and S being merely N-subrings, R and S are Z_d-subrings of T with distinguished sets $\mathcal{Q}_R$ and $\mathcal{Q}_S$ satisfying $\mathcal{Q}_R \subseteq \mathcal{Q}_S$. Therefore, most of the proof follows the proof of **[3,** Lemma 4], and we need only show that condition (iv) of Definition 3.1 holds for the S we construct, that we can choose $\mathcal{Q}_S$ so that $\mathcal{Q}_R \subseteq \mathcal{Q}_S$, and that $P \cap S = (0)$. We will proceed by inducting on the number of generators of $\mathfrak{a}$. Let $\mathfrak{a} = (a_1, \ldots, a_n)R$.

It is shown in the proof of **[3,** Lemma 4] that, without loss of generality, we may assume that $\mathfrak{a}$ is not contained in a height one prime ideal of R, and thus, we will assume this for the rest of our proof. If $n = 1$, this implies that $\mathfrak{a} = R$, so $S = R$ is the desired A_+-extension of R. As in the proof of Lemma 3.4, we define

$$C = \{\mathfrak{p} \in \operatorname{Spec} T \mid \mathfrak{p} \in \operatorname{Ass}(T/rT) \text{ with } 0 \neq r \in R\} \cup \operatorname{Ass} T \cup \mathcal{Q}_R \cup \{P\}.$$

Now, we consider the case $n = 2$. Then, $c = a_1t_1 + a_2t_2$ for some $t_1, t_2 \in T$, and thus, $c = (t_1 + a_2t)a_1 + (t_2 - a_1t)a_2$ for any $t \in T$. Following the proof of **[3,** Lemma 4], let $x_1 = t_1 + a_2t$ and $x_2 = t_2 - a_1t$ for some t to be carefully chosen later. We claim that, if $\mathfrak{p} \in C$, then at most one of a_1 or a_2 is contained in $\mathfrak{p}$. This is clear if $\mathfrak{p} \cap R = (0)$, and if $\operatorname{ht}(\mathfrak{p} \cap R) = 1$, then $\mathfrak{p} \cap R = pR$ for some prime element $p \in R$. If a_1 and a_2 are both in $\mathfrak{p}$, then $\mathfrak{a} \subset \mathfrak{p} \cap R = pR$, contradicting that $\mathfrak{a}$ is not contained in a height one prime ideal of R. Hence, we can assume that no $\mathfrak{p} \in C$ contains both a_1 and a_2. Define $\mathring{R} = R[a_2^{-1}, x_1] \cap R[a_1^{-1}, x_2]$ and $S = \mathring{R}_{\mathfrak{m} \cap \mathring{R}}$.

If $\mathfrak{p} \in C$, then $|R/\mathfrak{p} \cap R| \leq |R|$, and so, the algebraic closure of $R/\mathfrak{p} \cap R$ in $T/\mathfrak{p}$ has cardinality at most $|R|$ if R is infinite, and will be countable if R is finite. Now, let $\mathfrak{p} \in C$, and suppose that $a_2 \notin \mathfrak{p}$. Then, each choice of t modulo $\mathfrak{p}$ gives a different x_1 modulo $\mathfrak{sp}$, and thus, for all but $\max(\aleph_0, |R|)$ choices of t modulo $\mathfrak{p}$, the element $x_1 + \mathfrak{p}$ of $T/\mathfrak{p}$ will be transcendental over $R/\mathfrak{p} \cap R$. For each $\mathfrak{p} \in C$ with $a_2 \notin \mathfrak{p}$, define

$$G_{\mathfrak{p}} = \{t + \mathfrak{p} \in T/\mathfrak{p} \mid t + \mathfrak{p} \text{ yields an } x_1 + \mathfrak{p} \text{ that is algebraic over } R/\mathfrak{p} \cap R\} \subseteq T/\mathfrak{p}.$$

Now, for each element $t+\mathfrak{p}$ of $G_{\mathfrak{p}}$, choose exactly one element of $t+\mathfrak{p}$, and call that element $\alpha_{t+\mathfrak{p}}$. Define

$$D^1_{\mathfrak{p}} = \bigcup_{t+\mathfrak{p}\in G_{\mathfrak{p}}} \{\alpha_{t+\mathfrak{p}}\}.$$

Then, $D^1_{\mathfrak{p}} \subseteq T$ is a full set of coset representatives for those choices of $t+\mathfrak{p}$ such that $x_1+\mathfrak{p}$ is algebraic over $R/\mathfrak{p}\cap R$. Similarly, for each $\mathfrak{p} \in C$ that does not contain a_1, let $D^2_{\mathfrak{p}}$ be a full set of coset representatives for the elements t that make $x_2+\mathfrak{p}$ algebraic over $R/\mathfrak{p}\cap R$, and define

$$D = \bigcup \{D^1_{\mathfrak{p}} \cup D^2_{\mathfrak{p}} \mid \mathfrak{p} \in C\}.$$

Then, $|C \times D| \leq \max(\aleph_0, |R|)$, and thus, using Lemma 2.3 if R is countable and Lemma 2.4 otherwise, we can choose t such that, for $\mathfrak{p} \in C$, we have $x_1 + \mathfrak{p}$ is transcendental over $R/\mathfrak{p} \cap R$ if $a_2 \notin \mathfrak{p}$ and $x_2 + \mathfrak{p}$ is transcendental over $R/\mathfrak{p} \cap R$ if $a_1 \notin \mathfrak{p}$. It is shown in **[3,** Lemma 4] that, for such a choice of t, S is an N-subring, and prime elements in R are prime in S.

Suppose that $f \in P \cap \mathring{R}$. Then, $f \in R[a_1^{-1}, x_2]$, and thus, there is a positive integer r such that $a_1^r f \in R[x_2]$. By the way that t was chosen, $x_2 + P$ is transcendental over $R/R \cap P$, and so, the coefficients of $a_1^r f$ are in $R \cap P = (0)$. Since a_1 is not a zerodivisor, we have that the coefficients of f are all zero. It follows that $P \cap \mathring{R} = (0)$, and thus, we have $P \cap S = (0)$.

