JOURNAL OF COMMUTATIVE ALGEBRA
Volume 10, Number 3, Fall 2018

PROJECTIVE MODULES AND ORBIT SPACE OF UNIMODULAR ROWS OVER DISCRETE HODGE ALGEBRAS OVER A NON-NOETHERIAN RING

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ABSTRACT. For any commutative ring R of Krull dimension zero and for any discrete Hodge algebra D over R, it is proven that, if $n \geq 3$, the group $E_n(D)$ of $n \times n$ elementary matrices acts transitively on $Um_n(D)$, the set of unimodular rows of length n over D.

1. Introduction. Let R be a commutative Noetherian ring of dimension d and $D = R[X,Y]/(XY)$. Let $E_n(D)$ denote the group of $n \times n$ elementary matrices and $Um_n(D)$ the set of unimodular rows of length n over D. In **[2]**, Bhatwadekar and Roy proved the following results:

(i) *$E_n(D)$ acts transitively on $Um_n(D)$ for all $n \geq d+2$;*

(ii) *any projective D-module of rank $\geq d+1$ is cancellative*;

and

(iii) *any projective D-module of rank $\geq d+1$ contains a unimodular element.*

In this paper, we investigate extensions of Bhatwadekar and Roy's result to the case where the base ring R is commutative but not necessarily Noetherian. More precisely, we prove the following results (see Theorems 3.2, 4.7 and 5.1).

Theorem 1.1. *Let R be a finite-dimensional commutative ring and $D = R[X,Y]/(XY)$. Then, $E_n(D)$ acts transitively on $Um_n(D)$ for $n \geq \dim(R)+2$.*

2010 AMS *Mathematics subject classification.* Primary 13B25, 13C10.

Keywords and phrases. Unimodular rows, projective modules, discrete Hodge algebras.

Received by the editors on August 7, 2015, and in revised form on March 29, 2016.

DOI:10.1216/JCA-2018-10-3-435

Theorem 1.2. *Let R be a finite-dimensional commutative ring such that the total quotient ring of R_{red} is an arithmetical ring and $D = R[X,Y]/(XY)$. Then any projective D-module of* $\mathrm{rank} \geq \min\{2\dim(R), \dim(D)\}$ *is cancellative.*

Theorem 1.3. *Let R be a finite-dimensional commutative ring and $D = R[X,Y]/(XY)$. Then, any projective D-module of* $\mathrm{rank} \geq \dim(R[X])$ *contains a unimodular element.*

Note that $D = R[X,Y]/(XY)$ is one of the simplest examples of discrete Hodge algebra over R (D is called a *discrete Hodge algebra over R* if $D = R[X_0,\ldots,X_n]/J$ where J is an ideal of $R[X_0,\ldots,X_n]$ generated by monomials). In **[12, 20]**, Mandal and Wiemers extend the results of **[2]** to discrete Hodge algebras when the base ring is Noetherian. Therefore, it is very natural to ask whether we can extend Theorems 1.1, 1.2 and 1.3 to any discrete Hodge algebra over R. It is equivalent to ask whether Wiemers results (see **[20,** Corollary 4.3]) still hold for non-Noetherian rings R. Unfortunately, the answer to this question is not even known for polynomial algebras with more than one variable.

In **[21]**, Yengui tackled the following, interesting particular case: *For any finite-dimensional commutative ring R and $n \geq \dim(R) + 2$, the group $E_n(R[X])$ acts transitively on $Um_n(R[X])$.* Later, Abedelfatah **[1]** extended it to the polynomial rings $R[X_1,\ldots,X_n]$, when R is a zero-dimensional ring (see Theorem 2.7).

Motivated by the above discussion, we ask the following question.

Question 1.4. *Let R be a zero-dimensional ring and D a discrete Hodge algebra over R. Does $E_k(D)$ act transitively on $Um_k(D)$ for $k \geq 3$?*

In this article, we answer Question 1.4 in the affirmative (Theorem 3.7).

In Section 3, we also prove some results on stably free modules over discrete Hodge algebra when the base ring is non-Noetherian (see Theorems 3.10, 3.11 and Corollary 3.13).

1.1. Some historical motivation. Following the suggestion of the referee, we include in this subsection some historical motivation for some of the questions being considered in this paper.

Let R be a ring. A row $\underline{a} = (a_1, \ldots, a_n) \in R^n$ is said to be *unimodular* (of length n) if the ideal $(a_1, \ldots, a_n) = R$. The point of view of algebraic geometry is to think of elements of R as functions on $X = \text{Spec}(R)$. If R is an affine algebra over $\mathbb{R}$, then, from the topological point of view, elements of R yield functions from $X(\mathbb{R})$ to $\mathbb{R}$, where $X(\mathbb{R})$ denotes the set of real point of X.

If R is regular and X is nonempty, then $X(\mathbb{R})$ is a manifold of dimension equal to the Krull dimension of R. Therefore, R is the algebraic analogue of the ring of continuous functions from $X(\mathbb{R})$ to $\mathbb{R}$. There are, therefore, a large number of useful analogies between topology and algebra leading to questions on projective modules.

Observe that a unimodular row $\underline{a} = (a_1, \ldots, a_n) \in R^n$ yields a continuous function from $X(\mathbb{R})$ to $\mathbb{R}^n - (0, \ldots, 0)$. Two unimodular rows $\underline{a}$ and $\underline{b}$ in R are said to be *homotopic* if there exists a unimodular row $\underline{v}(X)$ in $R[X]$ such that $\underline{v}(0) = \underline{a}$ and $\underline{v}(1) = \underline{b}$. It is easy to see that, if the unimodular rows $\underline{a}$ and $\underline{b}$ in R are homotopic, then the corresponding functions from $X(\mathbb{R})$ to $\mathbb{R}^n - (0, \ldots, 0)$ are homotopic.