Now suppose that $\mathfrak{q} \in \mathcal{Q}_R$. If $a_1 \notin \mathfrak{q}$, then, by the way that t was chosen, $x_2 + \mathfrak{q}$ is transcendental over $R/\mathfrak{q} \cap R$. Let $f \in \mathfrak{q} \cap \mathring{R}$. Viewing f as an element of $R[x_2, a_1^{-1}]$, for sufficiently large k, we have that $(a_1)^k f \in R[x_2]$. It follows that the coefficients of $(a_1)^k f$ are in $\mathfrak{q} \cap R = pR$ for some prime element p of R. Hence, we have $f \in pR[x_2, a_1^{-1}]$. Similarly, if $a_2 \notin \mathfrak{q}$, then $f \in pR[x_1, a_2^{-1}]$. If $a_2 \in \mathfrak{q}$, then $a_2 \in R \cap \mathfrak{q} = pR$, and thus, p is a unit in $R[x_1, a_2^{-1}]$. Therefore,

$$pR[x_1, a_2^{-1}] = R[x_1, a_2^{-1}].$$

In any case,

$$f \in pR[x_2, a_1^{-1}] \cap pR[x_1, a_2^{-1}] = p\mathring{R},$$

and it follows that $\mathfrak{q} \cap \mathring{R} = p\mathring{R}$, and thus, $\mathfrak{q} \cap S = pS$. We will define $\mathcal{Q}_S$ to contain $\mathcal{Q}_R$; then, each height one prime ideal of S generated by a prime element of R will have a corresponding element in $\mathcal{Q}_S$.

For every height one prime ideal of S whose intersection with R is the zero ideal, we use the same procedure as in Lemma 3.4 to choose an appropriate corresponding $\mathfrak{q}_p \in \operatorname{Spec}_d T$. We define $\mathcal{Q}_S$ to be the union of the set of these prime ideals of T with $\mathcal{Q}_R$. Then, S is our desired A_+-extension of R, and the $n = 2$ case is shown.

For $n > 2$, we construct an A_+-extension R'' of R with

$$R \subseteq R'' \subset T, \qquad P \cap R'' = (0),$$

and such that there exists a $c^* \in R''$ and an $(n-1)$ generated ideal $\mathfrak{b}$ of R'' with $c^* \in \mathfrak{b}T$. By induction, there exists an A_+-extension S of R'' such that

$$c^* \in \mathfrak{b}S \quad \text{and} \quad P \cap S = (0).$$

We then show that $c \in \mathfrak{a}S$, which will complete the proof.

Let $\mathfrak{a} = (a_1, \ldots, a_n)R$, and let $\mathfrak{b} = (a_1, \ldots, a_{n-1})R$. We first assume that $\mathfrak{b}$ is not contained in a height one prime ideal of R. Since $c \in \mathfrak{a}T \cap R$, we can write $c = \sum a_i t_i$, where $t_i \in T$. We define

$$\widetilde{t} = t_n + \sum_{i=1}^{n-1} a_i u_i,$$

where $u_i \in T$ will be chosen later, and we define $c^* = c - \widetilde{t} a_n$. Note that $c^* \in \mathfrak{b}T$ as desired. Now, we work to define R'' so that it is an A_+-extension of R, $c^* \in R''$ and $P \cap R'' = (0)$. Since we are assuming that $\mathfrak{b}$ is not contained in a height one prime ideal of R, we cannot have a $\mathfrak{p} \in C$ that contains all of

$$a_1, a_2, \ldots, a_{n-1}.$$

Now, as in the $n = 2$ case, choose $u_1 \in T$ so that $t_n + a_1 u_1$ is transcendental over $R/\mathfrak{p} \cap R$ for all $\mathfrak{p} \in C$ satisfying $a_1 \notin \mathfrak{p}$.

We then repeat the process and choose u_2 so that $t_n + a_1 u_1 + a_2 u_2$ is transcendental over $R/\mathfrak{p} \cap R$ whenever $\mathfrak{p} \in C$ and $a_2 \notin \mathfrak{p}$. Now, if $a_2 \in \mathfrak{p}$ and $a_1 \notin \mathfrak{p}$, then $a_2 u_2 \in \mathfrak{p}$, and $t_n + a_1 u_1 + a_2 u_2 + \mathfrak{p}$ is transcendental over $R/\mathfrak{p} \cap R$; thus, transcendental elements will remain transcendental as additional terms are added. Continuing the process until all u_i have been chosen, we obtain $\widetilde{t}$ so that $\widetilde{t} + \mathfrak{p}$ is transcendental over $R/\mathfrak{p} \cap R$ for all $\mathfrak{p} \in C$. We then define

$$R'' = R[\widetilde{t}]_{\mathfrak{m} \cap R[\widetilde{t}]},$$

and we note that $c^* \in R''$. As in the $n = 2$ case, we have that $P \cap R'' = (0)$. Following the proof of Lemma 3.4, R'' can be shown to be an A_+-extension of R. By induction, there exists an A_+-extension S of R'' such that $c^* \in \mathfrak{b}S$ and $P \cap S = (0)$. Since $c = c^* + \widetilde{t}a_n$, we have that $c \in \mathfrak{a}S$ as desired.

Now, suppose that $\mathfrak{b}$ is contained in a height one prime ideal of R. By factoring out common divisors, we obtain an element $r \in R$ and an ideal $\mathfrak{b}^* = (z_1, \ldots, z_{n-1})$ of R such that $\mathfrak{b} = r\mathfrak{b}^*$, $a_i = rz_i$ for all $i = 1, 2, \ldots, n-1$, and $\mathfrak{b}^*$ is not contained in a height one prime ideal of R. Define

$$w = \sum_{i=1}^{n-1} t_i z_i.$$

Then, $c = rw + t_n a_n$. We now use the $n = 2$ case with $\mathfrak{a} = (r, a_n)$ to find an A_+-extension R''' of R such that $P \cap R''' = (0)$ and $c = v_1 a_n + v_2 r$ with v_1 and v_2 in R'''. Note that $\mathfrak{b}^* R'''$ is not contained in a height one prime ideal of R''', and $v_2 = w - ta_n$ for some $t \in T$. Hence, we can use the previous case with $\mathfrak{b} = \mathfrak{b}^* R'''$, $\mathfrak{a} = (z_1, z_2, \ldots, z_{n-1}, a_n)R'''$ and $c = v_2$ to obtain our A_+-extension R'' of R, and our element $c^* \in \mathfrak{b}^* T$. By induction, we get S. We need only show that $c \in \mathfrak{a}S$. Since $v_2 \in \mathfrak{b}^* S + a_n S$, we have that $v_2 r \in \mathfrak{b}S + r a_n S$, and it follows that $c = v_1 a_n + v_2 r \in \mathfrak{a}S$, as desired. □

Recall that, at the end of our construction, we want $\alpha(A, (0)) = t$ and $\alpha(A, pA) = d - 1$ for all prime elements p of A. We will construct A to be a UFD and to satisfy condition (iv) of Z_d-subrings; thus, $\alpha(A, pA) \geq d - 1$ for all prime elements p of A. For the case where $d - 1 = t$, we will construct A so that $\alpha(A, (0)) \leq d - 1$. In order to ensure this, we simply need A to contain a nonzero element of every height d prime ideal of T. The next lemma allows us to adjoin nonzero elements of the height d prime ideals of T to A. Then, by Matsumura's inequality, we have $\alpha(A, (0)) \geq \alpha(A, pA)$ for all prime elements p of A, and so, we get that

$$d - 1 \geq \alpha(A, (0)) \geq \alpha(A, pA) \geq d - 1.$$

We can conclude that

$$\alpha(A, (0)) = \alpha(A, pA) = d - 1,$$

as desired. For the case $d-1 \neq t$, we construct A so that there is a prime ideal P of T of height t such that $P \cap A = (0)$ so that $\alpha(A,(0)) \geq t$. In addition, if $\mathfrak{q}$ is a prime ideal of T whose height is greater than d and $\mathfrak{q} \not\subseteq P$, we use Lemma 3.7 to construct A so that $\operatorname{ht}(\mathfrak{q} \cap A) > 1$. This ensures that $\alpha(A,(0)) \leq t$ and that, if $\mathfrak{p}$ is a height one prime ideal of A, then $\alpha(A,\mathfrak{p}) \leq d-1$.

Lemma 3.6. *Under Assumption* 3.3*, let* $(T,\mathfrak{m})$ *be such that, for all* $\mathfrak{p} \in \operatorname{Ass} T$*,* $\operatorname{ht}\mathfrak{p} \leq d-1$*, and suppose that* T *satisfies the condition that, if* z *is a regular element of* T *and* $Q \in \operatorname{Ass}(T/zT)$*, then* $\operatorname{ht} Q \leq d$*. Let* P *be a nonmaximal prime ideal of* T *such that* $P \cap R = (0)$*, and let* $\mathfrak{q} \in \operatorname{Spec}_r T$ *with* $d \leq r \leq \dim T$ *and such that* $\mathfrak{q} \not\subseteq P$*, and* $\mathfrak{q} \cap R = (0)$*. Then, there exists an* A_+*-extension* S *of* R *such that* $S \cap P = (0)$ *and* $\mathfrak{q} \cap S \neq (0)$*.*

Proof. Let C be the same set detailed in the proof of Lemma 3.4. Our hypotheses yield that $\mathfrak{q} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in C$. Using Lemma 2.3 or Lemma 2.4, choose $x \in \mathfrak{q}$ such that $x+\mathfrak{p}$ is transcendental over $R/\mathfrak{p}\cap R$ for every $\mathfrak{p} \in C$. We let $S = R[x]_{\mathfrak{m}\cap R[x]}$. As in the proof of Lemma 3.4, S will be our desired A_+-extension of R. □

The following lemma allows us to control the formal fibers at height one prime ideals. We want to make sure that there are no prime ideals of T with height greater than d in the formal fiber of a height one prime ideal of our final ring. We accomplish this in the next lemma by adjoining elements of prime ideals of T of height greater than d.

Lemma 3.7. *Under Assumption* 3.3*, let* $(T,\mathfrak{m})$ *be such that, for all* $\mathfrak{p} \in \operatorname{Ass} T$*,* $\operatorname{ht}\mathfrak{p} \leq d-1$*, and suppose that* T *satisfies the condition that, if* z *is a regular element of* T *and* $Q \in \operatorname{Ass}(T/zT)$*, then* $\operatorname{ht} Q \leq d$*. Let* P *be a nonmaximal prime ideal of* T *such that* $P \cap R = (0)$*. Let* $\mathfrak{q} \in \operatorname{Spec}_r T$ *with* $d+1 \leq r \leq \dim T$*,* $\mathfrak{q} \not\subseteq P$*, and* $\operatorname{ht}(\mathfrak{q} \cap R) \leq 1$*. Then, there exists an* A_+*-extension* S *of* R *such that* $P \cap S = (0)$ *and* $\operatorname{ht}(\mathfrak{q} \cap S) > 1$*.*