It may be presumed that some sort of converse of this result holds, and this is used to relate topology to algebra. For example, suppose that R and $X(\mathbb{R})$ are as above and all continuous functions from $X(\mathbb{R})$ to $\mathbb{R}^n - (0, \ldots, 0)$ are homotopic. Then, it may be conjectured that any two unimodular rows over the corresponding ring R are homotopic.

If Y is a d-dimensional simplicial complex and $n \geq d + 2$, then any two continuous functions from Y to $\mathbb{R}^n - (0, \ldots, 0)$ are homotopic. Therefore, the following algebraic analogue may be considered.

Question 1.5. *Let R be a ring of dimension d. Then, are any two unimodular rows over R of length $n \geq d + 2$ homotopic? In particular, is any unimodular row over R homotopic to* $(1, 0, \ldots, 0)$*?*

It now may be observed that two unimodular rows over R of length n are homotopic, if one can be obtained from the other via the action of $E_n(R)$ (the group of elementary matrices). Therefore, we reformulate the above question as:

Question 1.6. *Let R be a ring of dimension d. Then, can any unimodular row of length $n \geq d + 2$ over R be transformed using the action of $E_n(R)$ to* $(1, 0, \ldots, 0)$*?*

2. Preliminaries.

Notation 2.1. All of the rings considered in this paper are assumed to be of finite dimension and commutative. By dimension of a ring A we mean its Krull dimension, denoted by $\dim(A)$. Modules are assumed to be finitely generated. Projective modules are assumed to have constant rank.

We begin with the following definition.

Definition 2.2. A row $(a_1, \ldots, a_n) \in R^n$ is said to be *unimodular* if there exist some elements $b_1, \ldots, b_n$ in R such that $a_1b_1+\cdots+a_nb_n = 1$. $Um_n(R)$ will denote the set of all unimodular rows $(a_1, \ldots, a_n) \in R^n$.

The group of *elementary matrices* is a subgroup of $Gl_n(R)$, denoted by $E_n(R)$, and it is generated by matrices of the form $E_{ij}(\lambda) = I_n+\lambda e_{ij}$, where $\lambda \in R$, $i \neq j$, $e_{ij} \in M_n(R)$, whose ijth entry is 1 and all other entries are 0.

$E_n(R)$ acts on $Um_n(R)$ in the natural way: if $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in Um_n(R)$, then $(a_1, \ldots, a_n) \sim_{E_n(R)} (b_1, \ldots, b_n)$ means $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)\varepsilon$ for some $\varepsilon \in E_n(R)$.

The next two results are due to Brewer and Costa [**3**] and Lequain and Simis [**9**]. These are generalizations of the Quillen-Suslin theorem [**15, 17**].

Theorem 2.3. *Let R be a commutative ring with* $\dim(R) = 0$. *Then, all projective modules over $R[X_1, \ldots, X_n]$ are free.*

Theorem 2.4. *Let R be a Bezout domain. Then, for every $n \geq 0$, all projective modules over $R[X_1, \ldots, X_n]$ are free.*

Definition 2.5. An integral domain is called a *Bezout domain* if every finitely generated ideal is principal.

The following result is due to Yengui [**21**].

Theorem 2.6. *Let R be a ring of dimension d and $n \geq d+2$. Then, $E_n(R[X])$ acts transitively on $Um_n(R[X])$.*

The next result is due to Abedelfatah [**1**].

Theorem 2.7. *Let R be a zero-dimensional ring and $A = R[X_1, \ldots, X_m]$. Then, for $n \geq 3$, the group $E_n(A)$ acts transitively on $Um_n(A)$.*

We now prove the following lemma. The lemma will be crucially used in Sections 3 and 6.

Lemma 2.8. *Let R be a ring and $I \subset R[X_0, \ldots, X_n]$ an ideal generated by square free monomials with at least one monomial containing X_n and some other variables. Let $J = I \cap R[X_0, \ldots, X_{n-1}]$. Let K be the ideal of $R[X_0, \ldots, X_{n-1}]$ generated by the monomials, say $X_{i_1}X_{i_2}\cdots X_{i_k}$, such that $X_{i_1}X_{i_2}\cdots X_{i_k}X_n$ is a generator of I where $0 \leq i_1 < i_2 < \cdots < i_k \leq n-1$ and L is the ideal of $R[X_0, \ldots, X_{n-1}]$ generated by J and K. Then,*

$$\begin{array}{ccc} R[X_0, \ldots, X_{n-1}]/J & \longrightarrow & R[X_0, \ldots, X_n]/I \\ \downarrow & & \downarrow \\ R[X_0, \ldots, X_{n-1}]/L & \longrightarrow & (R[X_0, \ldots, X_{n-1}]/L)[X_n] \end{array}$$

is the Cartesian diagram of rings.

Proof. Instead of writing the entire proof, we merely give a sketch. Note that the image of the ideal K in $R[X_0, \ldots, X_{n-1}]/J$ is the conductor ideal of the ring extension

$$R[X_0, \cdots, X_{n-1}]/J \hookrightarrow R[X_0, \cdots, X_{n-1}]/I.$$

Therefore, the given diagram in this lemma is a conductor diagram, and hence, it is Cartesian. $\square$

Example 2.9. Let R be a ring and $D = R[X, Y, Z]/(XY, YZ, XZ)$. Then, by Lemma 2.8, we have the following Cartesian diagram.

$$\begin{array}{ccc} R[X, Y]/(XY) & \longrightarrow & R[X, Y, Z]/(XY, YZ, XZ) \\ \downarrow & & \downarrow \\ R & \longrightarrow & R[Z]. \end{array}$$

In this example, we have $K = L = (X, Y)R[X, Y]$.