Proof. First, suppose that $\mathfrak{q} \cap R = (0)$. Then, use Lemma 3.6 to find an A_+-extension S_0 such that $\mathfrak{q} \cap S_0 \neq (0)$ and $P \cap S_0 = (0)$. Now, $\operatorname{ht}(\mathfrak{q} \cap S_0) \geq 1$. If

$$\operatorname{ht}(\mathfrak{q} \cap S_0) > 1,$$

then $S = S_0$, and we are done. Therefore, consider the case where $\mathrm{ht}(\mathfrak{q} \cap S_0) = 1$. Then, since S_0 is a UFD, $\mathfrak{q} \cap S_0 = pS_0$ for some prime element $p \in S_0$. Let $\mathfrak{q}_p$ be the element of $\mathcal{Q}_{S_0}$ corresponding to pS_0. Now, we let

$$C = \{\mathfrak{p} \in \operatorname{Spec} T \mid \mathfrak{p} \in \operatorname{Ass}(T/rT) \text{ with } 0 \neq r \in S_0\} \cup \operatorname{Ass} T \cup \mathcal{Q}_{S_0} \cup \{P\},$$

and, as before, adjoin some element $x \in \mathfrak{q}$ such that $x + \mathfrak{p}$ is transcendental over $S_0/\mathfrak{p} \cap S_0$ for every $\mathfrak{p} \in C$. We let $S = S_0[x]_{\mathfrak{m} \cap S_0[x]}$. Then, S is an A_+-extension of S_0, which is an A_+-extension of R, and so, S is an A_+-extension of R. Since $P \in C$, we have that $P \cap S = (0)$. Note that $pS \subseteq \mathfrak{q} \cap S$, and that pS is a height one prime ideal of S. Clearly, $x \in \mathfrak{q} \cap S$. If $x \in pS$, then $x \in \mathfrak{q}_p$, contradicting that $x + \mathfrak{q}_p$ is transcendental over $S_0/\mathfrak{q}_p \cap S_0$. It follows that pS is strictly contained in $\mathfrak{q} \cap S$, and thus, $\mathrm{ht}(\mathfrak{q} \cap S) > 1$. □

Lemma 3.8. *Under Assumption* 3.3*, let* $(T, \mathfrak{m})$ *be such that* $\operatorname{depth} T > 1$ *and, for all* $\mathfrak{p} \in \operatorname{Ass} T$*,* $\operatorname{ht} \mathfrak{p} \leq d - 1$*, and suppose that* T *satisfies the condition that, if* z *is a regular element of* T *and* $Q \in \operatorname{Ass}(T/zT)$*, then* $\operatorname{ht} Q \leq d$*. Let* P *be a nonmaximal prime ideal of* T *such that* $P \cap R = (0)$*, and let* $t \in T$*. If* $\mathfrak{q} \in \operatorname{Spec}_d T$ *with* $\mathfrak{q} \not\subseteq P$*, then there exists an* A_+*-extension* S *of* R *such that*

$$t + \mathfrak{m}^2 \in \operatorname{Image}(S \longrightarrow T/\mathfrak{m}^2),$$

$P \cap S = (0)$ *and* $\mathfrak{q} \cap S \neq (0)$*. If* $\mathfrak{q} \in \operatorname{Spec}_r T$ *with* $d + 1 \leq r \leq \dim T$*, and* $\mathfrak{q} \not\subseteq P$*, then there exists an* A_+*-extension* S *of* R *such that* $t + \mathfrak{m}^2 \in \operatorname{Image}(S \to T/\mathfrak{m}^2)$*,* $P \cap S = (0)$ *and* $\operatorname{ht}(\mathfrak{q} \cap S) > 1$*.*

Proof. If $\mathfrak{q} \in \operatorname{Spec}_d T$, use Lemma 3.4 to find an A_+-extension R' of R such that

$$t + \mathfrak{m}^2 \in \operatorname{Image}(R' \longrightarrow T/\mathfrak{m}^2)$$

and $P \cap R' = (0)$. If $\mathfrak{q} \cap R' \neq (0)$, then $S = R'$, and we are done. If $\mathfrak{q} \cap R' = (0)$, then we use Lemma 3.6 to find an A_+-extension S of R' such that $\mathfrak{q} \cap S \neq (0)$ and $P \cap S = (0)$.

If $\mathfrak{q} \in \operatorname{Spec}_r T$, where $d + 1 \leq r \leq \dim T$, then, as before, first use Lemma 3.4 to find an A_+-extension R' of R such that $t + \mathfrak{m}^2 \in \operatorname{Image}(R' \to T/\mathfrak{m}^2)$ and $P \cap R' = (0)$. If $\mathrm{ht}(\mathfrak{q} \cap R') > 1$, then $S = R'$, and we are done. If $\mathrm{ht}(\mathfrak{q} \cap R') \leq 1$, then use Lemma 3.7 to find an A_+-extension of R' such that $P \cap S = (0)$ and $\mathrm{ht}(\mathfrak{q} \cap S) > 1$. □

For the proof of our main theorem, we apply Lemmas 3.5 and 3.8 infinitely often. The next result, adapted from **[3,** Lemma 6], will allow us to do so.

Lemma 3.9. *Let $(T, \mathfrak{m})$ be a complete local ring and R_0 a Z_d-subring of T with distinguished set $\mathcal{Q}_{R_0}$. Let P be a nonmaximal ideal of T such that $P \cap R_0 = (0)$. Let Ω be a well-ordered set with least element 0, and assume either Ω is countable or, for all $\beta \in \Omega$,*

$$|\{\gamma \in \Omega \mid \gamma < \beta\}| < |T/\mathfrak{m}|.$$

Suppose that $\{R_\beta \mid \beta \in \Omega\}$ is an ascending collection of rings such that $R_\alpha \cap P = (0)$ for every $\alpha \in \Omega$ and such that, if β is a limit ordinal, then

$$R_\beta = \bigcup_{\gamma<\beta} R_\gamma,$$

with $\mathcal{Q}_{R_\beta}$ defined as

$$\bigcup_{\gamma<\beta} Q_{R_\gamma},$$

while, if $\beta = \gamma + 1$ is a successor ordinal, then R_β is an A_+-extension of R_γ.

Then,

$$S = \bigcup_{\beta\in\Omega} R_\beta$$

satisfies all of the conditions to be a Z_d-subring of T with distinguished set

$$\mathcal{Q}_S = \bigcup_{\beta\in\Omega} Q_{R_\beta},$$

except the cardinality condition. Instead, $|S| \leq \sup(\aleph_0, |R_0|, |\Omega|)$. Furthermore, $P \cap S = (0)$, elements which are prime in some R_β remain prime in S, and $\mathcal{Q}_{R_0} \subseteq \mathcal{Q}_S$.

Proof. First note that, since $R_\alpha \cap P = (0)$ for every $\alpha \in \Omega$, it is clear that $S \cap P = (0)$. We now follow the proof of **[3,** Lemma 6], adding steps where necessary. Define $\Omega' = \Omega \cup \{\delta\}$, and declare that $\delta > \alpha$ for all $\alpha \in \Omega$. Now, define $R_\delta = S$. We will show that, for all $\alpha \in \Omega'$, there is a set $\mathcal{Q}_{R_\alpha}$ such that R_α satisfies conditions (ii), (iii), and (iv) of Definition 3.1 and that

$$|R_\alpha| \leq \sup(\aleph_0, |R_0|, |\{\beta \in \Omega \mid \beta < \alpha\}|).$$

Furthermore, we will show that, for $\beta < \alpha$, we have $\mathcal{Q}_{R_\beta} \subseteq \mathcal{Q}_{R_\alpha}$ and prime elements of R_β remain prime in R_α. We proceed with transfinite induction, the base case being trivial.

Assume that $\alpha \in \Omega'$, and that the inductive hypotheses hold for every $\beta < \alpha$. In the proof of **[3,** Lemma 6], it is shown that R_α satisfies conditions (ii) and (iii) of Definition 3.1, that the cardinality condition given in the preceding paragraph holds, and that, if $\beta < \alpha$, every prime element of R_β is prime in R_α. If $\alpha = \gamma + 1$ is a successor ordinal, then R_α is an A_+-extension of R_γ, and so, R_α satisfies condition (iv) of Definition 3.1 and $\mathcal{Q}_{R_\gamma} \subseteq \mathcal{Q}_{R_\alpha}$.

If α is a limit ordinal, then

$$R_\alpha = \bigcup_{\gamma<\alpha} R_\gamma,$$

and we have

$$\mathcal{Q}_{R_\alpha} = \bigcup_{\gamma<\alpha} \mathcal{Q}_{R_\gamma}.$$

By definition, $\mathcal{Q}_{R_\gamma} \subseteq \mathcal{Q}_{R_\alpha}$ for all $\gamma < \alpha$. It remains to show that $\mathcal{Q}_{R_\alpha}$ is a distinguished set for R_α. Let $\mathfrak{q}_p \in \mathcal{Q}_{R_\alpha}$. Then, $\mathfrak{q}_p \in \mathcal{Q}_{R_\gamma}$ for some $\gamma < \alpha$. Therefore, $\mathfrak{q}_p \cap R_\gamma = pR_\gamma$ for some prime element p of R_γ. We will show that $\mathfrak{q}_p \cap R_\alpha = pR_\alpha$. Clearly, $pR_\alpha \subseteq \mathfrak{q}_p \cap R_\alpha$. If $x \in R_\alpha \cap \mathfrak{q}_p$, then $x \in R_\beta$ for some $\beta < \alpha$. Define $\lambda = \max\{\beta, \gamma\}$. Then, $p, x \in R_\lambda$ and $\mathfrak{q}_p \in \mathcal{Q}_{R_\lambda}$. It follows that

$$\mathfrak{q}_p \cap R_\lambda = pR_\lambda;$$

thus,

$$x \in \mathfrak{q}_p \cap R_\lambda = pR_\lambda \subseteq pR_\alpha,$$

and we have that $\mathfrak{q}_p \cap R_\alpha = pR_\alpha$ as desired. It is not difficult to show that, if $\mathfrak{q}_p, \mathfrak{q}'_p \in \mathcal{Q}_{R_\alpha}$ with

$$\mathfrak{q}_p \cap R_\alpha = \mathfrak{q}'_p \cap R_\alpha,$$

then $\mathfrak{q}_p = \mathfrak{q}'_p$, and that, if $\mathfrak{p} = pR_\alpha$ is a height one prime ideal of R_α, then there is a $\mathfrak{q}_p \in \mathcal{Q}_{R_\alpha}$ such that

$$\mathfrak{q}_p \cap R_\alpha = \mathfrak{p}.$$

Hence, $\mathcal{Q}_{R_\alpha}$ is a distinguished set for R_α.

By induction, then, we have that S satisfies all of the desired properties. □

Lemma 3.10. *Under Assumption* 3.3*, let* $(T, \mathfrak{m})$ *be such that* $\operatorname{depth} T > 1$ *and, for all* $\mathfrak{p} \in \operatorname{Ass} T$*,* $\operatorname{ht} \mathfrak{p} \leq d-1$*, and suppose that* T *satisfies the condition that, if* z *is a regular element of* T *and* $Q \in \operatorname{Ass}(T/zT)$*, then* $\operatorname{ht} Q \leq d$*. Let* P *be a nonmaximal prime ideal of* T *such that* $P \cap R = (0)$*, and let* $t + \mathfrak{m}^2 \in T/\mathfrak{m}^2$*. If* $\mathfrak{q} \in \operatorname{Spec}_d T$ *with* $\mathfrak{q} \not\subseteq P$*, then there exists an* A_+*-extension* S *of* R *such that*

$$t + \mathfrak{m}^2 \in \operatorname{Image}(S \longrightarrow T/\mathfrak{m}^2),$$