Definition 2.10. An R algebra D is said to be a *discrete Hodge algebra over R* if D is isomorphic to $R[X_0, \ldots, X_n]/J$, where J is an ideal of

$R[X_0, \ldots, X_n]$ generated by monomials. A discrete Hodge algebra over R is called *trivial* if it is a polynomial algebra over R. Otherwise, it is called a *non-trivial discrete Hodge algebra.*

Example 2.11. $D = R[X_0, \ldots, X_n]/(X_0X_1 \cdots X_n)$ is a discrete Hodge algebra over R.

Example 2.12. Given a simplicial subcomplex Σ of $\triangle_n$ and a ring R, a discrete Hodge algebra $R(\Sigma)$ may be constructed in the following way.

Let $I(\Sigma)$ be the ideal of $R[X_1, \ldots, X_n]$ generated by all square free monomials

$$X_{i_1}X_{i_2} \cdots X_{i_r},$$

with $0 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $\{i_1, \ldots, i_r\}$ not a face of Σ. By $R(\Sigma)$ we denote the discrete Hodge algebra $R[X_1, \ldots, X_n]/I(\Sigma)$.

The next lemma is due to Vorst **[19]**.

Lemma 2.13. *Let R be a ring and Σ a simplicial subcomplex of $\triangle_n$ which is not a simplex. Suppose that $\Sigma_1 = \Sigma \cap \triangle_{n-1}$. Then, there exist subcomplexes $\Sigma_0 \subseteq \Sigma_1$ and $\Sigma_2 \subseteq \Sigma$ such that we have a Cartesian square of rings*

$$\begin{array}{ccc} R(\Sigma) & \xrightarrow{i_1} & R(\Sigma_1) \\ \Big\downarrow{\scriptstyle i_2} & & \Big\downarrow{\scriptstyle j_1} \\ R(\Sigma_2) = R(\Sigma_0)[X_n] & \xrightarrow[j_2]{} & R(\Sigma_0), \end{array}$$

where all of the maps are natural surjections and j_2 is the retraction sending X_n to zero.

Lemma 2.14. *With notation as in Lemma* 2.13*, if P is a projective $R(\Sigma)$-module, then the following diagram*:

$$\begin{array}{ccc} P & \longrightarrow & P \otimes R(\Sigma_1) \\ \Big\downarrow & & \Big\downarrow \\ P \otimes R(\Sigma_0)[X_n] & \longrightarrow & P \otimes R(\Sigma_0) \end{array}$$

is a Cartesian diagram.

The next theorem is the non-Noetherian generalization of the affine Horrocks' theorem **[15, 17]**.

Theorem 2.15. *Let R be a ring and P a projective $R[T]$-module. Suppose that P_f is free for some monic polynomial $f \in R[T]$. Then, P is free.*

Proof. Since any commutative ring is a filtered union of Noetherian commutative rings, the result follows from the original proof. □

3. Main results. In this section, we prove our main results.

Remark 3.1. Let R be a ring and D the ring $R[X,Y]/(XY)$. We continue to denote the classes of X and Y in D by X and Y.

Proposition 3.2. *Let R be a ring and $D = R[X,Y]/(XY)$. Then, all projective D-modules are free in the following cases*:

(i) *R is a zero-dimensional ring.*
(ii) *R is a Bezout domain.*

Proof. Let P be a projective D-module. We consider the following Cartesian diagram:

$$\begin{array}{ccc} R[X] & \longrightarrow & D \\ \downarrow & & \downarrow \\ R & \longrightarrow & R[Y] \end{array}$$

where the horizontal arrows are the inclusion maps and the vertical arrows are the retractions sending X to 0.

(i) If R is a zero-dimensional ring, by Theorem 2.3, $P \otimes_D R[Y]$ $(=P/XP)$ is a free $R[Y]$-module. Therefore, $P \otimes_D R[Y]$ is extended from R, say $P' \otimes_R R[Y] \simeq P \otimes_D R[Y]$, for some projective R-module P'. Hence, the projective D-module P and the projective R-module P' will patch together to give a projective $R[X]$-module Q such that $Q \otimes_{R[X]} D \simeq P$. Since Q is a projective $R[X]$-module, applying Theorem 2.3, we conclude that Q is a free $R[X]$-module. Therefore, P is free.

(ii) In the case when R is a Bezout domain, we exactly follow the proof of (i), only replacing Theorem 2.3 with Theorem 2.4. □

Theorem 3.3. *Let R be a ring of dimension d and $D = R[X,Y]/(XY)$. Then, $E_n(D)$ acts transitively on $Um_n(D)$ for $n \geq d+2$.*

Proof. Let $(f_1, \dots, f_n) \in Um_n(D)$. We must find $\varepsilon \in E_n(D)$ such that $(f_1, \dots, f_n)\varepsilon = (1, \dots, 0)$. We consider the following Cartesian diagram:

$$\begin{array}{ccc} R[X] & \longrightarrow & D \\ \downarrow & & \downarrow \\ R & \longrightarrow & R[Y]. \end{array}$$

Let $(\overline{f}_1, \dots, \overline{f}_n)$ denote the image of $(f_1, \dots, f_n)$ in $(R[Y])^n$. Here, we note that $(\overline{f}_1, \dots, \overline{f}_n) \in Um_n(R[Y])$ with $n \geq \dim(R) + 2$. Therefore, Theorem 2.6 ensures that we indeed can find $\overline{\sigma} \in E_n(R[Y])$ such that $(\overline{f}_1, \dots, \overline{f}_n)\overline{\sigma} = (1, \dots, 0)$.

Let $\sigma \in E_n(D)$ be a lift of $\overline{\sigma}$ and $(f_1, \dots, f_n)\sigma = (g_1, \dots, g_n)$. Then, clearly, $(g_1, \dots, g_n) \in Um_n(D)$ such that $(\overline{g}_1, \dots, \overline{g}_n) = (1, \dots, 0)$ in $R[Y]$. Since the vertical maps are surjective in the above Cartesian square, the unimodular rows $(g_1, \dots, g_n)$ and $(1, \dots, 0)$ over D and R, respectively, will patch up together to give a unimodular row in $R[X]$, say $(h_1, \dots, h_n) \in Um_n(R[X])$.

Applying Theorem 2.6, we can find $\delta \in E_n(R[X])$ such that $(h_1, \dots, h_n)\delta = (1, \dots, 0)$. Therefore,

$$(f_1, \dots, f_n)\sigma\delta = (g_1, \dots, g_n)\delta = (h_n, \dots, h_n)\delta = (1, \dots, 0).$$

This completes the proof. □

We now derive some consequences of the above theorems.

Corollary 3.4. *Let R be a ring of dimension d and $D = R[X,Y]/(XY)$. Then, all finitely generated stably free D-modules of rank $> d$ are free.*

The proofs of the next two corollaries are standard. Here, we provide the proof for the sake of completeness.

Corollary 3.5. *Let R be a zero-dimensional ring and $D = R[X,Y]/(XY)$. Then:*

$$SL_n(D) = E_n(D)$$

for all $n \geq 2$.

Proof. Clearly, $E_n(D) \subset SL_n(D)$. Let $M \in SL_n(D)$. By Theorem 3.3, there exists an elementary matrix $\sigma \in E_n(D)$ such that

$$M\sigma = \begin{pmatrix} 1 & 0 \\ a & M_1 \end{pmatrix}.$$

Applying further a sequence of row transformations brings $M\sigma$ to

$$\begin{pmatrix} 1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where $M_2 \in SL_{n-1}(D)$. The proof now proceeds by induction on n. □

Corollary 3.6. *Let R be a ring of dimension d and $D = R[X,Y]/(XY)$. Then, the canonical map $GL_{d+1}(D) \to K_1(D)$ is surjective.*

Proof. The proof is along the same lines as that of Corollary 3.5. Let $[M] \in K_1(D)$. We want to show that $[M] = [M']$ in $K_1(D)$ for some $M' \in GL_{d+1}(D)$. Without loss of generality, we can assume that $M \in GL_{d+2}(D)$. Applying Theorem 3.3 and then a sequence of row transformations, we can bring M to

$$\begin{pmatrix} 1 & 0 \\ 0 & M' \end{pmatrix},$$

where $M' \in GL_{d+1}(D)$. Hence, $[M] = [M']$ in $K_1(D)$. This completes the proof. □

Now we answer the question raised at the beginning. The theorem is an improvement of Theorem 3.3 to all discrete Hodge algebras over a zero-dimensional ring.

Theorem 3.7. *Let R be a zero-dimensional ring and D a discrete Hodge algebra over R with* $\dim(D) > \dim(R)$. *Then, $E_k(D)$ acts transitively on $Um_k(D)$ for $k \geq 3$.*

Proof. Suppose that $D = R[X_0, X_1, \ldots, X_n]/\mathcal{I}$ where $\mathcal{I}$ is an ideal generated by monomials. Let I be the ideal generated by square free monomials $X_{i_1} \cdots X_{i_k}$, $0 \le i_1 < i_2 < \cdots < i_k \le n$, where $X_{i_1}^{l_1} \cdots X_{i_k}^{l_k} \in \mathcal{I}$ and $l_i \ge 1$.

By [**12,** Proposition 1.3] (also, see [**19**]), $I = I(\Sigma)$ for some simplicial subcomplex Σ of $\triangle_n$. Since ID is a nilpotent ideal of D, it is sufficient to assume that $D = R(\Sigma) = R[X_0, X_1, \ldots, X_n]/I$. We can assume that D is also reduced.

We shall prove that $E_k(D[T_1, \ldots, T_m])$ acts transitively on $Um_k(D[T_1, \ldots, T_m])$ for all $m \ge 0$ using induction on n. If $n = 0$, then D is a polynomial ring over R, and the theorem is due to Abedelfatah (Theorem 2.5). Thus, we will assume that $n \ge 1$. Let $(u_1, \ldots, u_k) \in Um_k(D[T_1, \ldots, T_m])$.

Suppose that D is a non-trivial discrete Hodge algebra. Then, we can assume that there is a monomial in I of the form $X_{i_1} \cdots X_{i_k} X_n$ with $0 \le i_1 < i_2 < \cdots < i_k \le n-1$. The following Cartesian diagram:

$$\begin{array}{ccc}
\frac{R[X_0,\ldots,X_{n-1}]}{J}[T_1,\ldots,T_m] & \longrightarrow & \frac{R[X_0,\ldots,X_n]}{I}[T_1,\ldots,T_m] \\
\downarrow & & \downarrow \\
\frac{R[X_0,\ldots,X_{n-1}]}{L}[T_1,\ldots,T_m] & \longrightarrow & \frac{R[X_0,\ldots,X_{n-1}]}{L}[X_n, T_1,\ldots,T_m]
\end{array}$$

is an extension of the diagram given in Lemma 2.8. Recall that $J = I \cap R[X_0, \ldots, X_{n-1}]$, K the ideal of $R[X_0, \ldots, X_{n-1}]$ generated by the monomials, say $X_{i_1} X_{i_2} \cdots X_{i_k}$, such that $X_{i_1} X_{i_2} \cdots X_{i_k} X_n$ is a generator of I where $0 \le i_1 < i_2 < \cdots < i_k \le n-1$, and L is the ideal of $R[X_0, \ldots, X_{n-1}]$ generated by J and K. We will denote $R[X_0, \ldots, X_{n-1}]/J$ by $R(\Sigma_1)$ and $R[X_0, \ldots, X_{n-1}]/L$ by $R(\Sigma_0)$.