$\mathfrak{q} \cap S \neq (0)$*,* $P \cap S = (0)$*, and, for every finitely generated ideal* $\mathfrak{a}$ *of* S*,* $\mathfrak{a}T \cap S = \mathfrak{a}$*. If* $\mathfrak{q} \in \operatorname{Spec}_r T$ *with* $d+1 \leq r \leq \dim T$ *and* $\mathfrak{q} \not\subseteq P$*, then there exists an* A_+*-extension* S *of* R *such that* $t + \mathfrak{m}^2 \in \operatorname{Image}(S \to T/\mathfrak{m}^2)$*,* $\operatorname{ht}(\mathfrak{q} \cap S) > 1$*,* $P \cap S = (0)$*, and, for every finitely generated ideal* $\mathfrak{a}$ *of* S*,* $\mathfrak{a}T \cap S = \mathfrak{a}$*.*

Proof. If $\mathfrak{q} \in \operatorname{Spec}_d T$, employ Lemma 3.8 to obtain an A_+-extension R_0 of R such that $t + \mathfrak{m}^2 \in \operatorname{Image}(R_0 \to T/\mathfrak{m}^2)$, $P \cap R_0 = (0)$, and $\mathfrak{q} \cap R_0 \neq (0)$. If $\mathfrak{q} \in \operatorname{Spec}_r T$ with $d+1 \leq r \leq \dim T$, employ Lemma 3.8 to obtain an A_+-extension R_0 of R such that $t + \mathfrak{m}^2 \in \operatorname{Image}(R_0 \to T/\mathfrak{m}^2)$, $P \cap R_0 = (0)$, and $\operatorname{ht}(\mathfrak{q} \cap R_0) > 1$. Let

$$\Omega = \{(\mathfrak{a}, c) \mid \mathfrak{a} \text{ is a finitely generated ideal of } R_0 \text{ and } c \in \mathfrak{a}T \cap R_0\}.$$

Then, $|\Omega| = |R_0|$, and so, either Ω is countable or $|\Omega| < |T/\mathfrak{m}|$. Well order Ω, letting 0 designate its initial element in such a way that Ω does not have a maximal element; then, it clearly satisfies the hypothesis of Lemma 3.9. We will recursively define an increasing chain of rings with one ring for every element of Ω. We begin with R_0. If $\beta = \gamma + 1$ is a successor ordinal and $\gamma = (\mathfrak{a}, c)$, then we choose R_β to be an A_+-extension of R_γ, given by Lemma 3.5 such that $c \in \mathfrak{a}R_\beta$ and $P \cap R_\beta = (0)$. If β is a limit ordinal, define

$$R_\beta = \bigcup_{\gamma<\beta} R_\gamma \quad \text{and} \quad \mathcal{Q}_{R_\beta} = \bigcup_{\gamma<\beta} \mathcal{Q}_{R_\gamma}.$$

Set $R_1 = \bigcup R_\beta$. From Lemma 3.9, we see that R_1 is an A_+-extension of R_0 and $P \cap R_1 = (0)$. Also, if $\mathfrak{a}$ is any finitely generated ideal of

R_0 and $c \in \mathfrak{a}T \cap R_0$, then $(\mathfrak{a}, c) = \gamma$ for some $\gamma \in \Omega$. Then, for some $\beta > \gamma$, $c \in \mathfrak{a}R_\beta \subseteq \mathfrak{a}R_1$. Thus, $\mathfrak{a}T \cap R_0 \subseteq \mathfrak{a}R_1$.

We repeat the process to obtain an A_+-extension R_2 of R_1 such that $P \cap R_2 = (0)$ and $\mathfrak{a}T \cap R_1 \subseteq \mathfrak{a}R_2$ for every finitely generated ideal $\mathfrak{a}$ of R_1. Continue recursively to obtain an ascending chain $R_0 \subseteq R_1 \subseteq \cdots$ such that $P \cap R_n = (0)$ and $\mathfrak{a}T \cap R_n \subset \mathfrak{a}R_{n+1}$ for every finitely generated ideal $\mathfrak{a}$ of R_n. Then, by Lemma 3.9,

$$S = \bigcup R_i \quad \text{with } \mathcal{Q}_S = \bigcup \mathcal{Q}_{R_i}$$

is an A_+-extension of R_0, and so, also of R, and $P \cap S = (0)$. Furthermore, if $\mathfrak{a}$ is a finitely generated ideal of S, then some R_n contains a generating set $\{a_1, \ldots, a_k\}$ for $\mathfrak{a}$. If $c \in \mathfrak{a}T \cap S$, then $c \in R_m$ for some $m \geq n$; thus,

$$c \in (a_1, \ldots, a_k)T \cap R_m \subseteq (a_1, \ldots, a_k)R_{m+1} \subseteq \mathfrak{a}.$$

Hence, $\mathfrak{a}T \cap S = \mathfrak{a}$. In order to see that $\operatorname{ht}(\mathfrak{q} \cap R_0) > 1$ implies $\operatorname{ht}(\mathfrak{q} \cap S) > 1$, let

$$(0) \subset pR_0 \subset \mathfrak{q} \cap R_0$$

be a strictly increasing chain of prime ideals of R_0, and let $x \in \mathfrak{q} \cap R_0$ with $x \notin pR_0$. Since prime elements in R_0 are prime in S, the prime factorization of x in R_0 is the prime factorization of x in S, and thus, $x \notin pS$, and we have

$$(0) \subset pS \subset \mathfrak{q} \cap S$$

is a strictly increasing chain of prime ideals of S. □

4. The main theorem and corollaries.

Theorem 4.1. *Let $(T, \mathfrak{m})$ be a complete local equidimensional ring such that $\dim T \geq 2$ and $\operatorname{depth} T > 1$. Suppose that no integer of T is a zerodivisor in T and $|T| = |T/\mathfrak{m}|$. Let d and t be integers such that $1 \leq d \leq \dim T - 1$, $0 \leq t \leq \dim T - 1$ and $d - 1 \leq t$. Assume that, for every $\mathfrak{p} \in \operatorname{Ass} T$, $\operatorname{ht} \mathfrak{p} \leq d-1$ and that, if z is a regular element of T and $Q \in \operatorname{Ass}(T/zT)$, then $\operatorname{ht} Q \leq d$. Then, there exists a local* UFD *A such that $\widehat{A} = T$, $\alpha(A, (0)) = t$, and, if $\mathfrak{p} \in \operatorname{Spec}_1 A$, then $\alpha(A, \mathfrak{p}) = d - 1$.*

Proof. First, assume $t = d - 1$. Let $\Omega_1 = \operatorname{Spec}_d T$, and recall that $|\Omega_1|$ is a cardinal number that we can think of as the set of ordinal

numbers less than $|\Omega_1|$. Well order Ω_1 using the elements in the cardinal $|\Omega_1|$. Then, each element of Ω_1 has fewer than $|\Omega_1|$ predecessors. Similarly, let $\Omega_2 = T/\mathfrak{m}^2$, well ordered so that each element of Ω_2 has fewer than $|\Omega_2|$ predecessors. Since $|\Omega_1| = |\Omega_2|$ and we are ordering both sets so that each element of Ω_i has fewer than $|\Omega_i|$ predecessors, we can use $\mathscr{B}$ as the index set for both of the Ω_is. Let

$$\Omega = \{(\mathfrak{q}_a, t_a) \mid \mathfrak{q}_a \in \Omega_1, t_a \in \Omega_2 \text{ where } a \in \mathscr{B}\},$$

well ordered using $\mathscr{B}$ as the index set. Then, Ω is the diagonal of $\Omega_1 \times \Omega_2$. Let 0 designate the first element of Ω. Now, Ω satisfies the hypothesis of Lemma 3.9, and we now recursively define a family of rings

$$\{R_\beta \mid \beta \in \Omega\}$$

which also satisfies the hypotheses of Lemma 3.9. As in the proof of **[3,** Theorem 8], let R_0 be the appropriate localization of the prime subring of T: either $\mathbb{Q}$, $\mathbb{Z}_p$ or $\mathbb{Z}_{(p)}$, where p is a prime integer. It is not difficult to verify that R_0 is an N-subring. Now, R_0 is either dimension 0 or 1. If it is dimension 0, define $\mathcal{Q}_{R_0}$ to be the empty set, and, if it is dimension 1, define $\mathcal{Q}_{R_0}$ to be any height d prime ideal of T that contains a minimal associated prime ideal of pT. Then, R_0 is a Z_d-subring of T with distinguished set $\mathcal{Q}_{R_0}$. We will use the shorthand $\beta = (\mathfrak{q}_\beta, t_\beta)$ for an element of Ω. Then, whenever $\beta = \omega + 1$ is a successor ordinal, we let R_β be an A_+-extension of R_ω, chosen in accordance with Lemma 3.10 so that $\mathfrak{q}_\omega \cap R_\beta \neq (0)$, $t_\omega \in \text{Image}(R_\beta \to T/\mathfrak{m}^2)$ and $\mathfrak{a}T \cap R_\beta = \mathfrak{a}$ for every finitely generated ideal $\mathfrak{a}$ of R_β. Note that, in this case (where $t = d - 1$), we do not need to use the P given in Lemma 3.10. If β is a limit ordinal, choose

$$R_\beta = \bigcup_{\gamma<\beta} R_\gamma.$$

We claim that

$$A = \bigcup_{\beta\in\Omega} R_\beta$$

is the desired example.

By construction, $A \to T/\mathfrak{m}^2$ is onto. Now, let $\mathfrak{a} = (a_1, \ldots, a_n)$ be a finitely generated ideal of A, and let $x \in \mathfrak{a}T \cap A$. Then, there is a $\beta \in \Omega$ such that β is a successor ordinal and $x, a_1, \ldots, a_n \in R_\beta$; thus,

$$x \in (a_1, \ldots, a_n)T \cap R_\beta = (a_1, \ldots, a_n)R_\beta \subseteq \mathfrak{a}.$$

It follows that $\mathfrak{a}T \cap A = \mathfrak{a}$ for all finitely generated ideals of A. Then, by Lemma 2.1, $\widehat{A} = T$ and A is Noetherian. From Lemma 3.9, except for the cardinality condition, A satisfies all of the conditions of Definition 3.1 for being a Z_d-subring of T with some distinguished set, which implies that, for any $\mathfrak{p} \in \operatorname{Spec}_1 A$,

$$\alpha(A, \mathfrak{p}) \geq d - 1.$$

Now, let $\mathfrak{q} \in \operatorname{Spec}_d T$. Then, by our construction, there is some R_β such that $\mathfrak{q} \cap R_\beta \neq (0)$, and hence, $\mathfrak{q} \cap A \neq (0)$. Thus, $\alpha(A, (0)) \leq d - 1$. Then, by [**6,** Theorem 1],

$$d - 1 \geq \alpha(A, (0)) \geq \alpha(A, \mathfrak{p}) \geq d - 1.$$

Therefore, for every $\mathfrak{p} \in \operatorname{Spec}_1 A$, $\alpha(A, \mathfrak{p}) = \alpha(A, (0)) = d - 1$.

Now, we consider the case $t \neq d - 1$. In this case, we adjust the construction described above. We first find a prime ideal P of T such that $\operatorname{ht} P = t$ and $P \cap R_0 = (0)$. If $R_0 = \mathbb{Q}$ or $R_0 = \mathbb{Z}_p$, then all of the nonzero elements of R_0 are units of T, and thus, we can choose P to be any prime ideal of T with height t. If $R_0 = \mathbb{Z}_{(p)}$ and $t = 0$, then choose P to be any minimal prime ideal of T. Since p is not a zerodivisor of T, we have $P \cap R_0 = (0)$. If $R_0 = \mathbb{Z}_{(p)}$, and $t > 0$, then note that p is contained in finitely many height one prime ideals of T, and hence, there is a height one prime ideal P_1 such that $P_1 \cap R_0 = (0)$. Now, the element $p + P_1$ in T/P_1 is contained in finitely many height one prime ideals of T/P_1, and thus, there is a height one prime ideal $\overline{P}_2$ of T/P_1 such that $p + P_1 \notin \overline{P}_2$. As T is universally catenary and equidimensional, there is a height two prime ideal P_2 of T such that $R_0 \cap P_2 = (0)$. Continue in this way to find a height t prime ideal P of T such that $P \cap R_0 = (0)$.