Let $(\overline{u}_1, \ldots, \overline{u}_k)$ be the image of $(u_1, \ldots, u_k)$ in $R(\Sigma_0)[X_n, T_1, \ldots, T_m]$. By the induction hypothesis, $E_k(R(\Sigma_0)[X_n, T_1, \ldots, T_m])$ acts transitively on $Um_k(R(\Sigma_0)[X_n, T_1, \ldots, T_m])$. Therefore, there exists $\overline{\theta} \in E_k(R(\Sigma_0)[X_n, T_1, \ldots, T_m])$ such that $(\overline{u}_1, \ldots, \overline{u}_k)\overline{\theta} = (1, 0, \ldots, 0)$. Let $\theta \in E_k(D[T_1, \ldots, T_m])$ be a lift of $\overline{\theta}$ and $(v_1, \ldots, v_k) = (u_1, \ldots, u_k)\theta$. Clearly, $(\overline{v}_1, \ldots, \overline{v}_k) = (1, 0, \ldots, 0)$, where $(\overline{v}_1, \ldots, \overline{v}_k)$ is the image of $(v_1, \ldots, v_k)$ in $R(\Sigma_0)[X_n, T_1, \ldots, T_m]$. Then, the unimodular rows $(v_1, \ldots, v_k)$ and $(1, 0, \ldots, 0)$ over $D[T_1, \ldots, T_m]$ and $R(\Sigma_0)[T_1, \ldots, T_m]$ will patch up together to give a unimodular row $(w_1, \ldots, w_k) \in Um_k(R(\Sigma_1)[T_1, \ldots, T_m])$.

Again, applying the induction hypothesis, we can find $\sigma \in E_k(R(\Sigma_1)[T_1, \ldots, T_m])$ such that

$$(w_1, \ldots, w_k)\sigma = (1, 0, \ldots, 0).$$

Therefore, $(u_1, \ldots, u_k) \sim_{E_k(D[T_1,\ldots,T_m])} (1, 0, \ldots, 0)$, and hence, we are done. □

Using the techniques of the proofs of Proposition 3.2, Theorem 3.7 and using Theorem 2.3, we can easily derive the next result.

Theorem 3.8. *Let R be a zero-dimensional ring and D a discrete Hodge algebra over R. Then, all projective D-modules are free.*

Corollary 3.9. *Let R be a zero-dimensional ring and D a discrete Hodge algebra over R. Then,*

$$SL_k(D) = E_k(D)$$

for all $k \geq 3$.

Now, we will generalize Corollary 3.4 to all discrete Hodge algebras with certain additional assumptions in the hypothesis.

Theorem 3.10. *Let R be ring of dimension d such that $\dim(R[X]) = d + 1$, and let D be a discrete Hodge algebra over R with $\dim(D) > \dim(R)$. Then, all stably free projective modules over D of rank $> d$ are free.*

Proof. By a similar argument as that in the previous theorem, we can assume that $D = R(\Sigma)$ for some simplicial subcomplex Σ of $\triangle_n$. We prove the theorem using induction on n. If $n = 0$, then D is a polynomial ring over R, and this case is covered by Yengui **[21]**. Therefore, we will assume that $n > 0$.

Let $\Sigma_1 = \Sigma \cap \triangle_{n-1}$ and Σ_0, Σ_2 are subcomplexes of Σ, as in Lemma 2.13. Now, consider the following Cartesian diagram:

$$\begin{array}{ccc} R(\Sigma) & \longrightarrow & R(\Sigma_1) \\ \downarrow & & \downarrow \\ R(\Sigma_0)[X_n] & \longrightarrow & R(\Sigma_0). \end{array}$$

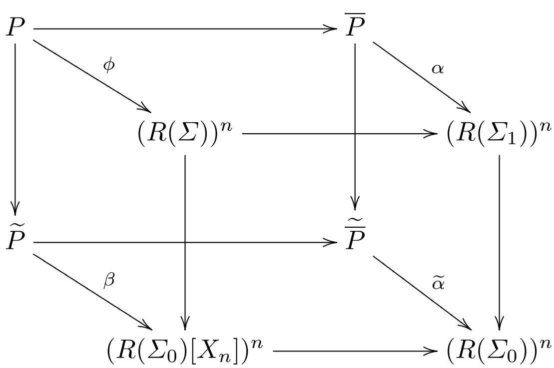
Let P be a stably free D module of rank $n > d$. Write $\overline{P} = P \otimes R(\Sigma_1)$ and $\widetilde{P} = P \otimes R(\Sigma_0)[X_n]$.

By the induction hypothesis, $\overline{P}$ is a free $R(\Sigma_1)$-module of rank n. Fix an isomorphism $\alpha : \overline{P} \to R(\Sigma_1)^n$. Consider the isomorphism $\widetilde{\alpha} : \widetilde{\overline{P}} \to R(\Sigma_0)^n$ induced by α.