Now suppose that $t = d = \dim T - 1$. Then, using the same construction described for the $t = d - 1$ case with the following adjustments, let $\Omega_1 = \operatorname{Spec}_d T - \{P\}$ where P is the height t prime ideal of T chosen in the above paragraph. Then, in the construction, whenever $\beta = \omega + 1$ is a successor ordinal, let R_β be an A_+-extension of R_ω, chosen in accordance with Lemma 3.10, so that $\mathfrak{q}_\omega \cap R_\beta \neq (0)$, $P \cap R_\beta = (0)$,

$$t_\omega \in \operatorname{Image}(R_\beta \longrightarrow T/\mathfrak{m}^2),$$

and $\mathfrak{a}T \cap R_\beta = \mathfrak{a}R_\beta$ for every finitely generated ideal $\mathfrak{a}$ of R_ω. Then,

using this adjusted construction, we obtain a UFD A such that $\widehat{A} = T$, $P \cap A = (0)$, and, for every $\mathfrak{p} \in \operatorname{Spec}_1 A$, there is a height $d = \dim T - 1$ prime ideal of T whose intersection with A is $\mathfrak{p}$. It follows that $\alpha(A, (0)) = t$ and $\alpha(A, \mathfrak{p}) = d - 1$ for all $\mathfrak{p} \in \operatorname{Spec}_1 A$.

Finally, consider all other cases, that is, suppose either that $t > d$ or $t = d$ with $t < \dim T - 1$. Then, use the same construction described for the $t = d - 1$ case with the following adjustments. Let

$$\Omega_1 = \{\mathfrak{q} \in \operatorname{Spec}_r T \mid d + 1 \leq r \leq \dim T \text{ and } \mathfrak{q} \not\subseteq P\},$$

where P is the height t prime ideal of T chosen previously. Then, in the construction, whenever $\beta = \omega + 1$ is a successor ordinal, let R_β be an A_+-extension of R_ω, chosen in accordance with Lemma 3.10 so that $\operatorname{ht}(\mathfrak{q}_\omega \cap R_\beta) > 1$, $P \cap R_\beta = (0)$, $t_\omega \in \operatorname{Image}(R_\beta \to T/\mathfrak{m}^2)$ and $\mathfrak{a}T \cap R_\beta = \mathfrak{a}R_\beta$, for every finitely generated ideal $\mathfrak{a}$ of R_β. Then, using this adjusted construction, we get a UFD A such that $\widehat{A} = T$, $P \cap A = (0)$, and, for every $\mathfrak{p} \in \operatorname{Spec}_1 A$, there is a height d prime ideal of T whose intersection with A is $\mathfrak{p}$. Moreover, if $\mathfrak{q} \in \operatorname{Spec}_r T$ with $d + 1 \leq r \leq \dim T$ and $\mathfrak{q} \not\subseteq P$, then $\operatorname{ht}(\mathfrak{q} \cap A) > 1$. It follows that

$$\alpha(A, (0)) = t \quad \text{and} \quad \alpha(A, \mathfrak{p}) = d - 1$$

for all $\mathfrak{p} \in \operatorname{Spec}_1 A$. □

Corollary 4.2. *Let $(T, \mathfrak{m})$ be a complete local equidimensional ring such that $\dim T \geq 2$, no integer is a zerodivisor in T, $\operatorname{depth} T > 1$ and $|T| = |T/\mathfrak{m}|$. Let d be an integer such that $1 \leq d \leq \dim T - 1$, and assume that, for every $\mathfrak{p} \in \operatorname{Ass} T$, $\operatorname{ht} \mathfrak{p} \leq d - 1$ and that, if z is a regular element of T and $Q \in \operatorname{Ass}(T/zT)$, then $\operatorname{ht} Q \leq d$. Then, there exists a local unique factorization domain A such that $\widehat{A} = T$ and, for every $pA \in \operatorname{Spec}_1 A$,*

$$\alpha(A, pA) = \alpha(A, (0)) = d - 1.$$

Note that Corollary 4.2 answers Question 1.1 stated in the introduction, posed by Heinzer, Rotthaus and Sally in the nonexcellent case. In particular, it shows that there are nonexcellent local integral domains where the dimension of the generic formal fiber ring is positive, and where the set

$$\Delta = \{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{ht} \mathfrak{p} = 1 \text{ and } \alpha(A, \mathfrak{p}) = \alpha(A)\}$$

is equal to the set of all of the height one prime ideals of the integral domain.

Example 4.3. Let $T = \mathbb{C}[[x_1, \ldots, x_n]]$. Then, for any $1 \leq d \leq n-1$, $0 \leq t \leq n-1$ and $d-1 \leq t$, we can find a local unique factorization domain A such that $\widehat{A} = T$, $\alpha(A, (0)) = t$, and $\alpha(A, pA) = d-1$ for every $pA \in \operatorname{Spec}_1 A$. Note that this will also hold if we replace $\mathbb{C}$ with any uncountable field.

Example 4.4. Let $T = \mathbb{C}[[x, y, z, w]]/(xy - zw)$. Then, there exists a local unique factorization domain A such that $\widehat{A} = T$ and $\alpha(A, pA) = \alpha(A, (0)) = 2$ for every $pA \in \operatorname{Spec}_1 A$.

Example 4.5. Let $T = \mathbb{C}[[v, w, x, y, z]]/(vx - w^2, xz - y^2)$. Then, there exists a local unique factorization domain A such that $\widehat{A} = T$, $\alpha(A, (0)) = 2$, and $\alpha(A, pA) = 0$ for every $pA \in \operatorname{Spec}_1 A$.

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