Now, consider the projective $R(\Sigma_0)[X_n]$-module $\widetilde{P}$. $R(X_n)$ denotes the localization of $R[X_n]$ at the multiplicative set of all monic polynomials of $R[X_n]$. By [**3,** Lemma 1], $\dim(R(X_n)) = \dim(R[X_n]) - 1 = d$. Then, by the induction hypothesis, the projective $R(X_n)(\Sigma_0)$-module $\widetilde{P} \otimes R(X_n)(\Sigma_0)$ is free. Now, using Theorem 2.15, $\widetilde{P}$ is free, and therefore, extended from $R(\Sigma_0)$, say $Q \otimes_{R(\Sigma_0)} R(\Sigma_0)[X_n] \simeq \widetilde{P}$. However, then $Q \simeq \overline{\widetilde{P}} = \widetilde{\overline{P}}$. Hence, $\widetilde{P} = \widetilde{\overline{P}} \otimes_{R(\Sigma_0)} R(\Sigma_0)[X_n]$. Therefore, the isomorphism $\widetilde{\alpha}$ induced an $R(\Sigma_0)[X_n]$-isomorphism

$$\beta : \widetilde{P} \longrightarrow (R(\Sigma_0)[X_n])^n$$

such that $\overline{\beta} = \widetilde{\alpha}$. Therefore, we have the following fiber product diagram:



with $\widetilde{\alpha} = \overline{\beta}$. By a standard patching argument we have a map $\phi : P \to (R(\Sigma))^n$. Since the inclined arrows α and β are isomorphisms, therefore, ϕ is also an isomorphism. Hence, P is free. $\square$

The next result shows that we can remove the condition $\dim(R[X]) = \dim(R)+1$ in Theorem 3.10, although with a stronger hypothesis on the rank. The proof is along the same lines as in Theorem 3.10; therefore, we do not repeat the proof here.

Theorem 3.11. *Let R be ring of dimension d and D a discrete Hodge algebra over R with* $\dim(D) > \dim(R)$*. Then, all stably free D-modules of rank* $\geq \dim(D)$ *are free.*

Definition 3.12. A ring is said to be *strong S-ring* if, for any two consecutive prime ideals $p \subset q$ in R, the prime ideals $p[X] \subset q[X]$ in $R[X]$ are consecutive.

Let R be of finite type over a Prüfer domain. Then, it was shown in [**11,** Corollary3.6] that R is a strong S-ring.

Corollary 3.13. *Let R be a ring of dimension d of finite type over a Prüfer domain and D a discrete Hodge algebra over R with* $\dim(D) > \dim(R)$*. Then, all stably free modules over D of rank $> d$ are free.*

Proof. Since R is a strong S-ring of dimension d, by [**7,** Theorem 39], $\dim(R[X]) = d + 1$. Therefore, the result follows from Theorem 3.10. □

Corollary 3.14. *Let R be a Bezout domain of dimension d and D a discrete Hodge algebra over R. Then, all projective modules are free.*

4. Cancellation of projective modules.

In this section, we discuss on the cancellative nature of projective modules over the ring $D = R[X,Y]/(XY)$. Note that, if $\dim(R) = n$, then $\dim(D) = \dim(R[X]) \leq 2n + 1$ [**7,** Theorem 38].

In order to prove our main theorem we need the following results. Note that the "bar" will always denote modulo the ideal (X, Y).

Lemma 4.1. *Let R be a ring and $D = R[X,Y]/(XY)$. If all projective $R[X]$-modules are extended from R, then all projective D-modules are extended from R.*

Proof. The proof follows from the standard patching argument. □

Lemma 4.2. *Let R be a ring of dimension n. Suppose that there exists an $a \in \mathrm{Jac}(R)$ such that $\dim(R/aR) < n$. Then*

$$D(a) := \{\mathbf{p} \in \mathrm{Spec}(D) \mid a \notin \mathbf{p}\}$$

and

$$V(a) := \{\mathbf{p} \in \mathrm{Spec}(D) \mid a \in \mathbf{p}\}$$

both have dimension $\leq 2n - 1$.

Proof. The proof follows from [**13,** Lemma 2.2] and the fact that any chain of prime ideals in D is actually a chain either in $R[X]$ or in $R[Y]$. $\square$

The next lemma was proven by Bhatwadekar and Roy [**2**] in the case when the base ring R is Noetherian. However, almost the identical proof works for non-Noetherian rings.

Lemma 4.3. *Let P and Q be two projective $D = R[X,Y]/(XY)$-modules. Let s be an element of R and $S = 1 + sR$. Assume that:*

(i) *there is an isomorphism $\alpha : P_s \to Q_s$ of D_s-modules;*
(ii) *there is an isomorphism $\beta : P_S \to Q_S$ of D_S-modules;*
(iii) *P_{sS} is extended from R_{sS} and $\overline{\alpha_S} = \overline{\beta_s}$.*

Then, there exists an isomorphism $\sigma : P \to Q$ such that $\overline{\sigma_s} = \overline{\alpha}$ and $\overline{\sigma_S} = \overline{\beta}$.

Proof. Since P_{sS} and Q_{sS} are extended from R_{sS} and there is an R-algebra homomorphism $D \to D[T]$ sending X to XT and Y to YT, we can define

$$\tau(X,Y,T) : P_{sS}[T] \longrightarrow P_{sS}[T]$$

by

$$\tau(X,Y,T) = \beta_s(X,Y)(\beta_s^{-1}\alpha_S)(XT,YT).$$

The map τ is an isomorphism and satisfies $\tau(0) = \beta_s$ and $\tau(1) = \alpha_S$. Now, the proof follows from [**14,** Section 2, Lemma 1]. $\square$

Definition 4.4. A commutative ring R is said to be *arithmetical* if, for all ideals I, J and K, $I \cap (J + K) = (I \cap J) + (I \cap K)$.

Remark 4.5. Known examples of arithmetical rings are Bezout rings, like principal or absolutely flat rings.

The next result follows from [**9,** page 171, Theorem B and Remark].

Theorem 4.6. *Let R be an arithmetical ring. Then, all finitely generated projective $R[X_1,\dots,X_n]$-modules are extended from R.*

Now, we are ready to prove our main theorem of this section. We closely follow the proof of Bhatwadekar and Roy [**2,** Theorem 5.3 (i)] and [**13,** Theorem 2.4].

Theorem 4.7. *Let R be a ring of dimension n such that the total quotient ring of $R_{\rm red}$ is arithmetical, and let $D = R[X,Y]/(XY)$. Then, any projective D-module of* rank $\geq \min\{2n, \dim(D)\}$ *is cancellative.*

Proof. We may assume D to be reduced. Let $t = X-Y$. Then, t is a non-zerodivisor of D and $\dim(D/tD) = \dim(R[X,Y]/(XY, X-Y)) = n$.

Let $\alpha : D\oplus P \overset{\sim}{\to} D\oplus Q$ be an isomorphism and $\alpha(1,0) = (a,q)$. Now, applying a result of Heitmann [**6**], we can assume that $(a,q) \equiv (1,0) \pmod{t}$. Therefore, going modulo t, α induces an isomorphism

$$\gamma' : P/tP \longrightarrow Q/tQ.$$

Then, γ' induces an isomorphism

$$\gamma : \overline{P} \longrightarrow \overline{Q}.$$

Let

$$S' = \{s \in S' \mid s \text{ is not contained in any minimal prime ideal of } R\}.$$

Now, it follows from Theorem 4.6 and Lemma 4.1 that $P_{S'}$ and $Q_{S'}$ are extended from $R_{S'}$. Furthermore, we can find an $s \in S'$ such that P_s and Q_s are extended from R_s, and we may lift γ to an isomorphism

$$\theta : P_s \longrightarrow Q_s$$

such that $\overline{\theta} = \gamma_s$.

Let $S = \{1 + sa \mid a \in R\}$. Now, s is in $\mathrm{Jac}(R_S)$ and, in addition, we have $\dim(R_S/sR_S) < n$. Now, using Lemma 4.2, and following the proof of [**13,** Theorem 2.4], we can find an isomorphism

$$\beta : P_S \longrightarrow Q_S$$

such that $\beta = \gamma'$ (mod tD_S), and hence, $\overline{\beta} = \gamma_S$. Note that, in the case of rank $\geq \dim(D)$, we can also conclude the above fact. Finally, the result follows from Lemma 4.3. $\square$

5. Existence of unimodular elements and number of generators of projective modules. In this section, we derive a result on the existence of unimodular elements in projective $R[X, Y]/(XY)$-modules. We also discuss the number of generators for projective modules over the rings $R[X, Y]/(XY)$.

Theorem 5.1. *Let R be a ring and $D = R[X, Y]/(XY)$. Then, any projective D-module of* rank $\geq \dim(R[X])$ *contains a unimodular element.*

Proof. Let P be a projective D-module of rank $\geq \dim(R[X])$. Let bar and tilde denote reduction modulo X and Y, respectively. By [**5,** Theorem 4.6], the projective $R[Y]$-module P/XP has a unimodular element. Let p_1 be a unimodular element of P/XP. Then, $\widetilde{p}_1$ is a unimodular element of projective R-module $P/(X, Y)P$. Again, by [**5,** Theorem 4.6], the map

$$Um(P/YP) \twoheadrightarrow Um(P/(X, Y)P)$$

is surjective. Therefore, there exists a $p_2 \in Um(P/YP)$ such that $\overline{p}_2 = \widetilde{p}_1$.

Since the following square of rings

$$\begin{array}{ccc} D & \longrightarrow & R[X] \\ \downarrow & & \downarrow \\ R[Y] & \longrightarrow & R \end{array}$$

is Cartesian with surjective vertical maps, the unimodular elements p_1 and p_2 of P/XP and P/YP, respectively, will patch up together to give a unimodular element of P. Hence, we are done. $\square$

The next result is a consequence of Corollary 3.4 and Theorem 5.1. The proof is along the same lines as that in [**8,** Theorem 2.6]. We give the proof for the sake of completeness.

Theorem 5.2. *Let R be a ring and $D = R[X,Y]/(XY)$. Let P be a projective D-module. Then, P can be generated by* $(\mathrm{rank}(P) + \dim(R[X]) - 1)$ *elements.*

Proof. Since P is a projective D-module, therefore, P is a direct summand of a free module, say D^n with $n = \mu(P)$ (clearly n is the least one). Write $P \oplus Q_1 \simeq D^n$. Note that $\mathrm{rank}(Q_1) = n - \mathrm{rank}(P)$.

Now, if $\mathrm{rank}(Q_1) \geq \dim(R[X])$, then, by Theorem 5.1, Q_1 has a unimodular element, and therefore, $Q_1 \simeq Q_2 \oplus D$ for some projective D-module Q_2. However, then $P \oplus Q_2 \oplus D \simeq D^n$, which shows that $P \oplus Q_2$ is a stably free module of rank $\geq \dim(R[X]) \geq \dim(R) + 1$. Applying Corollary 3.4, we have $P \oplus Q_2$ is free, which contradicts the minimality of n. Therefore, we have $\mathrm{rank}(Q_1) < \dim(R[X])$. Then, $\mu(P) = \mathrm{rank}(P) + \mathrm{rank}(Q_1) \leq \mathrm{rank}(P) + \dim(R[X]) - 1$. This completes the proof. □

Finally, we have the following consequence.

Corollary 5.3. *Let R be a ring of finite type dimension n over a Prüfer domain and $D = R[X,Y]/(XY)$. Then, the following hold.*

(i) *If P is a projective D-module of* rank $\geq n+1$, *then P has a unimodular element.*
(ii) *If P is a projective D-module of* rank r, *then P can be generated by $n + r$ elements.*

6. Some results in the Noetherian case. In this section, we derive some known results in the circumstance when the ring is Noetherian. In [**20**], Wiemers proved the following result: *Let R be a commutative Noetherian ring of dimension d. Let D be a discrete Hodge algebra over R with* $\dim(D) > \dim(R)$. *Then, $E(D \oplus P)$ acts transitively on $Um(D \oplus P)$.*

Next, we give a proof of his theorem using our technique.

Lemma 6.1 ([**4,** Remark 2.2]). *Let R be a ring and I an ideal of R. Let P be a projective R-module. Then, the natural map*

$$E(R \oplus P) \longrightarrow E\left(\frac{R \oplus P}{I(R \oplus P)}\right)$$

is surjective.

The following result is due to Lindel [**10,** Lemma 1.1].

Lemma 6.2. *Let R be a ring and P a projective R-module of rank r. Then, there exists an $s \in R$ such that the following hold*:

(i) *P_s is free*;
(ii) *there exist $p_1, \ldots, p_r \in P$, $\phi_1, \ldots, \phi_r \in P^*$ such that $(\phi_i(p_j))_{1 \le i,j \le r}$ $=$ diagonal$(s, \ldots, s)$*;
(iii) *$sP \subset p_1R + \cdots + p_rR$*;
(iv) *the image of s in R_{red} is a non-zerodivisor and*
(v) *$(0 : sR) = (0 : s^2R)$.*

The next lemma is due to Dhorajia and Keshari [**4,** Lemma 3.3].

Lemma 6.3. *Let R be a ring and P a projective R-module of rank r. Choose $s \in R$ satisfying the conditions in Lemma* 6.2. *Assume that, if $A = R[X]/(X^2 - s^2X)$, then $E_{r+1}(A)$ acts transitively on $Um_{r+1}(A)$. Then, $E(R \oplus P)$ acts transitively on $Um(R \oplus P, s^2R)$.*

We first prove the following theorem.

Theorem 6.4. *Let R be a commutative Noetherian ring of dimension d. Let D be a discrete Hodge algebra over R with* $\dim(D) >$ $\dim(R)$. *Then, $E_n(D)$ acts transitively on $Um_n(D)$ for all $n \ge d + 2$.*

Proof. The proof of the theorem is similar to that of Theorem 3.7; therefore, we merely give a sketch. Suppose that $D = R[X_0, X_1, \ldots, X_n]/I$, where I is an ideal generated by monomials. Let J be the ideal generated by square free monomials $X_{i_1} \cdots X_{i_k}$, $0 \le i_1 < i_2 < \cdots < i_k \le n$, where $X_{i_1}^{l_1} \cdots X_{i_k}^{l_k} \in I$ and $l_i \ge 1$.

By [**12,** Proposition 1.3], $J = I(\Sigma)$ for some simplicial subcomplex Σ of $\triangle_n$. Since JD is a nilpotent ideal of D, it is sufficient to assume that $D = R(\Sigma)$. In addition, we can assume that D is also reduced.

We shall prove the result for the polynomial ring $D[T_1, \ldots, T_m]$ for all $m \geq 0$ using induction on n. If $n = 0$, then D is merely R, due to Suslin [**18**]. Thus, we will assume $n \geq 1$. Let $(u_1, \ldots, u_k) \in Um_k(D)$.

Now, consider the following Cartesian diagram.

$$\begin{array}{ccc} R(\Sigma_1)[T_1, \ldots, T_m] & \longrightarrow & R(\Sigma)[T_1, \ldots, T_m] \\ \downarrow & & \downarrow \\ R(\Sigma_0)[T_1, \ldots, T_m] & \longrightarrow & R(\Sigma_0)[X_n, T_1, \ldots, T_m] \end{array}$$

The remainder is similar to Theorem 3.7. □

Now, using the technique of [**4,** Theorem 3.4] we prove the following result.

Theorem 6.5. *Let R be a commutative Noetherian ring of dimension d. Let D be a discrete Hodge algebra over R with* $\dim(D) >$ $\dim(R)$. *Let P be a projective D-module of* rank $n \geq d + 1$. *Then, $E(D \oplus P)$ acts transitively on $Um(D \oplus P)$.*

Proof. We may assume D to be reduced. We prove the theorem by induction on d. If $d = 0$, then P is free. Thus, we are done by the previous theorem. Let $d > 0$ and $(a, p) \in Um(D \oplus P)$.

Choose a non-zerodivisor $s \in R$ satisfying the conditions in Lemma 6.2. Let bar denote reduction modulo s^2D. Since $\dim(R/s^2R) \leq d - 1$, by the induction hypothesis, there exists a $\overline{\sigma} \in E(\overline{D} \oplus \overline{P})$ such that $(\overline{a}, \overline{p})\overline{\sigma} = (1, 0)$. From Lemma 6.1, we can lift $\overline{\sigma}$ to $\sigma \in E(D \oplus P)$. If $(a, p)\sigma = (b, q)$, then $(b, q) \in Um(D \oplus P, s^2D)$.

Let $R_1 = R[T]/(T^2 - s^2T)$ and $S = R(\Sigma)[T]/(T^2 - s^2T)$. Then, $\dim(R_1) = d$ and $S = R_1(\Sigma)$. Now, by Theorem 6.5, $E_{d+2}(S)$ acts transitively on $Um_{d+2}(S)$.

Finally, applying Lemma 6.3, there exists a $\theta \in E(D \oplus P)$ such that $(b, q)\theta = (a, p)\sigma\theta = (1, 0)$. This proves the theorem. □

Corollary 6.6. *Let R be a Noetherian commutative ring of dimension d. Let D be a discrete Hodge algebra over R with* $\dim(D) > \dim(R)$*. Then, any projective D-module of* $\operatorname{rank} n \geq d+1$ *is cancellative.*

Acknowledgments. I sincerely thank Mrinal Kanti Das for many useful discussions. I sincerely thank the referee for the historical motivation in the introduction.

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